New Robust Stability and Performance Conditions based on Parameter Dependent Multipliers

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Abstract. This paper introduces a new robust stability and performance characterization for parameter dependent systems. The main idea is to jointly use parameter dependent Lyapunov functions and parameter dependent multipliers. It is shown that the novel tests can be easily specialized to results that have been recently proposed in the literature and that have been derived in an independent fashion for discrete- and continuous-time systems. It is revealed that these characterization apply to system descriptions with rational parameter dependence, and we discuss in how far they extend to robust state-feedback controller synthesis. Various examples demonstrate the benefit of the proposed algorithms over previous approaches.

Keywords. Robust stability, uncertain systems, parameter dependent Lyapunov functions, linear matrix inequalities, multipliers.

1 Introduction

Robust analysis and synthesis problems for systems depending on uncertain parameters are among the most classical problems in control theory and they have attracted an intensive research effort. The proposed techniques to tackle such problems can be divided in three groups. The frequency domain-based methods, like μ analysis [12], are quite general and allow to treat both dynamic and parametric uncertainty. They are, however, computationally quite costly since they require a gridding over frequencies. Moreover, they can suffer from considerable inaccuracies, especially in the case of real parametric uncertainty. The second group of methods are algebraic in nature and have been specifically developed to deal with parametric uncertainty. They have their origin in the work of Kharitonov on interval polynomials [10]. As a main drawback, they give rise to quite complex stability conditions that are rarely useful for synthesis. The last group is based on Lyapunov methods and has witnessed considerable progress in recent years, in correspondence to the development of Linear Matrix Inequality computational techniques.

One of the first important contributions in this group of methods has been the introduction of the concept of quadratic stability [3] where stability of a convex polytope of matrices is assessed through the use of a parameter-independent Lyapunov function. Subsequently it has been tried to reduce the conservatism of the quadratic stability approach by allowing for Lyapunov functions that depend on the uncertain parameters. As an alternative, it has been proposed to reduce the conservatism through extra variables that introduce more degrees of freedom in the analysis tests. These extra variables, called multipliers or scalings, describe the nature of the uncertainty and how it affects the behavior of the system. The more precise this description, the sharper is the corresponding stability characterization. In [7] we proposed robust analysis tests based on a Lyapunov function that depends rationally on the uncertain parameters and on a class of general constant full block multipliers. As a main weakness, these Lyapunov-based methods do not lead to convex optimization problems for controller synthesis, due to the interaction of all the variables (Lyapunov matrices, multipliers and controller parameters) that are involved in the corresponding inequalities.

A first important contribution to resolve this problem is given in [6]. In this work, using some structural characteristics of the discrete-time Lyapunov inequality, the Lyapunov matrix and the plant parameters are decoupled through the introduction of a particular multiplier. Among other advantages, this allows the translation of controller synthesis to convex optimization. An attempt to derive equivalent results for the continuous-time case has been undertaken in [1], although the authors could not overcome some difficulties, such as only obtaining a sufficient characterization for nominal H_∞ performance which is not necessary.

In this paper we introduce a general framework based on parameter dependent Lyapunov functions and a family of parameter dependent multipliers that lead, as shown by examples, to less conservative results both for analysis and synthesis if compared to [6, 5, 1]. As the main advantage, the procedure applies to both discrete- and continuous-time systems, the results are easily specialized to those in [6, 5], and they allow to recover an exact test for nominal continuous-time H_∞-performance. As a side-aspect, we point out how different but equivalent LFT representations of uncertain systems lead to different non-equivalent versions of robustness tests. The relation among these tests, and in particular the question of which involves the least degree of conservatism, appears to be an interesting subject for further research.

