A Fake Algebraic Riccati Technique for Tracking Superimposed Signals

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Abstract. This paper describes an adaptive nonlinear filter for tracking superimposed signals. The filter gain is selected via a fake algebraic Riccati equation. A passivity approach is applied to deduce stability conditions for the filter error system. The performance is compared with an extended Kalman filter for tracking multiple frequency modulated signals.

I. INTRODUCTION

The extended Kalman filter (EKF) [1] is used ubiquitously for state estimation within many communication and aerospace applications. However the EKF can exhibit suboptimal performance when there are errors or uncertainties in the problem assumptions. We seek to develop a filter which exhibits performance benefits in the presence of problem uncertainties.

The nonlinear state estimation problem is formulated along similar lines to the application of the EKF described in [1,2]. The EKF uses the nonlinear plant update and measurement function to compute a prediction error, which is then multiplied by the Kalman gain matrix derived from the linearised system and added to the state estimate. The EKF is not guaranteed to be stable and a so-called robust filter is desired in which optimality is traded off in return for increased stability. In a previous approach to robust nonlinear filtering [3], we have retained the structure of the EKF and sought a solution to a Riccati equation that achieves a compromise between least squares and H_∞ optimality criteria. In this paper optimality is pursued via the fake algebraic Riccati techniques of [4-7] and stability conditions are obtained in a passivity framework [8].

This paper generalises the fake algebraic Riccati approach from a one signal application [9] to a superimposed signal case. The nonlinear filter is developed in Section II. Section III describes the application of the fake algebraic Riccati technique to select the filter gain. The stability conditions for the error system are set out in Section IV. The superimposed signals application and an example making use of the stability conditions are discussed in Sections V and VI respectively.

II. DEVELOPMENT OF A NONLINEAR FILTER

Consider the following model, comprising a stable, linear state evolution and a nonlinear output mapping

$$x_{k+1} = Ax_k + w_k, \;\; y_k = c_k(x_k) + v_k, \qquad (1,2)$$

where w_k and v_k are uncorrelated, zero mean, white, l and p order processes with known covariances Q and R respectively. The matrix A and the matrix function $c_k(.)$ are of appropriate dimensions. It is assumed that the components of $c_k(.)$ are continuous and differentiable. A recursive filter is desired which yields estimates of x_k, denoted by $\hat{x}_k$, given measurements y_k for each k > 0. A nonlinear observer may be constructed having the form

$$\hat{x}_{k+1} = A\hat{x}_k + g_k(y_k - c_k(\hat{x}_k)), \qquad (3)$$

where $g_k(.)$ is a nonlinear gain function to be designed. From (1) – (3), the state prediction error may be written as

$$\tilde{x}_{k+1} = A\tilde{x}_k - g_k(\varepsilon_k) + w_k, \qquad (4)$$

where $\tilde{x}_{k+1} = x_k - \hat{x}_k$ and $\varepsilon_k = y_k - c_k(\hat{x}_k)$ is the output prediction error. The Taylor series expansion of the output mapping $c_k(.)$ to terms linear in the state error yields $c_k(x_k) = c_k(\hat{x}_k) - C_k\tilde{x}_k$ where $C_k = [\partial c_k(x)/\partial x]_{x=\hat{x}_k}$. It follows that the prediction error is approximately $\varepsilon_k \approx C_k\tilde{x}_k + v_k$. The objective here is to design the $g_k(.)$ to be a linear function of $\tilde{x}_k$ to first order terms. It will be shown that for certain classes of problems this can be achieved by a suitable choice of a nonlinear matrix function D_k resulting in the adaptive gain function

$$g_k = K_k D_k \varepsilon_k, \qquad (5)$$

where K_k is a gain matrix of appropriate dimensions. The observer system (3) is adaptive because the $g_k(.)$ is a function of the time-varying state estimates. In view of (5), the locally linearised error (4) may be written as

$$\tilde{x}_{k+1} = (A - K_k\overline{C}_k)\tilde{x}_k - K_k D_k v_k + w_k, \qquad (6)$$

where $\overline{C}_k = D_k C_k$. Suppose that a bounded matrix function of the states, D_k, can be found so that $\overline{C}_k = D_k C_k$ is a constant matrix. If $|\lambda(A)| < 1$ and if the pair $[\overline{C}_k, A]$ is completely observable, then the asymptotic stability of (6) can be guaranteed by placing the eigenvalues arbitrarily to ensure $|\lambda(A - K_k\overline{C}_k)| < 1$. A method for choosing the gain K_k is described in the next section.

