# Quantum Control Techniques for Switched Electrical Networks

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**Abstract**

We illustrate two techniques for specifying piecewise constant controls for a switched electrical network used in converting power in a dc-dc converter. Both procedures make use of decompositions of $SU(2)$ to obtain controls that satisfy pulse area constraints. One of the methods depends intrinsically on the network and we show that ideal controls can be obtained with this technique. The other approach is general, but requires more computational effort. The systems being studied are single input systems with drift. In this paper no approximations or other artifices are used to remove the drift. Instead, the drift is important in the determination of the controls. Piecewise contant controls, such as the ones obtained in this work, are closer to physical reality than other types of controls. We also show how to obtain controls with values of either 0 or 1 which represent the position of the switch.

## 1. Introduction

Decompositions of an element of $SU(2)$, the set of $2 \times 2$ unitary matrices with determinant equal to one, can be used to control various systems[1], [2]. Examples of such systems include two-level systems controlled by piecewise constant controls, two-level and N-level systems controlled using sinusoidal controls combined with approximations, and mechanical systems such as satellites, underwater vehicles, and switched electrical networks. This paper focuses on switched electrical networks having state spaces on $SO(3)$, the set of $3 \times 3$ orthogonal matrices with determinant one[3], [4], [5].The Lie groups $SO(3)$ and $SU(2)$ are closely related and we use this fact to convert a system on $SO(3)$ to a system on $SU(2)$, and this explains the title of this paper [1]. By using methods adapted for quantum control, we manipulate a classical system.

The switched electrical network example discussed in this paper is the one studied by Leonard and Krishnaprasad[3]. This switched electrical network has its state transition matrix in $SO(3)$. Given a target $G \in SO(3)$,we translate $G$ to a target $S$ for the associated system on $SU(2)$, and illustrate two techniques that determine decompositions of $S$ for obtaining controls for the switched electrical network. Both methods represent $S$ as a product of factors. Each factor is the exponential of a linear combination of $A$ and $B$, two linearly independent elements of $su(2)$, the set of $2 \times 2$ anti-Hermitian matrices with zero trace. The coefficients of $A$ and $B$, which describe the time and control inputs of the system, have constraints placed on them. With the use of decompositions of $S$, piecewise constant controls for the system are obtained.

The first method depends intrinsically on the structure of the matrices $A$ and $B$. It turns out that the associated system in $SU(2)$ has its drift matrix of the form $A = ai\sigma_x + bi\sigma_y$ and its control vector field $B = ci\sigma_x + di\sigma_y$, where $a, b, c, d \in \mathbb{R}$ and $\sigma_x, \sigma_y$ are the Pauli matrices given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In the second method, a further translation is carried out from $S$ to a target for a new system in $SU(2)$ that has drift $\hat{A} = \frac{i}{2}\sigma_z$, where

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and control vector field $\hat{B} = \frac{i}{2}\sigma_y$. Although the second method involves more arduous calculations than the first method, it applies to any $A, B \in su(2)$.

Systems such as dc-dc switchmode power converters, in which switched electrical networks have a significant part, can be implemented in communication and data handling sytems, portable battery-operated equipment and other applications. Thus, the results of this paper have useful consequences for these applications.

The methods for finding controls for switched electrical networks described in this paper provide piecewise constant controls without using approximations or optimization calculations. Piecewise constant controls are closer to actual physical reality than other types of controls. Other strategies for controlling switched electrical networks use state-space averaging, for example, Leonard and Krishnaprasad[3] transform these systems into 'drift free' systems and then apply

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averaging theory on Lie groups to specify small amplitude, periodic, open-loop controls. The approach of Sira-Ramirez[6], based on variable structure systems theory and sliding regimes, provides feedback controls for switched electrical networks. In contrast, we obtain piecewise constant controls by construction that can be taken to be the ideal controls of 0 and 1 which represent the position of the switch. From the results of Jurdjevic and Sussmann[7] we know that the controls can be exactly 0 and 1 to prepare $G$. Furthermore, we do not resort to techniques for driftless systems by either, one, removing the drift via approximations or other methods which work only in fortuitous situations or, two, by making use of periodicity. Arguments relying on periodicity are invalid in general[1] and can lead to expensive controls even when valid. In this paper the only time periodicity is used is to rewrite free evolution terms with negative drift coefficients as free evolution terms with positive drift coefficients.

