Robustness of a relaxation oscillator [1]

Tryphon T. Georgiou [2] and Malcolm C. Smith [3]

Abstract

For a relaxation oscillator which consists of a relay-hysteresis in feedback with negative integral action we prove that the oscillatory behaviour is robust to perturbations in the dynamical component of the feedback loop which are sufficiently small in a gap sense.

1 Introduction

A central theme in George Zames' investigations was to quantify the role of feedback in combating modelling uncertainty. To this end, in his joint work with Ahmed K. El-Sakkary [30], he set the goal of seeking in full generality a "*description of the tolerable uncertainties.*" This work gave rise to a suitable metric topology having the desired property that feedback stability is maintained in a small neighbourhood of a nominally stable feedback system. The metric used is known as the *gap metric.*

To date, this paradigm has been studied for both linear and nonlinear systems in the neighbourhood of a fixed operating condition (see [5–11, 15, 22, 25, 26] and the references therein). In the present work we seek to extend this paradigm to a new situation—a nonlinear feedback oscillator. Such systems are not globally stable in an input-output sense. Moreover, the closeness between responses of such systems is not conveniently assessed with the usual norm-based measures. Nevertheless, with appropriate modifications, we are able to show that the basic ideas of the paradigm generalize to this new context.

Nonlinear oscillations are encountered in a large variety of physical phenomena from chemical reactions and interacting populations [21, pp. 154, 180], to circadian processes [28, pp. 169, 173], to neurosciences [3, p. 41], and to the dynamics of Cepheid variables in Astrophysics [18, p. 106]. Although the phenomenon of limit cycle oscillation appears ubiquitous, and therefore it *must be* fairly robust, there is little known about robustness of the respective mathematical models. In fact, the extensive mathematical literature on relaxation oscillators focuses on conditions for limit cycles to exist in a given system, on entrainment by external signals, and on the effects of parametric or state-equation uncertainty in fixed-order models (see [1, 2, 13, 14, 16, 17, 19, 20, 24] and the references therein). In contrast the gap approach to uncertainty allows for changes in the dynamic order of the system including the possibility of infinite-dimensional elements such as time-delays.

In our analysis we focus on a common type of oscillator, sometimes called a relaxation oscillator, where negative integral action drives a bistable system via a feedback interconnection. The bistable element in the feedback path is in general a dynamic hysteresis-type of nonlinearity. When the element is fixed in either of its two states, there is a build-up by the integral action in a direction which forces it into the other state, and so on. The most widely referenced example in the engineering literature is the van der Pol oscillator (see [16, p. 288], [13]). This paper considers the simple relaxation oscillator of Figure 1 where the hysteresis is an ideal relay with infinitely fast transition between the two states, while the dynamical component is a simple integrator. However, the authors believe that the approach of the paper is amenable to generalization to other nonlinear feedback oscillators.

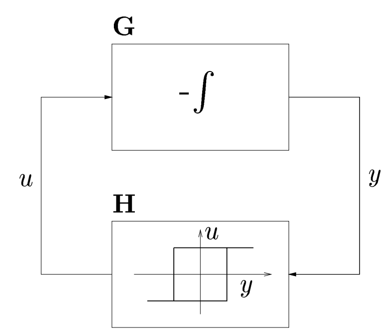

Figure 1: Relay-relaxation oscillator

The paper is a summary of [12] where we refer for proofs and detailed discussion. Below, in Section 2 we describe the mathematical framework used for analyzing the relaxation oscillator. In Section 3 we present a notion of distance between oscillatory signals. Section 4 presents the main result (Theorem 1) which gives a bound on the amount of modelling uncertainty, measured in the gap metric, which guarantees that oscillatory behaviour

[1] Research supported by the NSF, AFOSR, and EPSRC.
[2] Department of Electrical and Computer Engineering, University of Minnesota, MN 55455, U.S.A.
[3] Department of Engineering, University of Cambridge, Cambridge, CB2 1PZ, U.K.

persists for the uncertain system. Section 5 considers a specific class of perturbations of the negative integrator in the relay oscillator and computes robustness bounds by applying Theorem 1.

2 Feedback systems with relay-hysteresis

A relay-hysteresis $\mathbf{H}(\cdot)$ is defined for a continuous input $y(t)$ for which $y(0) = 0$ (e.g., [23, p. 66]). The output $u(t)$ takes values from the set $\{-1, +1\}$ and can be determined from:

(i) $u(0) = 1$.