2 Robust stability tests with parameter dependent multipliers

Consider the uncertain system $\dot{x} = A(\delta)x$ where $A(\delta)$ is a continuous function of the parameter δ which belongs to the set $\boldsymbol{\delta} = \{\delta = (\delta_1, ..., \delta_k) : \delta_j \in [\underline{\delta}_j, \bar{\delta}_j],\ \ j = 1, ..., k\}$. Notice that $\boldsymbol{\delta}$ is the convex hull of the finite set $\boldsymbol{\delta^0} = \{\delta = (\delta_1, ..., \delta_k) : \delta_j \in \{\underline{\delta}_j, \bar{\delta}_j\},\ \ j = 1, ..., k\}$. We assume that $0 \in \boldsymbol{\delta}$ (w.l.o.g.) and that $A(0)$ is Hurwitz. In this paper we focus on parameter-dependence that allows to represent $A(\delta)$ as a linear fractional transformation (LFT)

$$A(\delta) = F_a + F_b\Delta(\delta)(I - F_d\Delta(\delta))^{-1}F_c \tag{1}$$

with a continuous $\Delta(\delta)$. It is well-known that such a representation is not unique, and we will briefly address the effect of using different LFT descriptions.

It is easy to see that $A(\delta)$ is Hurwitz iff there exists a continuous symmetric-vlaued matrix function $X(\delta)$ such that

$$A(\delta)'X(\delta) + X(\delta)A(\delta) < 0 \tag{2}$$

holds for all $\delta \in \boldsymbol{\delta}$. Note that, since $A(0)$ is Hurwitz and $\boldsymbol{\delta}$ is path-connected, $X(\delta)$ is positive definite such that it is not required to explicitly include this constraint. As the main trouble in directly verifying this condition, (2) is in general not convex in δ and it has to be valid in an infinite number of points. The key point of this paper is a sufficient condition to guarantee the global validity of the Lyapunov inequality (2) on $\boldsymbol{\delta}$ in terms of a family of parameter dependent multipliers.

Theorem 1 *Suppose there exist symmetric-valued continuous $X(\delta)$ and $P(\delta)$ with*

$$\begin{bmatrix} \Delta(\delta) \\ I \end{bmatrix}' P(\delta) \begin{bmatrix} \Delta(\delta) \\ I \end{bmatrix} \geq 0, \tag{3}$$

$$\begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix}' \left[\begin{array}{cc|c} 0 & X(\delta) & \\ X(\delta) & 0 & \\ \hline & & P(\delta) \end{array}\right] \left[\begin{array}{cc} I & 0 \\ F_a & F_b \\ \hline 0 & I \\ F_c & F_d \end{array}\right] < 0 \tag{4}$$

for all $\delta \in \boldsymbol{\delta}$. Then $I - F_d\Delta(\delta)$ is invertible and (2) holds for all $\delta \in \boldsymbol{\delta}$.

The proof is a matter of direct verification.

The search for solutions of the functional inequalities (3) and (4) is, in general, an infinite dimensional semi-infinite feasibility problem. In order to transform it into a standard LMI problem, we should specify a class of multipliers and a structure for the function $X(\delta)$. Several choices are possible for the class of multipliers. The use of constant block-diagonal multipliers for robust stability problems has been exploited in [8]. In [7] we chose a more general class of full-block multipliers. As a novel ingredient of the present work, we propose a family of multipliers that depends on the uncertain parameters as

$$P(\delta) = \begin{bmatrix} S_1 + S_1' & -S_0 - S_1\Delta(\delta) \\ -S_0' - \Delta(\delta)'S_1' & S_0'\Delta(\delta) + \Delta(\delta)'S_0 \end{bmatrix} \tag{5}$$

with arbitrary S_0 and S_1. This class of multipliers renders the left-hand side of (3) identically zero, so that only (4) remains to be guaranteed. If the left-hand side of (4) is a partially convex function on $\boldsymbol{\delta}$, the inequality holds throughout $\boldsymbol{\delta}$ iff it is satisfied in the finitely many generators $\boldsymbol{\delta^0}$.