In section 2 we describe the model for switched electrical networks evolving on $SO(3)$, give the details about the network example, and explain the translation of a system on $SO(3)$ to a system on $SU(2)$. We obtain controls for our example in section 3 using method 1(section 3.1) with which general piecewise constant controls are provided(section 3.2) as well as ideal controls, i.e., controls with values of 0 and 1,(section 3.3) and also method 2 is discussed(section 3.4). Conclusions are given in section 4.

## 2. Switched Electrical Networks

In this paper we consider a switched electrical network with three circuit elements and no external constant power sources. Such systems can be modeled by the following[3]:

$$\frac{d}{dt}\mathbf{x} = \left(\tilde{A} + \tilde{B}u\right)\mathbf{x}, \qquad (1)$$

where $\mathbf{x} \in \mathbb{R}^3$ is the state vector of the system and the control input, $u$, is a scalar signal that takes the values of 0 or 1. Matrices $\tilde{A}$ and $\tilde{B}$ describe the drift and control coupling parts of the system respectively and are elements of the Lie algebra $so\,(3)$ which is defined by

$$so\,(3) = \left\{V \in \mathbb{R}^{3\times 3} \left| V^T = -V \right.\right\}. \qquad (2)$$

As long as $u$ remains constant, at a time $t$, the state of the system can be written as

$$\mathbf{x}\,(t) = e^{\left(\tilde{A}+\tilde{B}u\right)t}\mathbf{x}\,(0)\,. \qquad (3)$$

We assume that $\tilde{A}$, $\tilde{B}$ and the Lie bracket $\left[\tilde{A}, \tilde{B}\right]$ form a basis of $so\,(3)$, where $[\cdot,\cdot]$ is the matrix commutator. Thus, the Lie algebra generated by $\tilde{A}$ and $\tilde{B}$ is $so\,(3)$. Hence, this system's state transition matrix $e^{\left(\tilde{A}+\tilde{B}u\right)t}$ is an element of the Lie group $SO\,(3)$. The orthogonal state transition matrix acting on the initial state vector of the system $\mathbf{x}(0) \in \mathbb{R}^3$ maintains the vector's length so that the evolution of $\mathbf{x}$ lies on a sphere in $\mathbb{R}^3$. For our circuit this corresponds to energy conservation.

The switched electrical network that we examine is identical to the one used by Leonard, Krishnaprasad and Wood[3], [4]. This network consists of two capacitors $C_1$ and $C_2$ with corresponding voltages $V_1$ and $V_2$. These capacitors are connected by a switch $u$ and an inductor $L_3$ with current $I_3$. The control objective is to transfer energy from $C_1$ to $C_2$ by means of the inductor while satisfying performance specifications.

The following are the equations for the network:

$$\begin{aligned} C_1\frac{d}{dt}V_1 &= (1-u)\,I_3 \\ C_2\frac{d}{dt}V_2 &= uI_3 \\ L_3\frac{d}{dt}I_3 &= -(1-u)\,V_1 - uV_2. \end{aligned} \qquad (4)$$

Define the network state vector $\mathbf{x} = (x_1, x_2, x_3)^T$ by $x_1 = \sqrt{C_1}V_1$, $x_2 = \sqrt{C_2}V_2$, and $x_3 = \sqrt{L_3}I_3$. Then the above equations become

$$\begin{aligned} \frac{d}{dt}x_1 &= \frac{(1-u)}{\sqrt{C_1L_3}}x_3 \\ \frac{d}{dt}x_2 &= \frac{u}{\sqrt{C_2L_3}}x_3 \\ \frac{d}{dt}x_3 &= -\frac{(1-u)}{\sqrt{L_3C_1}}x_1 - \frac{u}{\sqrt{L_3C_2}}x_2. \end{aligned} \qquad (5)$$