(ii) $u(t)$ is $+1$ if $y(t) \geq +1$ and -1 if $y(t) \leq -1$.

(iii) Suppose $y(t_0) > -1$ and $u(t_0) = +1$ for some $t_0 \geq 0$. Then $u(t) = +1$ on any interval $[t_0, t_1)$ for which $y(t) > -1$.

(iv) Suppose $y(t_0) < +1$ and $u(t_0) = -1$ for some $t_0 \geq 0$. Then $u(t) = -1$ on any interval $[t_0, t_1)$ for which $y(t) < +1$.

Condition (i) is due to the inherent "memory" of the system which requires that the "state" $u(\cdot)$ is specified at the initial time $t = 0$.

The analysis of feedback systems with relay elements requires care due to the discontinuous nature of the outputs of such elements. To establish well-posedness of the nominal and perturbed feedback loops, i.e. existence and uniqueness of solution in the presence of a suitable class of external disturbance signals, the classical approach of using Banach's contraction mapping theorem [4, 27, 29] is not applicable. On the other hand, providing arbitrarily fast switching can be avoided, existence and uniqueness of solution can follow in a straightforward way by integrating the dynamic element of the feedback loop over successive intervals where the output of the relay-hysteresis element is constant. To ensure that this is the case, we will consider dynamical elements of the nominal and perturbed feedback loops whose outputs have a Lipschitz property. This allows a lower bound on the time between switches of the relay to be established.

2.1 Choice of Signal Spaces and Systems

Let $\mathcal{L}_\infty[0, \infty)$ denote the Lebesgue space with the usual sup norm, and $C[0, \infty)$ its subspace of continuous functions. Let

$$\begin{aligned}\mathrm{Lip}[0,\infty) \;=\; \{&y(t),\, t \in [0,\infty) : y(0) = 0, \text{ and satisfies} \\ &\text{a Lipschitz condition } \mathrm{C}_T = \sup\{\frac{|y(s) - y(t)|}{|s-t|} : \\ &s \neq t, \text{ and } s, t \in [0, T)\} < \infty, \\ &\text{for all } 0 < T < \infty\}.\end{aligned}$$

The Lipschitz constant C_T may depend on y and the length T of the interval, but is finite. We now define the following input and output spaces:

$$\begin{aligned}\mathcal{U} &= \mathcal{L}_\infty[0, \infty), \\ \mathcal{Y} &= \{y(t) \in C[0, \infty) \;:\; y(0) = 0\}.\end{aligned}$$

The choice of $\mathcal{U}$ and $\mathcal{Y}$ is dictated by the fact that the output of the relay-hysteresis is discontinuous while the input is only required to be continuous.

We consider linear dynamical systems defined by an integral operator

$$\mathbf{G} \;:\; u(t) \mapsto y(t) = \int_0^t g(t-\tau) u(\tau) d\tau,$$

where $u(t) \in \mathcal{U}$ and the kernel $g(t)$ is piecewise Lipschitz, i.e., for any $T > 0$ there are finitely many intervals $[0, \tau_1)$, ..., $[\tau_m, T)$ such that $|g(s) - g(t)| < C|s-t|$ where s, t belong to the same subinterval and C is a constant which may depend on T. The class of such systems will be designated by $\mathbb{G}$. It can be shown that the outputs of such systems belong to $\mathrm{Lip}[0, \infty)$, as stated below.

Proposition 1 *[12, Proposition 1]* The range of $\mathbf{G} \in \mathbb{G}$ is a linear submanifold of $\mathrm{Lip}[0, \infty)$.

2.2 Well-posedness of relay feedback systems

We consider the relay feedback system of Figure 2 where external disturbances are added at each node. This feedback system will be denoted by $[\mathbf{G}, \mathbf{H}]$. The feedback equations consist of the algebraic equations:

$$y_2 = y_0 - y_1, \tag{1}$$

$$u_2 = \mathbf{H}(y_0 - y_1), \tag{2}$$

$$u_1 = u_0 - u_2, \tag{3}$$

which determine y_2, u_2, u_1 in terms of y_1 and the external disturbances $y_0 \in \mathcal{Y}$, $u_0 \in \mathcal{U}$. The integral equation

$$y_1(t) = \int_0^t g(t-\tau) \left(u_0(\tau) - \mathbf{H}\left(y_0 - y_1\right)(\tau)\right) d\tau \tag{4}$$

expresses the output of the dynamical subsystem in terms of its inputs. Because the output of the dynamical subsystem is sufficiently well behaved (Proposition 1), the feedback equations are well-posed as stated below.