In order to enforce partial convexity, let us assume that $\Delta(\delta)$ is multi-affine in δ, and let us search for Lyapunov functions $X(\delta)$ that are multi-affine functions of δ as well. Let us finally recall that any $X(.)$ defined on the finite set $\boldsymbol{\delta^0}$ admits a multi-affine extension to the whole set $\boldsymbol{\delta}$ [9]. We conclude that the stability test based on (4) then amounts to solving 2^k LMIs (one for each element in $\boldsymbol{\delta^0}$) in the $2^k + 2$ unknown matrices S_0, S_1, $X(\delta)$, $\boldsymbol{\delta} \in \boldsymbol{\delta^0}$. At the expense of conservatism, it would of course be possible to confine the search to affine Lyapunov functions.

Let us now distinguish two specific LFT representations of the uncertain system. As we will see, they lead to two different robustness test, each with its own benefit. As a first choice we take

$$A(\delta) = A + B\Delta_d(\delta)(I - D\Delta_d(\delta))^{-1}C \tag{6}$$

in which $\Delta_d(\delta)$ is of the form $\mathrm{diag}(\delta_1 I, ..., \delta_k I)$. It is well known that this representation is always possible if $A(\delta)$ is a rational function of δ without pole in 0. To guarantee stability of $A(\delta)$, one has to satisfy (4) where $F_a = A$, $F_b = B$, $F_c = C$, $F_d = D$ and with suitable $X(\delta)$, and S_0, S_1 defining (5). This leads to a finite LMI condition to test robust stability for rational $A(\delta)$ in terms of a multi-affine $X(\delta)$ and, as such, it constitutes an extension of the test presented in [7]. As the main difference, we use parameter dependent multipliers whereas the results in [7] were based on parameter-independent multipliers. Unfortunately, the underlying structures are not easily comparable such that it is a priori unclear which of the two tests is preferable. The fundamental issue of qualifying the particular benefit of each of these structures remains to be a challenging but important subject of future research.

The other LFT representation of interest can be obtained by "pulling out" the whole matrix $A(\delta)$ as an uncertainty, through writing the system $\dot{x} = A(\delta)x$ as

$$\dot{x} = 0x + Iw,\ z = Ix + 0w,\ w = A(\delta)z \tag{7}$$

and thus identifying $F_a = 0$, $F_b = I$, $F_c = I$, $F_d = 0$, $\Delta(\delta) = A(\delta)$. In this case the stability test of Theorem 1, using the class of multipliers (5), amounts to finding $X(\delta)$, S_0, S_1 such that

$$\begin{bmatrix} S_0'A(\delta) + A(\delta)'S_0 & X(\delta) - S_0' - A(\delta)'S_1' \\ X(\delta) - S_0 - S_1A(\delta) & S_1 + S_1' \end{bmatrix} < 0 \tag{8}$$

for all $\delta \in \boldsymbol{\delta}$. Note that this family of inequalities reduces to a system of finitely many LMIs at the extreme points $\boldsymbol{\delta^0}$ if $A(\delta)$ is a multi-affine function of δ. In the case of rational parameter dependence, (8) is in general not easily verifiable such that one has to resort to the previous test. Note that (8) resembles the robust stability conditions presented in [1] which have been derived using variants of the Projection Lemma. Instead, the test (8) results in a straightforward fashion from the more general Theorem 1 and has the further advantage of leading to exact nominal H_∞ performance characterizations, as we will see in the next section.

To conclude, we stress again that the above two tests are obtained from Theorem 1 on the basis of two different LFT representations of the system $\dot{x} = A(\delta)x$. Unfortunately, it cannot be a priori decided whether the two conditions are equivalent and, if not, which is the least conservative. It is, however, clear that the first one leads to a convex stability test even if $A(\delta)$ depends rationally on δ whereas the second requires multi-affine parameter dependence. Moreover, the latter will turn out to have an advantageous structure that can be exploited to arrive at convex optimization based robust controller synthesis algorithms.