Let $\omega_1 = 1/\sqrt{C_1L_3}$ and $\omega_2 = 1/\sqrt{C_2L_3}$. Thus in matrix form the system can be written as

$$\frac{d}{dt}\mathbf{x} = \begin{pmatrix} 0 & 0 & \omega_1(1-u) \\ 0 & 0 & \omega_2 u \\ -\omega_1(1-u) & -\omega_2 u & 0 \end{pmatrix}\mathbf{x}. \qquad (6)$$

Now we consider two elements of the standard basis of $so\,(3)$ given by

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Thus equation (6) can be written as

$$\begin{aligned} \frac{d}{dt}\mathbf{x} &= \left[\omega_1A_2 + (-\omega_1A_2 - \omega_2A_1)\,u\right]\mathbf{x} \\ &= \left(\tilde{A} + \tilde{B}u\right)\mathbf{x}, \end{aligned} \qquad (7)$$

where $\tilde{A} = \omega_1A_2$ and $\tilde{B} = -(\omega_1A_2 + \omega_2A_1)$.

To use $SU\,(2)$ control techniques for this network system, the following facts are used:

- Since an element of $su\,(2)$ is a linear combination of the Pauli matrices $\sigma_x$,$\sigma_y$, and $\sigma_z$, the transformation $R_U : su(2) \longrightarrow su(2)$ defined by $R_U(A) = UAU^{-1}$, where $U \in SU(2)$, can be regarded as a transformation from $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$. Under this consideration, $R_U$ is an element of $SO(3)$ which brings about a Lie group homomorphism $\phi : SU(2) \longrightarrow SO(3)$.

- The map $\psi : su(2) \longrightarrow so(3)$ defined by

$$\psi(L) = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}, \qquad (8)$$

for $L = -\frac{i}{2}(a\sigma_x + b\sigma_y + c\sigma_z)$, $a, b, c \in \mathbb{R}$, is a Lie algebra isomorphism.

- It can be shown[1] that $\phi\left(e^K\right) = e^{\psi(K)}$, where $K \in su(2)$ is given by $K = i(a\sigma_x + b\sigma_y + c\sigma_z)$.

We associate a system (1) with a given objective final state matrix $G$ to a system evolving on $SU(2)$

$$\frac{d}{dt}\mathbf{x} = (A + Bu)\mathbf{x}, \qquad (9)$$

where $A, B \in su(2)$, with a final state matrix $S \in SU(2)$. The matrices $\tilde{A}$ and $A$ are related by $A = \psi^{-1}\left(\tilde{A}\right)$. Because $\tilde{A}$ and $\tilde{B}$ are linearly independent, $A$ and $B$ are linearly independent as well.

Factoring $S$ into a product $\prod_{k=1}^{Q} e^{a_k A + b_k B}$ leads to an analogous factorization of $G$ as shown by the following relations:

$$\begin{aligned} G &= \phi(S) = \phi\left(\prod_{k=1}^{Q} e^{a_k A + b_k B}\right) & (10) \\ &= \prod_{k=1}^{Q} \phi\left(e^{a_k A + b_k B}\right) & (11) \\ &= \prod_{k=1}^{Q} e^{a_k \tilde{A} + b_k \tilde{B}}. & (12) \end{aligned}$$

Hence the controls that achieve $S$ do likewise for $G$.

The Lie algebra isomorphism $\psi$ of equation (8) maps $-\frac{i}{2}\sigma_x \longrightarrow A_1$ and $-\frac{i}{2}\sigma_y \longrightarrow A_2$. So the objective is to express the state transition matrix in $SO(3)$ of the switched electrical network example as equation (12) which is associated to $S \in SU(2)$ given by

$$S = \prod_{k=1}^{Q} e^{a_k A + b_k B}, \qquad (13)$$

where $a_k > 0$ is the time that the $k$th control pulse is applied and $b_k = 0$ or $b_k = a_k$. Here we have $A = -\frac{i}{2}\omega_1\sigma_y$ and $B = \frac{i}{2}(\omega_1\sigma_y + \omega_2\sigma_x)$. The number of factors $Q$ depends on the technique used to arrive at equation (13).