Proposition 2 *[12]* For any $u_0 \in \mathcal{U}$ and $y_0 \in \mathcal{Y}$, (4) has a unique solution $y_1 \in \mathcal{Y}$. The remaining signals in the feedback loop satisfy: $u_1, u_2 \in \mathcal{U}$ and $y_2 \in \mathcal{Y}$.

3 When are two oscillatory signals close?

A new feature in the problem of robustness of limit cycle oscillations is the difficulty of using the norm of

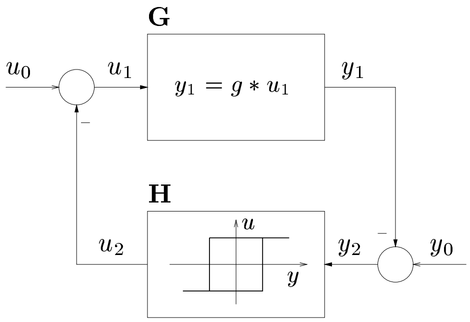

Figure 2: Relay feedback system with external disturbances

the difference of two signals over the semi-infinite time-axis to quantify closeness. This is because oscillatory trajectories can get "out of step" in time due to perturbations. Two possible ways of dealing with this—restriction to compact time intervals and analysis in a phase space—have drawbacks. In the first case, allowable perturbations need to become smaller and smaller as the time-interval is increased. In the second case, there may not be a common phase-space for the nominal and perturbed oscillator, e.g. when there is a difference in model order or when time-delays are introduced. Accordingly we introduce the device of allowing the time-axis to be re-scaled for one of the signals to be compared. Then, a notion of distance between oscillatory signals can be defined by combining the norm of their difference with the size of the chosen scaling. We now formalize such a distance measure.

Let $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$ and define:

$$d(w_1(t), w_2(t)) := \inf\{\|w_1(t) - w_2(\sigma(t))\|_\infty + \sup_t \frac{|\sigma(t) - t|}{t} \;:\; \text{for } \sigma \in \mathcal{K}_\infty\},$$

where $\mathcal{K}_\infty$ denotes the set of continuous monotonically non-decreasing functions σ of $t \in [0, \infty]$ such that $\sigma(0) = 0$ and $\sigma(\infty) = \infty$. For convenience, in the sequel, we use the notation $\sigma w(t) := w(\sigma(t))$. We note that a similar notion of distance has also been considered by S. Varigonda (personal communication).

4 Robustness analysis

Our analysis relies on the formalism developed in [9–11]. In particular, we consider the relay oscillator in the standard feedback interconnection of Figure 2. We denote by $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$ the "ambient space" where input/output signals reside and consider the graphs of systems as subsets of $\mathcal{W}$, i.e.,

$$\mathcal{M} := \text{graph}(\mathbf{G}) := \{\begin{pmatrix} u \\ y \end{pmatrix} \;:\; y = \mathbf{G}u,\ u \in \mathcal{U}, y \in \mathcal{Y}\},$$

$$\mathcal{N} := \text{graph}(\mathbf{H}) := \{\begin{pmatrix} u \\ y \end{pmatrix} \;:\; u = \mathbf{H}y,\ y \in \mathcal{Y}, u \in \mathcal{U}\}.$$

The equations specifying the feedback interconnection can be written as $w_0 = w_1 + w_2$ where

$$w_0 := \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \in \mathcal{W},$$
$$w_1 := \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} \in \mathcal{M},$$
$$w_2 := \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \in \mathcal{N}.$$

The theory [10, Sections II and VIII] makes extensive use of the feedback map from external disturbances to the input and output of one of the two components of the feedback loop; e.g., in the case $\mathbf{G}$, of the map

$$\mathbf{\Pi}_{\mathcal{M}\|\mathcal{N}} : \mathcal{W} \to \mathcal{M} \;:\; w_0 \mapsto w_1.$$

This is often referred to as a *parallel projection* operator—a terminology which reflects a geometric interpretation discussed in [5] and [10, Equation 1]. There is also a complementary parallel projection

$$\mathbf{\Pi}_{\mathcal{N}\|\mathcal{M}} : \mathcal{W} \to \mathcal{N} \;:\; w_0 \mapsto w_2,$$

and moreover

$$\mathbf{\Pi}_{\mathcal{M}\|\mathcal{N}} + \mathbf{\Pi}_{\mathcal{N}\|\mathcal{M}} = \mathbf{I}.$$