3 Robust performance analysis

The robust stability tests of the previous section admit a straightforward extension to robust performance. Let us consider the uncertain input-output system

$$\begin{aligned} \dot{x} &= A(\delta)x + B(\delta)w, \quad x(0) = 0 \\ z &= C(\delta)x + D(\delta)w \end{aligned} \tag{9}$$

with continuous functions $A(\delta)$, $B(\delta)$, $C(\delta)$, $D(\delta)$ in $\delta \in \boldsymbol{\delta} = \mathrm{co}(\boldsymbol{\delta^0})$ and with $A(0)$ being Hurwitz. For fixed Q_p, S_p, $R_p \geq 0$, this system is said to be robustly performing if it is robustly stable and if there exists an $\epsilon > 0$ such that

$$\int_0^\infty \begin{bmatrix} w(t) \\ z(t) \end{bmatrix}' \begin{bmatrix} Q_p & S_p \\ S_p' & R_p \end{bmatrix} \begin{bmatrix} w(t) \\ z(t) \end{bmatrix} dt \leq -\epsilon \|w\|_2^2$$

for every $w \in \mathcal{L}_2$. As special cases we obtain the standard L_2-gain specification for $Q_p = -\gamma^2 I$, $S_p = 0$, $R_p = I$ and the 'strict positive real' specification for $Q_p = 0$, $S_p = -I$, $R_p = 0$.

Again it is not difficult to prove that robust performance is equivalent to the existence of a continuous $X(\delta)$ that satisfies

$$\begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix}' \left[\begin{array}{cc|cc} 0 & X(\delta) & & \\ X(\delta) & 0 & & \\ \hline & & Q_p & S_p \\ & & S_p' & R_p \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A(\delta) & B(\delta) \\ \hline 0 & I \\ C(\delta) & D(\delta) \end{array}\right] < 0. \tag{10}$$

For an LFT description

$$\begin{bmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{bmatrix} = \\ = \begin{bmatrix} A & B_1 \\ C_1 & D_1 \end{bmatrix} + \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} \Delta(\delta)(I - D_2\Delta(\delta))^{-1} \begin{bmatrix} C_2 & D_{21} \end{bmatrix}$$

with continuous $\Delta(\delta)$ one can assure well-posedness and (10) if there exist continuous $X(\delta)$, $P(\delta)$ with (3) and

$$\begin{bmatrix} * \\ * \\ * \\ * \\ * \\ * \end{bmatrix}' \left[\begin{array}{cc|cc|c} 0 & X(\delta) & & & \\ X(\delta) & 0 & & & \\ \hline & & Q_p & S_p & \\ & & S_p' & R_p & \\ \hline & & & & P(\delta) \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ A & B_1 & B_2 \\ \hline 0 & I & 0 \\ C_1 & D_1 & D_{12} \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_2 \end{array}\right] < 0 \tag{11}$$

identically in $\delta \in \boldsymbol{\delta}$.

If $A(\delta)$, $B(\delta)$, $C(\delta)$, $D(\delta)$ are rational functions without pole in 0, one can determine an LFT representation with diagonal affine $\Delta_d(\delta) = \mathrm{diag}(\delta_1 I, ..., \delta_k I)$. For the specific class of parameter dependent multipliers (5) this leads to a novel LMI algorithm for the search of a multi-affine Lyapunov matrix $X(\delta)$ that guarantees robust performance.

If the system's parameter dependence is multi-affine, one can work instead with the LFT representation

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \end{bmatrix} = \left[\begin{array}{c|c|cc} 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ w_1 \\ w_2 \end{bmatrix},\ w_2 = \underbrace{\begin{bmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{bmatrix}}_{M(\delta)} z_2.$$

With the particular family (5), inequality (11) leads after simple rearrangements to

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & Q_p \end{bmatrix} + S_0'M(\delta) + M(\delta)'S_0 & * \\ \begin{bmatrix} X(\delta) & 0 \\ 0 & S_p' \end{bmatrix} - S_0 - S_1M(\delta) & \begin{bmatrix} 0 & 0 \\ 0 & R_p \end{bmatrix} + S_1 + S_1' \end{bmatrix} < 0. \tag{12}$$

We stress that both the derivations and the whole discussion in the previous section for robust stability extend to these robust performance tests. Let us now prove that the test based on (12) comprises any alternative that guarantees (10) with a parameter-independent Lyapunov matrix.