## 3. Obtaining Controls

In finding controls for our switched electrical network to prepare $S$ as in equation (13), we are mainly concerned with satisfying $a_k > 0$, $k = 1, \ldots, Q$. This is because $a_k$ represents the application time of the $k$th control pulse $u_k$. For bang-bang controls the control pulse $u_k$ is either 0 or 1 so that $b_k = 0$ or $b_k = a_k$.

Suppose that the state vector $\mathbf{x}$ lies on the unit sphere. Then the initial state vector $\mathbf{x}(0) = (1, 0, 0)^T$ and $\mathbf{x}(t) = (0, -1, 0)^T$. For instance if $C_1 = 0.1$, $C_2 = 0.2$ and $L_3 = 0.5$, then $A = -i\sqrt{5}\sigma_y$ and $B = \frac{i}{2}\left(2\sqrt{5}\sigma_y + \sqrt{10}\sigma_x\right)$. The problem that Leonard and Krishnaprasad [3] describe in designing controls for the system with constant current $I_3$ such that $x_3 = -1/\sqrt{2}$ is, for our techniques, the equivalent of specifying controls for the three problems 1.) $\mathbf{x}(0) = (1, 0, 0)^T$ and $\mathbf{x}(t1) = (1/\sqrt{2}, 0, -1/\sqrt{2})^T$, 2.) $\mathbf{x}(t1) = (1/\sqrt{2}, 0, -1/\sqrt{2})^T$ and $\mathbf{x}(t2) = (0, -1/\sqrt{2}, -1/\sqrt{2})^T$ and 3.) $\mathbf{x}(t2) = (0, -1/\sqrt{2}, -1/\sqrt{2})^T$ and $\mathbf{x}(t) = (0, -1, 0)^T$. In this paper we provide controls for the problem of taking the state from $\mathbf{x}(0)$ to $\mathbf{x}(t)$ without specifying any intermediate points. However, if desired, our methods can be easily adapted to make the state sojourn through the points $\mathbf{x}(t1)$ and $\mathbf{x}(t2)$ before reaching $\mathbf{x}(t)$. See [8].

The state transition matrix $S \in SU(2)$ in equation (9) can be written as

$$S = \exp i\left[a\sigma_x + b\sigma_y + c\sigma_z\right], \qquad (14)$$

where $a, b, c \in \mathbb{R}$. Let $\lambda = \sqrt{a^2 + b^2 + c^2}$. An expression for the Lie group homomorphism $\phi$, described above, acting on $S$ is obtained from equation (8) and Rodrigues' formula[9]. The numbers $a, b$, and $c$ are determined by the relation $\mathbf{x}(t) = \phi(S)\mathbf{x}(0)$ and the fact that $\phi(S) \in SO(3)$. These are satisfied by letting $a = b = 0$ and $c = \frac{\pi}{4}$. The target $G$ is not unique and there is an infinite family of vectors $(a, b, c)$ in $\mathbb{R}^3$ for which $\mathbf{x}(t) = \phi(S)\mathbf{x}(0)$ and $\phi(S) \in SO(3)$. Now finding controls for the network has been converted into obtaining controls that prepare $S = \exp(ic\sigma_z)$. Therefore, we concentrate only on preparing $S = \exp\left(i\frac{\pi}{4}\sigma_z\right)$. For the general problem see [1], [8].

### 3.1 Method 1

This method depends on the fact that $A$ and $B$ are linear combinations of $i\sigma_x$ and $i\sigma_y$. As has been proven in[2], the exponential of the third Pauli matrix can be expressed as a product of two factors

$$e^{iL\sigma_z} = V(\gamma_1)V(\gamma_2), \qquad (15)$$

where $L \in \mathbb{R}$ and

$$\begin{aligned} V(\gamma) &= \exp\begin{pmatrix} 0 & i\gamma \\ i\bar{\gamma} & 0 \end{pmatrix} \\ &= \exp\left[(-\operatorname{Im}\gamma)\, i\sigma_y + (\operatorname{Re}\gamma)\, i\sigma_x\right] \quad (16) \end{aligned}$$

for $\gamma \in \mathbb{C}$.