Next, the "distance" between dynamical systems is quantified by the distance to the identity of a suitable map which relates the input-output trajectories of the two systems. More precisely, if $\mathcal{M}_1$ denotes the graph of a perturbed system $\mathbf{G}_1$, we search over all causal maps $\mathbf{\Phi}_\mathcal{M}$ which map $\mathcal{M}$ bijectively onto $\mathcal{M}_1$ with $\mathbf{\Phi}_\mathcal{M} 0 = 0$ and select one which differs least from the identity. A nonlinear generalization of the gap metric can be based on the quantity $\inf_{\mathbf{\Phi}_\mathcal{M}}\{\|(\mathbf{I} - \mathbf{\Phi}_\mathcal{M})|_\mathcal{M}\|\}$ (see [10]). Our main result is stated below.

Theorem 1 Let $\mathbf{G}$ be the negative integrator (i.e., $g(t - \tau) = 1$ for $t \geq \tau$ and 0 otherwise), $\mathbf{G}_1$ be an arbitrary element in $\mathbb{G}$, and $\mathbf{H}$ be the relay-hysteresis defined in Section 2, and denote their graphs by $\mathcal{M}, \mathcal{M}_1, \mathcal{N}$, respectively. If there exists a surjective map $\mathbf{\Phi}_\mathcal{M} : \mathcal{M} \to \mathcal{M}_1$ such that

$$\|(\mathbf{I} - \mathbf{\Phi}_\mathcal{M})|_\mathcal{M}\| \leq \epsilon < \frac{1}{3},$$

then there exists a function $\sigma \in \mathcal{K}_\infty$ such that

$$\sup_t \frac{|\sigma(t) - t|}{t} \leq \frac{4\epsilon(1-\epsilon)}{(1-2\epsilon)^2}, \tag{5}$$

and the response of the two feedback systems $[\mathbf{G}, \mathbf{H}]$ and $[\mathbf{G}_1, \mathbf{H}]$ with zero external excitation signals satisfy

$$\|\sigma \mathbf{\Pi}_{\mathcal{M}\|\mathcal{N}} 0 - \mathbf{\Pi}_{\mathcal{M}_1\|\mathcal{N}} 0\|_\infty \leq \frac{2\epsilon}{1-\epsilon}. \tag{6}$$

Equation (6) shows that the trajectories of the nominal and perturbed systems are close in a peak sense, when one of the two is scaled appropriately in time. This shows that oscillations persist in the perturbed system, albeit with different and possibly varying periods.

The proof of the theorem is given in [12].

5 Example

We study the behaviour of the relaxation oscillator of Figure 2 when the nominal integrator, i.e. with transfer function $P(s) = -1/s$, is replaced by a system with transfer function

$$P'(s) = \frac{-ae^{-hs}}{s+b}$$

where $a > 0$. We first study the case of $h = 0$ since this is amenable to explicit calculation. We denote the solutions of the perturbed system by u_1', y_1', etc. Before the first hysteresis switch, the system evolves according to: $\dot{y}_1' + by_1' = a$, which has solution

$$y_1'(t) = (1 - e^{-bt})\frac{a}{b},$$

providing $b \neq 0$. If $b \geq a$, then the hysteresis never switches and the relaxation oscillation breaks down. So let us assume that $b < a$. After the switch the system evolves according to: $\dot{y}_1' + by_1' = -a$. In order that $\dot{y}_1'$ becomes negative just after the switch we require that $b > -a$, otherwise $\dot{y}_1'(t)$ remains equal to one or escapes to $+\infty$, and again the relaxation oscillation breaks down. So let us assume that $|b| < a$. Under such a condition the hysteresis continues to switch at times $0 < t_1' < t_2' < \ldots$, and in each interval $[t_k', t_{k+1}']$ the solution is:

$$y_1'(t) = (-1)^k \left(\frac{a}{b} - (1 + \frac{a}{b})e^{-b(t-t_k')}\right).$$

We now define a suitable scaling function $\sigma(t)$. On the interval $[t_k', t_{k+1}']$ we define

$$\sigma(t) = t_k + (-1)^k \left(y_1'(t) - y_1'(t_k')\right)$$

while on $[0, t_1']$, $\sigma(t) = y_1'(t)$. As before, observe that σ is monotonically increasing, $\sigma(t_k') = t_k$ for all k and $\sigma u_1(t) - u'(t) = 0$. Furthermore, on $[t_k', t_{k+1}']$:

$$\sigma y_1(t) - y_1'(t) = (-1)^k(-1 + \sigma(t) - t_k) - y_1'(t) = 0$$

since $y_1'(t_k') = (-1)^{k+1}$. The same fact holds on $[0, t_1']$, so we have shown that

$$\sigma \mathbf{\Pi}_{\mathcal{M}\|\mathcal{N}} 0 = \mathbf{\Pi}_{\mathcal{M}_1\|\mathcal{N}} 0.$$

