Synchronization Stability Analysis of Chaotic Oscillators

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Abstract

This paper proposes the use of symbolic computation tools for the parametric analysis of the synchronization stability of chaotic oscillators. Moreover, some concepts of synchronization recently proposed in the literature are briefly presented.

1 Introduction

The study of synchronization of coupled systems dates from the works of Huygens on coupled pendulums (1673). Since then, these concepts have been applied to a broad class of systems. In the beginning of the 20th century the occurrence of synchronization phenomena in electric and electromechanical systems has been reported. Recently, it has been shown how chaotic oscillators could be synchronized [2]. Nowadays, there are many applications of synchronized chaotic oscillators, such as: secure communication [6], control of chaotic systems [1], biological systems [7].

Synchronization could be roughly defined as the mutual time conformity of two or more processes characterized by the relations of some functionals of the referred processes. The synchronization phenomena can be viewed as the tendency toward self-organization in complex systems [2]. This self-organization could assume different forms, implying a diversity of synchronization types. Once the synchronization is defined the major problem is to establish the existence and the stability of the solutions of the respective synchronous motion. In the design of coupled systems, this implies the determination of the connection parameters that lead to the required motion.

In this sense, symbolic computation tools can be used to the determination of a range of parametric variation that assures the stability of synchronized chaotic oscillators. This can be done through the eigenvalue analysis of matrices obtained from invariant manifolds of the synchronous motion, as proposed in [4]. This paper is organized as follows: in section 2 a definition of synchronization is stated, in section 3 the analysis of the stability is addressed and the use of symbolic computation tools is proposed; an application is presented in section 4, leading to the conclusions in section 5.

2 Synchronization

The earlier studies on the synchronization of oscillators were restricted to the analysis of periodic motions. As the analysis of complex behavior (chaos, quasiperiodic oscillations) become current, different types of synchronization have been investigated: phase, identical, generalized, lag [3]. In despite of these many types of synchronization, some basic requirements are common:

- division of the global system in subsystems;
- measurement method for the specific properties of the subsystems trajectories;
- a method for the comparison of these properties;
- a criterion that allows the verification of the mutual time conformity.

Consider the following dynamical system:

$$\begin{cases} \dot{x} &= F_1(x,y,t) \\ \dot{y} &= F_2(x,y,t) \end{cases} \tag{1}$$

where $x \in X \subset \Re^{d_1}$, $y \in Y \subset \Re^{d_2}$ and $t \in \Re$. The space of all trajectories is defined by $Z = X \times Y$ and the global trajectory is denoted by $\Phi(z)$. The trajectory properties of each subsystem are defined by the functionals:

$$g_x : X \times \Re \rightarrow \Re^k \;\; ; \;\; g_y : Y \times \Re \rightarrow \Re^k$$

An example of functional commonly used is: $g_x = x(t)$. The comparison of the functionals is made by the function $h : \Re^k \times \Re^k \rightarrow \Re^k$, that is called comparison function. By using these functionals a synchronization definition can be given [3]:

Definition: The subsystems of equation (1) are synchronized on the trajectory $\Phi(z)$, with respect to the properties g_x, g_y and synchronization norm $\| \bullet \|_s$ if

$$\|h[g_x, g_y]\|_s = 0$$

The major point about this definition is the appropriated choice of the functionals g_x, g_y and the synchronization norm. The appropriated choice of the functionals and synchronization norm depends upon the required type of synchronization. Transients could occur on the process, making necessary to take into account the effect of perturbations and, in certain cases, a tolerance of deviation must be included. In the majority of chaotic applications the required synchronization is the one referred as identical. In this case the functionals are given by $g_x = x(t)$ and $g_y = y(t)$. The comparison function normally used is given by:

$$h[g_x, g_y] = g_x - g_y$$

and the synchronization norm is:

$$\|h[g_x, g_y]\|_s = \lim_{t \to \infty} \|h[g_x, g_y]\|$$

where $\| \bullet \|$ is the Euclidian norm. This choice of norm is suitable to the case of identical systems with chaotic behavior, such as the systems treated in this paper.