Lemma 2 *If the constant Lyapunov matrix $X(\delta) = X$ satisfies (10), there exist S_0, S_1 such that (12) is satisfied with the same $X(\delta) = X$.*

Proof. Let us choose

$$S_0 = \begin{bmatrix} X & 0 \\ 0 & S_p' \end{bmatrix},\ S_1 = \begin{bmatrix} -\alpha X & 0 \\ 0 & -R_p - \alpha I \end{bmatrix}$$

with some scalar parameter $\alpha > 0$. It then suffices to show that (12) holds with R_p replaced by $R_p + \alpha I$. Since $X > 0$ and $R_p + \alpha I > 0$, this inequality reads (after taking Schur complement) as

$$\begin{bmatrix} 0 & 0 \\ 0 & Q_p \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & S_p \end{bmatrix} M(\delta) + M(\delta)' \begin{bmatrix} X & 0 \\ 0 & S_p' \end{bmatrix} + \\ + M(\delta)' \begin{bmatrix} \frac{1}{2}\alpha X & 0 \\ 0 & R_p + \alpha I \end{bmatrix} M(\delta) < 0.$$

Clearly, this latter inequality is implied by (10) (with X replacing $X(\delta)$) for $\alpha = 0$. Hence, by continuity, the inequality persists to hold for all small $\alpha > 0$. ■

Remarks.

• Lemma 2 reveals that the robust performance test based on (12) is certainly not more conservative (and actually less conservative as confirmed by examples) than any alternative test that guarantees (10) with a parameter independent Lyapunov matrix. As the proof shows, this is even true if we restrict the scalings to

$$S_0 = \begin{bmatrix} S & 0 \\ S_{01} & S_{02} \end{bmatrix},\ S_1 = \begin{bmatrix} -\alpha S' & S_{11} \\ 0 & S_{12} \end{bmatrix} \quad \text{with} \quad \alpha > 0 \tag{13}$$

whose specific structure will allow the extension to robust state-feedback synthesis as seen below. Consequently, these synthesis conditions are also guaranteed to be not more conservative than those in [3] based on a parameter independent Lyapunov function.

• If considering nominal performance $\boldsymbol{\delta} = \{0\}$, Lemma 2 implies that the solvability of (10) and (12) are equivalent, even if restricting the scalings as in (13). This is in stark contrast to the conditions suggested in [1] that are only sufficient for H_∞ nominal performance.

4 Robust state-feedback synthesis

We have seen that the use of different LFT representations of the uncertain system lead to structurally different robustness tests. Unfortunately, the analysis tests based on (4),(5) or on (11),(5) cannot be applied to arrive at convex synthesis conditions since the controller parameters typically multiply the Lyapunov matrix $X(\delta)$.

As the main benefit of (8) and (12), the Lyapunov matrix $X(\delta)$ does not multiply the parameters of the system description what renders the corresponding analysis tests amenable to synthesis. This structural phenomenon has been observed for the first time in [6] for discrete-time systems and it has been partially extended to continuous-time systems in [1].

The robust performance design problem by static state-feedback amounts to finding a gain F such that

$$\dot{x} = [A(\delta) + \widehat{B}(\delta)F]x + B(\delta)w$$
$$z = [C(\delta) + \widehat{D}(\delta)F]x + D(\delta)w$$

is robustly performing. This is guaranteed by (12) with

$$M(\delta)' = \begin{bmatrix} A(\delta) + \widehat{B}(\delta)F & B(\delta) \\ C(\delta) + \widehat{D}(\delta)F & D(\delta) \end{bmatrix}$$

for some $X(\delta)$ and for multipliers S_0, S_1 which admit the structure (13). If we perform the substitution $K := FS$, we infer

$$M(\delta)' \left[S_0 \ S_1' \right] = \\ \begin{bmatrix} A(\delta)S + \widehat{B}(\delta)K & B(\delta) \\ C(\delta)S + \widehat{D}(\delta)K & D(\delta) \end{bmatrix} \left[\begin{bmatrix} I & 0 \\ S_{01} & S_{02} \end{bmatrix} \ \begin{bmatrix} \alpha I & 0 \\ S_{11}' & S_{12}' \end{bmatrix} \right].$$