The switching times are given by

$$\begin{aligned} t_1' &= \frac{1}{b}\ln\frac{a}{a-b}, \\ t_{k+1}' &= t_k' + \frac{1}{b}\ln\frac{a+b}{a-b}, \end{aligned}$$

and we easily check that $t_k' \to t_k$ as $a \to 1$ and $b \to 0$. It can also be shown that

$$\sup_t \frac{|\sigma(t) - t|}{t} \to 0$$

as $a \to 1$ and $b \to 0$.

We now turn to the question of the "gap" between $\mathbf{G}$ and $\mathbf{G}'$. We find that

$$\begin{aligned} \mathcal{M} &= \begin{bmatrix} M \\ N \end{bmatrix} \mathcal{L}_\infty = \begin{bmatrix} \frac{s}{s+1} \\ \frac{-1}{s+1} \end{bmatrix} \mathcal{L}_\infty, \\ \mathcal{M}' &= \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} \mathcal{L}_\infty = \begin{bmatrix} \frac{s+b}{s+1} \\ \frac{-ae^{-hs}}{s+1} \end{bmatrix} \mathcal{L}_\infty, \end{aligned}$$

where, as customary, we represent by $G(s)\mathcal{L}_\infty$ the image of $\mathcal{L}_\infty[0,\infty)$ under the action of a convolution operator whose kernel is the inverse Laplace transforms of $G(s)$. Define $(V, U) = (1, -1)$ and note that $(V, U)\begin{pmatrix} M \\ N \end{pmatrix} = 1$. Thus,

$$\mathbf{\Phi} := \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} \begin{pmatrix} V & U \end{pmatrix}$$

maps $\mathcal{M}$ onto $\mathcal{M}_1$ and we find that

$$\begin{aligned} (\mathbf{I} - \mathbf{\Phi})|_{\mathcal{M}} &= \begin{pmatrix} M \\ N \end{pmatrix} \begin{pmatrix} V & U \end{pmatrix} \\ &\quad - \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} \begin{pmatrix} V & U \end{pmatrix} \\ &= \begin{pmatrix} \frac{-b}{s+1} \\ \frac{ae^{-hs}-1}{s+1} \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix}. \end{aligned}$$

It follows that

$$\|(\mathbf{I} - \mathbf{\Phi})|_{\mathcal{M}}\| \leq 2\max\left\{\left\|\frac{b}{s+1}\right\|, \left\|\frac{1 - ae^{-hs}}{s+1}\right\|\right\}$$

where $\|\cdot\|$ in the righthand side of the above equation denotes the induced norm of the relevant operators in an $\mathcal{L}_\infty$ sense, i.e.,

$$\begin{aligned} \left\|\frac{b}{s+1}\right\| &= \|be^{-t}\|_{\mathcal{L}_1} = |b|, \text{ while} \\ \left\|\frac{1 - ae^{-hs}}{s+1}\right\| &= \int_0^h e^{-t}dt + \int_h^\infty |e^{-h} - a|e^{h-t}dt \\ &= 1 - e^{-h} + |e^{-h} - a|, \end{aligned}$$

being in both cases the $\mathcal{L}_1$ norm of the respective convolution kernels. Hence, Theorem 1 predicts that oscillations will not break down providing

$$\max\{|b|, 1 - e^{-h} + |e^{-h} - a|\} < \frac{1}{6}. \qquad (7)$$

In contrast, by direct calculation we were able to show for the case $h = 0$, that oscillations do not break down if $|b| < a$, which is consistent with condition (7).

6 Summary

Theorem 1 gives a sufficient condition for robustness of oscillatory behaviour of the relay relaxation oscillator. It states that, as long as a perturbation of the dynamical component is sufficiently small ($< 1/3$) in a gap sense, oscillations persist.

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