When the initial conditions of the subsystems are distinct and there is no coupling, the oscillations are not correlated. However, for a suitable choice of coupling parameters, the oscillations tend to be identical, thus occurring the synchronization.

3 Stability

The synchronization stability of identical subsystems coupled in a master-slave configuration exhibiting chaotic behavior is now analized through the method presented in [4], based on a criterion that assures the stability of the synchronous motion under small perturbations.

3.1 Stability Criterion

Two identical subsystems coupled in a master-slave configuration can be described by the following set of differential equations:

$$\begin{cases} \dot{x} &= F(x,t) \\ \dot{y} &= F(y,t) + E(x-y) \end{cases} \tag{2}$$

where $x, y \in \Re^n$ and E is a function that represents the coupling.

When synchronization occurs, for distinct initial conditions, the trajectory $y(t)$ converges to $x(t)$, that is $w(t) \triangleq y(t) - x(t) \rightarrow 0$. To evaluate the stability under small perturbations, a linearization around $w = 0$ is performed, leading to:

$$\dot{w} = [D_F(x,t) - D_E(0)]w$$

where $D_F(x,t)$ is the Jacobian of $F(x,t)$ evaluated at a trajectory x and $D_E(w)$ is the Jacobian of the coupling E. This time variant linear system is stable if

$$\lim_{t \to \infty} \|w(t)\| = \lim_{t \to \infty} \|y(t) - x(t)\| = \|h\|_s = 0 \tag{3}$$

holds for all possible trajectories $x(t)$ of the master system. As proposed in [4], this condition can be investigated through the following matrix A

$$A \triangleq < D_F > - D_E(0) \tag{4}$$

where $< \bullet >$ is the time average measured on a synchronous trajectory, given by:

$$< D_F > = \lim_{t \to \infty} \frac{1}{t} \int_0^t D_F[x(\beta); \beta] \, d\beta$$

In the case of chaotic oscillators a suitable choice of invariant manifold measure is the Sinai-Bowen-Ruelle (**SBR**). This measure is computed from an arbitrary trajectory embedded in the chaotic attractor. Another option is the computation of trajectories measures from fixed points and Unstable Periodic Orbits (**UPO**) on the chaotic attractor, that could suply informations about the stability of synchronization. This approach overcomes some numerical problems that could occur from the computation of the **SBR** measure. On the other hand, it presents the difficulty of the determination and localization of appropriated **UPO**'s [4].

In this paper the **SBR** measure is applied because it is generic and dismiss the application of **UPO** localization tools [11]. Once the measure to be used is defined, the analysis relies on the computation of the eigenvalues of A as a function of the coupling parameters. Note that the **SBR** measures exist whenever the system is ergodic ($< x >$ is invariant with t) [9], which is assumed in the paper.

It has been shown in [4] that equation (3) holds if the real part of the eigenvalues of A is negative. To establish a parameter range in which the synchronization stability is assured some methods originally used to local bifurcation analysis (associated to eigenvalues with null real part) can be applied. With these methods, based on symbolic computation, it is possible to compute in the parameter space the points for which the real part of the eigenvalues changes its sign. By determining these turning points the parameter space can be divided into regions for which the sign of the real part of the eigenvalues is invariant.

3.2 Parametric Analysis

The method proposed is based on the analysis of the characteristic polynomial coefficients of the matrix A. Let the characteristic polynomial of A be denoted by:

$$p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

As presented in [8], it is possible to derive necessary and sufficient conditions for the existence of eigenvalues with null real part by manipulating the coefficients of $p(\lambda)$. By applying these conditions it is possible to determine, in the parameter space, the edges (eigenvalues with null real part) between regions that correspond to eigenvalues with distinct sign of the real part. To determine the sign of the real part of the eigenvalues inside a region, the computation of the eigenvalues of A at a single point in the parameter space is required, assuring that the sign of the real part of the eigenvalues inside the interval is the same. There are two situations for which the real part of the eigenvalue is null: a null eigenvalue and a pair of purely imaginary eigenvalues. In the case of the null eigenvalue the necessary and sufficient condition is given by $c_0 = 0$. In the case of a purely imaginary pair the conditions are related to the order of the system n, as presented in table 1.