For a fixed $\alpha > 0$, it turns out that the resulting inequality is affine in all the variables K, S, S_{01}, S_{02}, S_{11}, S_{12}. In the same fashion as for analysis, we can hence test the existence of K, S, S_{01}, S_{02}, S_{11}, S_{12} and of a multi-affine Lyapunov matrix $X(\delta)$ that render the synthesis inequalities satisfied. If having found a solution, S can be assumed nonsingular (after perturbation, if necessary) and then $F = KS^{-1}$ is a controller that guarantees robust performance. With a line-search procedure, we can exploit the extra degree of freedom in the parameter $\alpha > 0$ to render the synthesis inequalities satisfied.

Remarks.

• Recall that the analysis test could be performed for general S_0, S_1. To render the synthesis inequalities convex, we need to impose the extra structure (13). Nevertheless, due to Lemma 2, the resulting synthesis algorithm is not more conservative than those based on the search for a parameter-independent Lyapunov function. This is considered to be the remedy to the main drawback of the performances inequalities suggested in [1], and a numerical example will as well confirm this benefit.

• If the parameter is varying with time, we can resort to the framework developed in [7] and conclude that all our results do admit immediate extensions to this type of uncertainties.

• It is straightforward to extend our results to multi-objective output feedback synthesis along the lines of [13, 11, 5, 1]. As for state-feedback synthesis, the presence of extra multipliers and the line-search parameter α provides extra freedom to arrive at less conservative results.

5 Discrete-time systems

The discrete time versions of the suggested robust analysis and synthesis tests are obtained in a straightforward way, by just performing the substitution

$$\begin{bmatrix} 0 & X(\delta) \\ X(\delta) & 0 \end{bmatrix} \to \begin{bmatrix} -X(\delta) & 0 \\ 0 & X(\delta) \end{bmatrix} \quad (14)$$

in (4) and (11) and the corresponding adjustments in all test derived from these inequalities.

After this substitution, (8) reads as

$$\begin{bmatrix} -X(\delta) + S_0'A(\delta) + A(\delta)'S_0 & -S_0' - A(\delta)'S_1' \\ -S_0 - S_1A(\delta) & X(\delta) + S_1 + S_1' \end{bmatrix} < 0 \quad (15)$$

and the performance inequality (12) is given by

$$\begin{bmatrix} \begin{bmatrix} -X(\delta) & 0 \\ 0 & Q_p \end{bmatrix} + S_0'M(\delta) + M(\delta)'S_0 & * \\ \begin{bmatrix} 0 & 0 \\ 0 & S_p' \end{bmatrix} - S_0 - S_1M(\delta) & \begin{bmatrix} X(\delta) & 0 \\ 0 & R_p \end{bmatrix} + S_1 + S_1' \end{bmatrix} < 0. \quad (16)$$

One can directly verify that the tests in [6, 5] correspond to the specializations

$$S_0 = 0,\ S_1 = -G \ \text{ or } \ S_0 = \begin{bmatrix} 0 & 0 \\ 0 & S_p' \end{bmatrix},\ S_1 = \begin{bmatrix} -G & 0 \\ 0 & -R_p \end{bmatrix} \quad (17)$$

respectively. With the choice

$$S_0 = \begin{bmatrix} \alpha X & 0 \\ 0 & S_p' \end{bmatrix} \ \text{ and } \ S_1 = \begin{bmatrix} -X & 0 \\ 0 & -R_p - \alpha I \end{bmatrix}$$

we recover the nominal performance inequalities for $\alpha \to 0$, similarly as in the proof of Lemma 2. Finally, the state-feedback synthesis inequalities can be rendered convex by a controller parameter transformation for the multiplier family

$$S_0 = \begin{bmatrix} \alpha S & 0 \\ S_{01} & S_{02} \end{bmatrix} \ \text{ and } \ S_1 = \begin{bmatrix} -S' & S_{11} \\ 0 & S_{12} \end{bmatrix} \ \text{ with } \ \alpha > 0.$$

Since this latter choice of multipliers involves more degrees of freedom, the resulting synthesis procedure leads to results that are less conservative than those in [6, 5] as seen by simple examples.