Let $\gamma_k = \frac{\pi}{2} e^{i\theta_k}$, $k = 1, 2$, then $L = \theta_1 - \theta_2 + \pi$. Since $L$ can be taken as an element of $[0, 2\pi)$, we have that $|L - \pi| < \pi$, $L \neq 0$. Thus for $L \neq 0$, $\gamma_1$ and $\gamma_2$ can be chosen so that $|\theta_1 - \theta_2| < \pi$. In other words, we may choose $\gamma_1$ and $\gamma_2$ to lie any open half plane. For our network example,

$$V(\gamma_k) = \exp\left[b_k \frac{\omega_2}{2} i\sigma_x + (b_k - a_k) \frac{\omega_1}{2} i\sigma_y\right], \quad (17)$$

so that the open half plane that is of interest to us is $\operatorname{Im}\gamma_k > -\frac{\omega_1}{\omega_2} \operatorname{Re}\gamma_k$, since this precisely ensures that $a_k > 0$.

### 3.2 Example Calculations for Method 1

By equations (16) and (17), the real and imaginary parts of $\gamma_k$, $k = 1, \ldots, Q$ are linear combinations of $a_k$ and $b_k$. For the network example, each of the factors of $S$ in equation (13) can be represented by

$$\exp\left[b_k \sqrt{\frac{5}{2}} i\sigma_x + (b_k - a_k)\sqrt{5} i\sigma_y\right] \quad (18)$$

for $k = 1, \ldots, Q$. From this relation and $a_k > 0$, we find that for each $k$, $\operatorname{Im}\gamma_k > -\sqrt{2}\operatorname{Re}\gamma_k$. So our choice of $\gamma_k$ must be above the line $\operatorname{Im}\gamma_k = -\sqrt{2}\operatorname{Re}\gamma_k$. This means that $\theta_k \in (-0.3041\pi, 0.6959\pi)$. This is in keeping with the fact that $\theta_k = \frac{\pi}{2}$ corresponds to free evolution and $\theta_k = 0$ when $u_k = 1$.

Let $c = \frac{\pi}{4}$. Since we are preparing $S = \exp(ic\sigma_z)$, it follows from equation(15) that we can write $S$ as

$$S = V(\gamma_1) V(\gamma_2), \quad (19)$$

where $\gamma_k = \frac{\pi}{2} e^{i\theta_k}$, $k = 1, 2$, and the phases of $\gamma_1$ and $\gamma_2$ must satisfy $\theta_2 - \theta_1 = \frac{3\pi}{4}$. Choose $\theta_1 = -\frac{\pi}{8}$ and $\theta_2 = \frac{5\pi}{8}$. Using equation (18), we get the following coefficients:

| $k$ | $a_k$ | $b_k$ |
|---|---|---|
| 1 | 0.649 | 0.917 |
| 2 | 0.269 | −0.380 |

(20)

These coefficients, when placed in equation (12), indeed drive the system in $SO(3)$ from $\mathbf{x}(0)$ to $\mathbf{x}(t)$ as shown in Figure 1. In Figure 1 the lines indicate that the network system is taken from a point in $\mathbb{R}^3$ to another along some undetermined path on the unit sphere. Because $b_2$ is a negative number, $u_2$ is negative and thus is not ideal. This may be remedied by an appropriate choice of $S$, see [8]. However, in the next section we show that even for this $S$ controls of 0 and 1 can be obtained.

### 3.3 Bang-Bang Controls via Method 1

The position of the switch is either 0 or 1 and we desire $u_k$ to represent the switch position. From Jurdjevic and Sussmann[7] we know that this can be

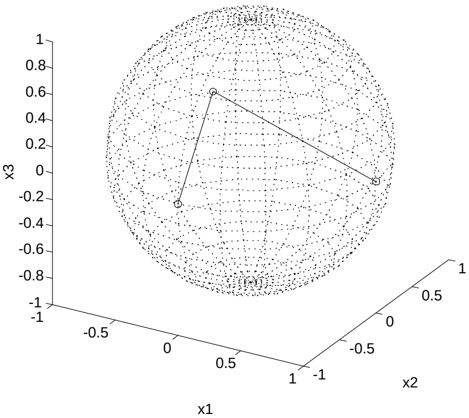


**Figure 1:** States Obtained by Method 1

done. Below we explain how to do this constructively for our network.