The application of the method can be summarized in the following steps:

$n = 3$	$c_0 - c_1c_2 = 0$ $c_1 > 0$
$n = 4$	$c_0c_3^2 - c_1c_2c_3 + c_1^2 = 0$ $c_1c_3 > 0$
$n = 5$	$(c_2 - c_3c_4)(c_1c_2 - c_0c_3)$ $+c_1c_4(c_1c_4 - 2c_0) + c_0^2 = 0$ $(c_2 - c_3c_4)(c_0 - c_1c_4) > 0$

Table 1: Necessary and sufficient conditions for the occurrence of a pair of purely imaginary eigenvalues as a function of the coefficients of the characteristic polynomial of matrix A for $n = 3$, $n = 4$ and $n = 5$ (order of matrix A).

1. Construction of matrix A
2. Computation of de **SBR** measures of $< D_F >$ obtained from the time simulation of the master subsystem.
3. Determination of the coefficients of the characteristic polynomial $p(\lambda)$ as a function of the coupling parameters.
4. Determination of the edges between regions by applying the conditions to the occurrence of eigenvalues with null real part.
5. Numeric computation of the sign of the real part of the eigenvalues of A in the interior of each region.

The method proposed here shows to be more suitable than the Routh-Hurwitz criterion for the analysis of multiparametric systems with order greater then $n = 3$ and eventually with nonlinear coupling, since only two constraints must be satisfied. The multiparametric analysis carried out through the Routh-Hurwitz criterion requires the examination of all coefficients of the Routh table [5].

3.3 Example of Application

In this section, the parametric analysis of the synchronization stability of two identical coupled subsystems is addressed. The subsystem (the Rössler oscillator [10]) is described by the following set of differential equations:

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& -x_2 - x_3 \\ \dot{x}_2 &=& x_1 + ax_2 \\ \dot{x}_3 &=& b + x_3(x_1 - c) \end{array} \right. \quad (5)$$

where a, b and c are the system parameters. In the application addressed in this paper these parameters are $a = 0.2$, $b = 0.2$ and $c = 9$ (these values assure that the system behaves chaotically [4]). The phase planes for these values are presented in figures 1, 2 and 3, with $x(0) = 0$.

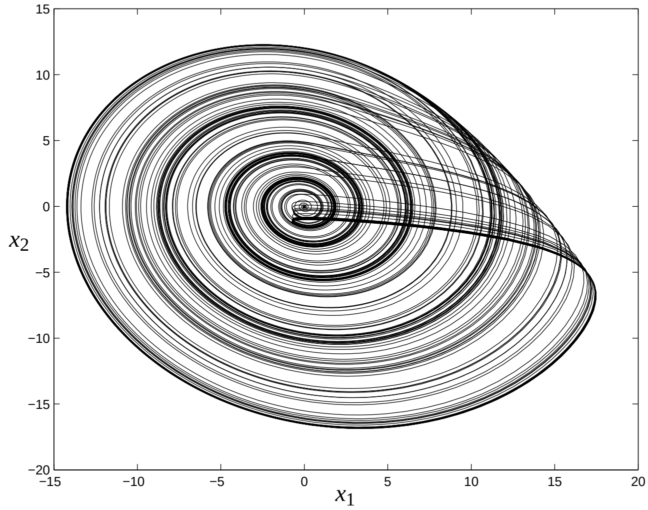

Figure 1: Phase plan $x_1 \times x_2$.