6 Numerical examples

Example 1. Consider the uncertain system

$$\dot{x} = \begin{bmatrix} -1 & \delta_1 & 0 & \delta_2 \\ 0.5\delta_1 & -2 & 0.5\delta_2 & 0 \\ 4\delta_1 & 0 & -3+2\delta_2 & 0 \\ 0 & -4\delta_1 & 0 & -4-2\delta_2 \end{bmatrix} x + \begin{bmatrix} \delta_1+\delta_2 \\ 0.5\delta_1+0.5\delta_2 \\ 4\delta_1+2\delta_2 \\ -4\delta_1-2\delta_2 \end{bmatrix} w$$
$$z = \begin{bmatrix} \delta_1 & \delta_1 & \delta_2 & \delta_2 \end{bmatrix} x + \begin{bmatrix} 2\delta_1 + 2\delta_2 \end{bmatrix} w$$

with $(\delta_1, \delta_2) \in \{(\delta_1, \delta_2) : \ \delta_1, \delta_2 \in [-r, r]\}$. For various values r of the diameter of the parameter box, we compute the minimal value of the worst L_2-gain of $w \to z$ that can be guaranteed with different tests. For reasons of comparison we plot in Figure 1 the percentage loss of the various test in comparison to that based on (12). Curve C shows the smallest loss and results from constraining the multipliers as in (13) (including a line search over $\alpha > 0$). Curve B results from using a parameter-independent Lyapunov matrix in (12), whereas Curve A results from applying the test in [1]. The figure shows an increase in conservatism for increasing diameters of the parameter box, reaching levels of about 8%, 20% and more than 100% for the tests C, B and A respectively.

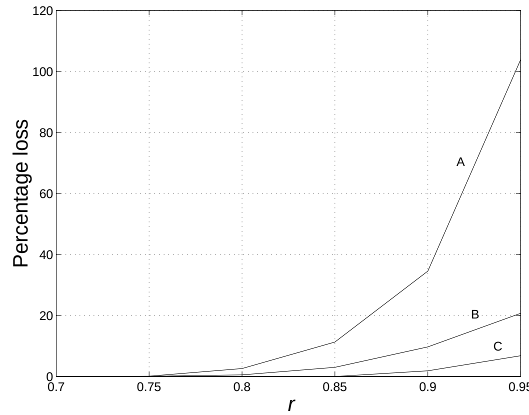

Figure 1: *Percentage losses of guaranteed L_2-gain levels in Example 1.*

Example 2. Consider the discrete-time uncertain system

$$x_{k+1} = \begin{bmatrix} -0.1 & \delta_1 & 0 & \delta_2 \\ 0.5\delta_1 & -0.2 & 0.5\delta_2 & 0 \\ 4\delta_1 & 0 & -0.3+2\delta_2 & 0 \\ 0 & -4\delta_1 & 0 & -0.4-2\delta_2 \end{bmatrix} x_k + \begin{bmatrix} \delta_1+\delta_2 \\ 0.5\delta_1+0.5\delta_2 \\ 4\delta_1+2\delta_2 \\ -4\delta_1-2\delta_2 \end{bmatrix} w_k$$
$$z(k) = \begin{bmatrix} \delta_1 & \delta_1 & \delta_2 & \delta_2 \end{bmatrix} x_k + \begin{bmatrix} 2\delta_1 + 2\delta_2 \end{bmatrix} w_k$$

with the same parameter range as in Example 1. As before let us compare the l_2-gain levels of $w \to z$ that can be guaranteed with (16) for free S_0, S_1 with the test obtained after the specialization (17) what corresponds to the characterization suggested in [5]. In Figure 2 we observe again an increase in conservatism for increasing r, reaching a level of more than 30% for $r = 0.6$.