To get bang-bang controls with values of 0 and 1 for $S = \exp\left(i\frac{\pi}{4}\sigma_z\right)$, use the fact that

$$e^{i\frac{\pi}{4}\sigma_z} = e^{-i\frac{7\pi}{4}\sigma_y} e^{i\frac{\pi}{4}\sigma_x} e^{-i\frac{\pi}{4}\sigma_y}. \quad (21)$$

The first and third factors of the above equation are free evolution and the second factor is obtained with a control pulse of 1. Each of the three factors are of the form of (18), so that we have the following coefficients:

| $k$ | $a_k$ | $b_k$ |
|---|---|---|
| 1 | $\frac{\pi\sqrt{5}}{20}$ | 0 |
| 2 | $\frac{\pi\sqrt{10}}{20}$ | $\frac{\pi\sqrt{10}}{20}$ |
| 3 | $\frac{7\pi\sqrt{5}}{20}$ | 0 |

(22)

These coefficients represent bang-bang controls that drive the system as shown in Figure 2. As in Figure 1, each line in Figure 2 represents the system being taken from one point in $\mathbb{R}^3$ to another along an undetermined path on the unit sphere. It turns out that the second and third factors in equation (21), i.e., $e^{i\frac{\pi}{4}\sigma_x} e^{-i\frac{\pi}{4}\sigma_y}$, translate to elements of $SO(3)$ whose product forms another element of $SO(3)$ that takes $\mathbf{x}(0)$ to $\mathbf{x}(t)$. Thus, we have "discovered" another element of $SO(3)$ which achieves the desired transfer! The first factor when multiplied with the product of the other two, yields the $SO(3)$ element that is associated to $S$. See [8] for how bang-bang controls(with values of 0 and 1) can be obtained for any target $S$ in $SU(2)$.

### 3.4 Method 2

This approach can be used for any $A$ and $B$ in $su(2)$ that are orthonormal. Let $A = \frac{i}{2}(a\sigma_x + b\sigma_y + c\sigma_z)$ and $B = \frac{i}{2}(\alpha\sigma_x + \beta\sigma_y + \gamma\sigma_z)$.

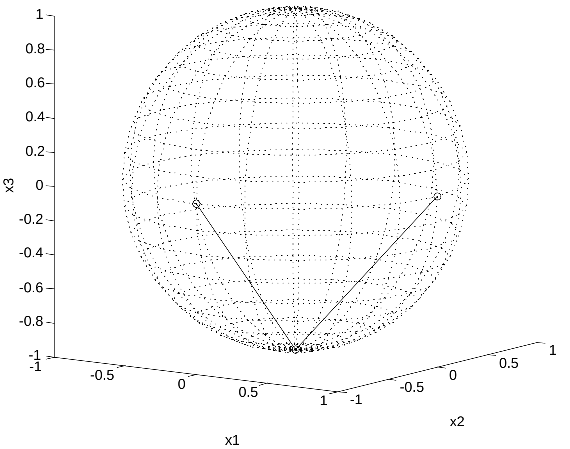


**Figure 2:** States Obtained with Bang-Bang Controls

Here orthonormality of $A$ and $B$ means that the vectors $(a, b, c)$ and $(\alpha, \beta, \gamma)$ in $\mathbb{R}^3$ are orthonormal. Orthonormalization of $A$ and $B$ can be achieved by introducing preliminary controls and scaling if necessary. Application of preliminary controls causes this method to be unfavorable. The idea is to map $A$, $B$, and $[A, B]$ via a Lie algebra isomorphism $\psi$ to $\hat{A}$, $\hat{B}$, and $[\hat{A}, \hat{B}]$ for a pair $\hat{A}, \hat{B}$ for which suitable controls have been obtained. To ensure that $\psi$ is a Lie algebra isomorphism, orthonormality is required for $A$ and $B$ and for $\hat{A}$ and $\hat{B}$. Choose the latter pair to be $\hat{A} = \frac{i}{2}\sigma_z$ and $\hat{B} = \frac{i}{2}\sigma_y$. Correspondingly translate the problem of preparing $S$ to that of preparing a related matrix $T$ for a system with drift $\hat{A}$ and control $\hat{B}$.