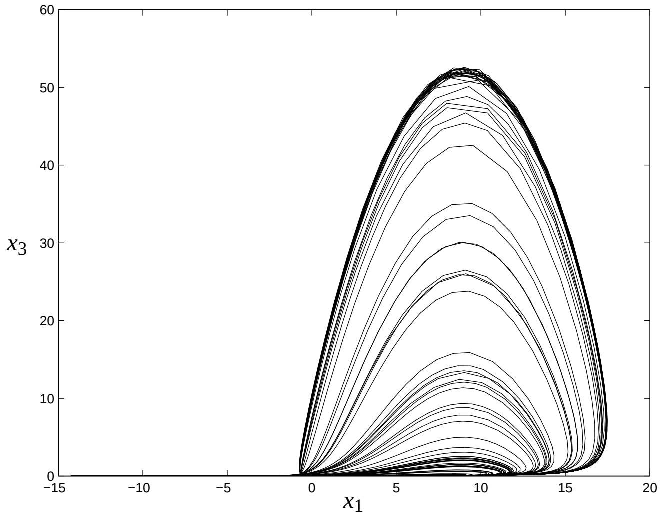

Figure 2: Phase plan $x_1 \times x_3$.

A linear coupling is considered

$$E(x-y) = \begin{bmatrix} e_1(x_1-y_1) \\ e_2(x_2-y_2) \\ e_3(x_3-y_3) \end{bmatrix}$$

yielding e_1, e_2 and e_3 as the parameters under analysis.

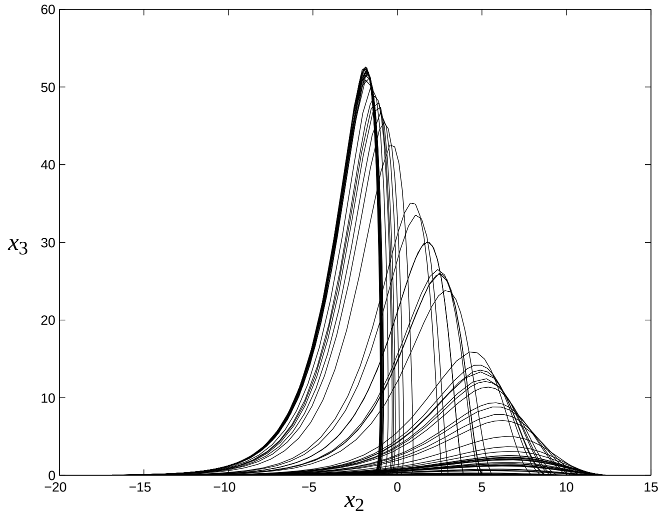

Figure 3: Phase plan $x_2 \times x_3$.

3.4 Application of the Procedure

1. The first step consists on the determination of matrix A. By applying (4), one gets:

$$A = \begin{bmatrix} -e_1 & -1 & -1 \\ 1 & a-e_2 & 0 \\ <x_3> & 0 & <x_1> - c - e_3 \end{bmatrix}$$

2. From the numerical integration of the master system, the following **SBR** measures are obtained for the state variables:

$$<x_1>= 0.163\ ;\ \ <x_2>= -0.815$$

$$<x_3>= 0.815$$

3. Determination of the characteristic polynomial coefficients as a function of the coupling parameters (performed by using the software Maple V).

4. The conditions for the existence of eigenvalues with null real part are:

   - Null eigenvalue:

$$e_3 + 8.84e_1e_2 + e_1e_2e_3 + 8.67 - 0.2e_1e_3$$
$$+ 0.815e_2 - 1.77e_1 = 0$$

   - Purely imaginary pair:

$$e_3 + 8.84e_1e_2 + e_1e_2e_3 + 8.67 - 0.2e_1e_3$$
$$+ 0.815e_2 - 1.77e_1 - (8.84e_2 + e_1e_2$$