Example 3. In this example we want to design a state-feedback controller $u = Fx$ that minimizes the worst L_2 gain of $w \to z$ for

$$\dot{x} = \begin{bmatrix} -1 & \delta_1 & 0 & \delta_2 \\ 0.5\delta_1 & -2 & 0.5\delta_2 & 0 \\ 4\delta_1 & 0 & -3+2\delta_2 & 0 \\ 0 & -4\delta_1 & 0 & -4-2\delta_2 \end{bmatrix} x + \begin{bmatrix} \delta_1+\delta_2 \\ 0.5\delta_1+0.5\delta_2 \\ 4\delta_1+2\delta_2 \\ -4\delta_1-2\delta_2 \end{bmatrix} w + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u$$
$$z = \begin{bmatrix} \delta_1 & \delta_1 & \delta_2 & \delta_2 \end{bmatrix} x + \begin{bmatrix} 2\delta_1 + 2\delta_2 \end{bmatrix} w$$

and for different values r of the radius of the parameter box $\{(\delta_1, \delta_2) : \ \delta_1, \delta_2 \in [-r, r]\}$. Figure 3 displays the percentage losses in the guaranteed L_2-gain synthesis levels for the algorithm from [1] (Curve A), for synthesis with a parameter-independent Lyapunov matrix (Curve B) if compared to our synthesis algorithm based on (12) with the triangular multipliers (13) and including a line-search over $\alpha > 0$.

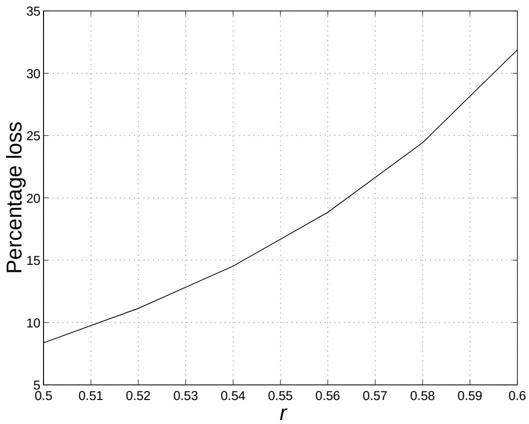

Figure 2: *Percentage losses of guaranteed l_2-gain levels in Example 2.*

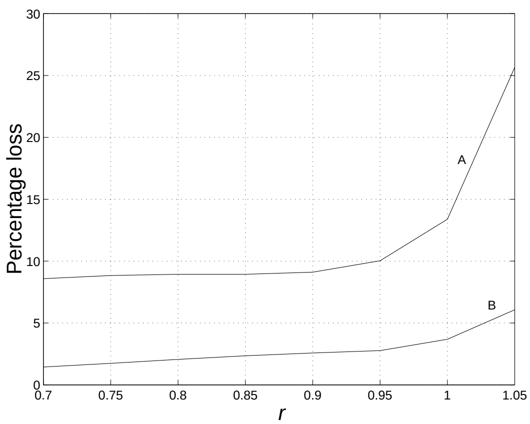

Figure 3: *Percentage losses of different algorithms in the determination of the synthesis L_2 gain in Example 3.*

7 Conclusions

In this paper we have introduced a novel characterization for robust stability and performance of parameter dependent systems. As the main feature we have shown how to derive in a straightforward fashion both for continuous- and discrete-time systems tests that are less conservative if compared to results recently proposed in the literature. Moreover, we have briefly discussed that different LFT representations of the parameter dependent systems lead to different versions of the robustness tests, one allowing for rational parameter dependence and the other being more suitable for controller synthesis. The benefit of the new tests has been illustrated by means of various examples.

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