To determine $T$, find the logarithm of the target $S \in SU(2)$ by

$$\log S = P^* \begin{pmatrix} i\nu & 0 \\ 0 & -i\nu \end{pmatrix} P, \qquad (23)$$

where $\nu \in [0, 2\pi)$ and $P$ is a unitary matrix such that

$$PSP^* = \begin{pmatrix} e^{i\nu} & 0 \\ 0 & e^{-i\nu} \end{pmatrix}. \qquad (24)$$

The matrices $A$ and $B$ are linearly independent and so $\{A, B, [A, B]\}$ forms a basis of $su(2)$. As $\log S$ is an element of $su(2)$, we can express this logarithm as

$$\log S = c_1 A + c_2 B + c_3 [A, B], \qquad (25)$$

for some $c_1,\ c_2, c_3 \in \mathbb{R}$.

The Lie algebra isomorphism $\psi : su(2) \longrightarrow su(2)$ is defined by

$$\psi\left(r_1\hat{A} + r_2\hat{B} + r_3\left[\hat{A}, \hat{B}\right]\right) = r_1 A + r_2 B + r_3 [A, B], \qquad (26)$$

for any $r_1, r_2, r_3 \in \mathbb{R}$. This isomorphism is associated with the Lie group homomorphism $\phi : SU(2) \longrightarrow SU(2)$ that satisfies the relation $\phi\left(e^K\right) = e^{\psi(K)}$. Hence the associated target $T$ for the $\hat{A}, \hat{B}$ system is

$$T = \exp\left(c_1\hat{A} + c_2\hat{B} + c_3\left[\hat{A}, \hat{B}\right]\right). \qquad (27)$$

In otherwords, $\phi(T) = S$.

Now the target $S$ is converted to the target $T$. And the objective is to express $T$ as the product

$$T = \prod_{k=1}^{Q} e^{a_k\hat{A} + b_k\hat{B}}, \qquad (28)$$

for $a_k > 0$ and $b_k = 0$ or $b_k = a_k$ as in equation (13). The controls that prepare $T$ also prepare $S$ and hence $G$. See [1] for the details of preparing $T$ for the $\hat{A}$, $\hat{B}$ system.

Although this method can be used for any $A$, $B$ system, it does not perform as well as method 1 because linear algebraic calculations are involved in its execution, it reflects the structure of the matrices $\hat{A}$ and $\hat{B}$ more than that of $A$ and $B$ and also, the use of preliminary controls, to achieve orthonormalization, makes placing bounds on $b_k$ difficult via this method. Therefore we do not discuss this method further.

## 4. Conclusions

The control of switched electrical networks has useful applications in areas such as power conversion. Switched networks are modeled as bilinear systems having drift and control vector fields. We have used an example of a switched electrical network evolving on $SO(3)$ to illustrate the implementation of two quantum control techniques which provide piecewise constant controls. Both approaches use a translation of the system on $SO(3)$ to a related system on $SU(2)$. The target of the system on $SU(2)$ is decomposed into a product of factors. These $SU(2)$ methods also extend to the case of four circuit elements. For such circuits the associated system evolves on $SO(4)$ which is related to $SU(2) \times SU(2)$. Preliminary calculations indicate that it is possible to obtain piecewise constant controls for these sytems. For details see [10].

The first method we described is intrinsic to the structure of the drift and control vector fields of the network. We have shown that this first technique can provide bang-bang controls with values of 0 and 1. The second approach is independent of the structure of the drift and control of the network and uses a further translation of the system on $SU(2)$ to another system on $SU(2)$. Nevertheless, this second method requires linear algebraic computations, reflects the structure of the second system on $SU(2)$ more than the first and uses preliminary controls. Thus the first method is more practical than the second.


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