$$+8.64e_1+e_1e_3+e_2e_3$$
$$+0.0476-0.2e_3)(8.64+e_3+e_1+e_2)=0$$

$$8.84e_2+e_1e_2+8.64e_1+e_1e_3$$
$$+e_2e_3+0.0476-0.2e_3>0$$

An analysis of the system behavior could be carried out by assuming that the coupling is performed through only one of the state variables. In this case, one parameter is chosen as a free variable while the others are fixed in zero. Thus, the following turning points of the signs of the real part of the eigenvalues of A are obtained:

$$e_2=e_3=0 \;\begin{cases} e_1=0.107 \\ e_1=4.91 \end{cases}$$

$$e_1=e_3=0\;;\; e_2=0.108$$

$$e_1=e_2=0 \;\begin{cases} e_3=-8.59 \\ e_3=-4.81 \\ e_3=-8.67 \end{cases}$$

5. By computing numerically the eigenvalues of A for points inside each region and verifying the sign of the real part of each eigenvalue, the following parameter ranges for which all the eigenvalues have negative real part are obtained:

   (a) $e_1 \in (0.107, 4.91)\;;\; e_2=e_3=0$

   (b) $e_2 \in (0.108, \infty)\;;\; e_1=e_3=0$

   (c) $e_3 \in (-8.59, -4.81)\;;\; e_1=e_2=0$

3.5 Analysis of the Results

The method of analysis presented here is based on first order approximation, indicating that the conditions do not assure the synchronization stability inside the chaotic attractor for an arbitrary initial condition of the subsystems. The parameter ranges obtained were tested by numerical integration. From the simulation results, it has been verified that the system does not reach a stable synchronous motion for the case **(c)** ($e_3 \in (-8.5, -4.8)$, $e_1=e_2=0$). Inside this interval the feedback through the state variable y_3 induces the destruction of the chaotic attractor of the slave subsystem (see figure 4, bottom). In cases **(a)** and **(b)** the results obtained from the simulation confirm that the

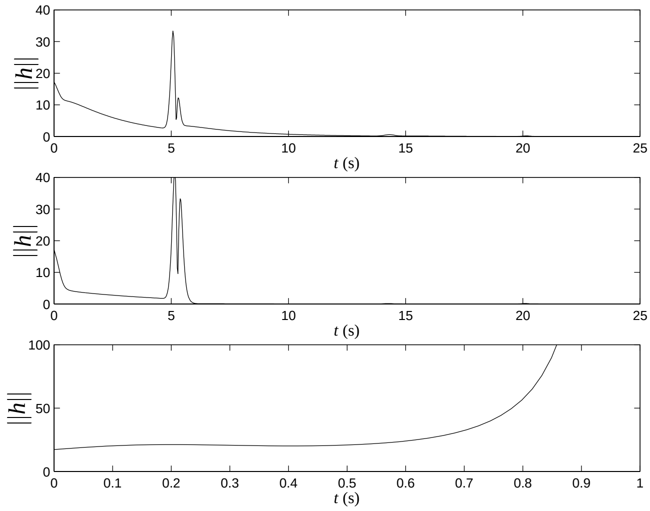

Figure 4: Time evolution of $||h||=||y-x||$ with initial conditions $x(0)=[\,10\;10\;10\,]'$, $y(0)=0$, in the cases **(a)** (top, with $e_2=e_3=0$ and $e_1=3$), **(b)** (middle, with $e_1=e_3=0$ and $e_2=5$) and **(c)** (bottom, with $e_1=e_2=0$ and $e_3=-6$).

synchronization is attained (see figure 4, top and middle).

In figure 5 the phase planes $x\times y$ are represented for the initial conditions $x(0)=[\,10\;10\;10\,]'$ and $y(0)=0$, with the coupling parameters $e_1=e_3=0$ and $e_2=5$. It can be verified that the trajectories of the slave subsystem converge to the trajectories of the master system (i.e. to the straight line $y=x$ in the phase plan).

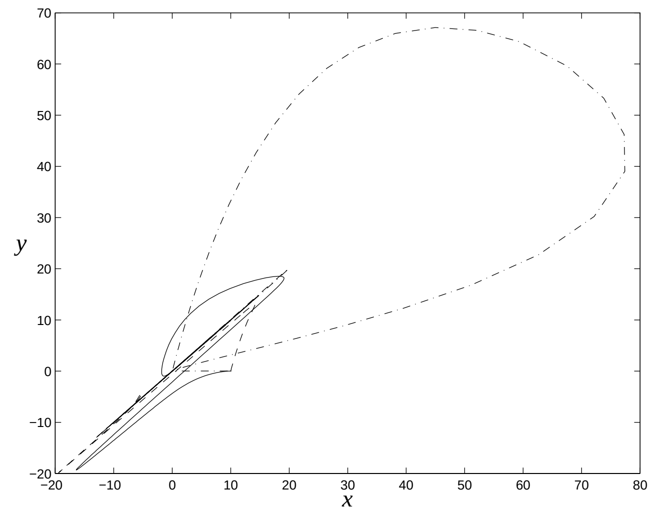

Figure 5: Trajectories in the phase plan $x\times y$ with the initial conditions $x(0)=[\,10\;10\;10\,]'$, $y(0)=0$, and the parameters $e_1=0$, $e_2=5$ and $e_3=0$. $x_1\times y_1$ (solid), $x_2\times y_2$ (dashed) and $x_3\times y_3$ (dash-dot).

The absolute value of the real part of the eigenvalues of matrix A can be used as a first criterion for the suitable choice of the coupling parameters, since the eigenvalues determine the rate of convergence of the synchronous motion.

4 Conclusion

The use of symbolic computation in association with standard analysis methods has proved to be a powerful tool for the study of synchronization of chaotic oscillators connected in a master-slave configuration.

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References

[1] E. Barreto, J. Kostelich, C. Grebogi, E. Ott, and J. A. Yorke. Efficient switching between controlled unstable periodic orbits in higher dimensional chaotic systems. *Physical Review*, 55(5):4169–4172, 1995.

[2] I. I. Blekhman, P. S. Landa, and M. G. Rosenblum. Synchronization and chaotization in interacting dynamical systems. *Applied Mechanics Review*, 48(11):733–752, November 1995.

[3] R. Brown and L. Kocarev. A unifying definition of synchronization for dynamical systems. Research Report 9811013, URL = http://xxx.if.usp.br/chao-dyn, Los Alamos National Laboratory, November 1998.

[4] R. Brown and N. F. Rulkov. Synchronization of chaotic systems: transverse stability of trajectories in invariant manifolds. *Chaos*, 7(3):395–413, 1997.

[5] C. T. Chen. *Linear System Theory and Design*. Oxford University Press, Oxford, 1999. 3rd Edition.

[6] L. O. Chua, T. Yang, G. Zhong, and C. W. Wu. Synchronization of Chua's circuits with time-varying channels and parameters. *IEEE Transactions on Circuits and Systems — I: Fundamental Theory and Applications*, 43(10):862–868, October 1996.

[7] A. S. Dmitriev, M. Shirikov, and S. O. Starkov. Chaotic synchronization in ensembles of couple maps. *IEEE Transactions on Circuits and Systems — I: Fundamental Theory and Applications*, 44(10):918–926, October 1997.

[8] J. Guckenheimer, M. Myers, and B. Sturmfels. Computing Hopf bifurcations I. *SIAM Journal on Numerical Analysis*, 34(1):1–21, February 1997.

[9] D. Ruelle. *Chaotic evolution and strange attractors: the statistical analysis of time series for deterministic nonlinear systems*. Cambridge University Press, New York, 1989.

[10] R. Seydel. *Practical Bifurcations and Stability Analysis*. Springer-Verlag, New York, 1994. 2nd Edition.

[11] S. M. Zoldi and H. S. Greenside. Spatially localized unstable periodic orbits of a high-dimensional chaotic system. *Physical Review E*, 57(3):2511–2514, March 1998.