III. GAIN SELECTION VIA FAKE ALGEBRAIC RICCATI EQUATIONS

In the case of the EKF, the gain is specified by the solution to a Riccati difference equation (RDE). We propose a procedure here which retains the familiar gain structure of the EKF, but with a “covariance” matrix that is choosen by the filter designer rather than by the solution of an RDE.

From (6), an approximate equation for the estimation error covariance $P_k = E\{\tilde{x}_{k+1}\tilde{x}_{k+1}^T\}$, neglecting all interdependencies, may be written as

$$P_{k+1} = (A - K_k\bar{C}_k)P_k(A - K_k\bar{C}_k)^T + K_k D_k R D_k^T K_k^T + Q. \qquad (7)$$

The optimal K_k which minimises P_{k+1} is given by

$$K_k = P_k\bar{C}_k^T(\bar{C}_k P_k \bar{C}_k^T + D_k R D_k^T)^{-1}, \qquad (8)$$

where P_k is found by solving the RDE

$$P_{k+1} = AP_kA^T - P_k\bar{C}_k^T(\bar{C}_k P_k\bar{C}_k^T + D_k R D_k^T)^{-1}\bar{C}_k P_k + Q. \quad (9)$$

In general the solutions P_k, while positive definite, need not be stabilizing because of the impact of the nonlinearities $c_k(.)$, $g_k(.)$ and therefore the resulting error system can lack stability. The fake ARE technique, also known as “covariance setting”, relies on connections between the RDE and algebraic Riccati equation (ARE) stability results [4–7]. Using the approach of [4–7], the RDE (9) may be masqueraded by the fake ARE

$$\Sigma = A\Sigma A^T - \Sigma\bar{C}_k^T(\bar{C}_k\Sigma\bar{C}_k^T + D_k R D_k^T)^{-1}\bar{C}_k\Sigma + Q. \quad (10)$$

That is, rather than finding a solution to (9), instead an arbitrary fixed positive definite solution Σ in (10) is assumed and then the gain may be calculated from (8), using Σ in place of P_k.

IV. STABILITY CONDITIONS

In this section we seek to identify conditions for a system such as (4) to be asymptotically stable. The problem is recast in a passivity framework in which there is a cascade of a linear system and a block of memoryless nonlinearities shown in Figure 1. This requires that the error system (4) is reformulated as

$$e = u - G\gamma(e), \qquad (11)$$

where G is a stable, linear system, $\gamma(.)$ is a nonlinear function matrix satisfying specified sector conditions, e and u denote the vectors $[e_k^{(1)}, e_k^{(2)}, e_k^{(3)}, \ldots, e_k^{(q)}]^T$ and $[u_k^{(1)}, u_k^{(2)}, u_k^{(3)}, \ldots, u_k^{(q)}]^T$ respectively. We set out the generalisation of the discrete-time Popov criterion [8] for the multiple-input-multiple-output case. Since we are motivated by the problem of tracking superimposed signals, our attention is confined to $\gamma(.)$ consisting of identical noninteracting nonlinearities. Namely for $i = 1, 2, \ldots, n$, the $\gamma(e_k^{(i)})$ depend only on $e_k^{(i)}$. Let $< a, b > = \sum_{i=1}^{n}(a_i b_i)$ denote the inner product of $a, b \in \mathbf{R}^n$.

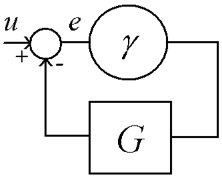

Fig. 1. Nonlinear system model.

Lemma: Consider the system (11) where u, e map $R^q \to R^q$. Suppose that $\gamma(.)$ consists of identical, noninteracting nonlinearities, with $\gamma(e^{(i)})$ monotonically increasing and in the sector [0,β], i.e.,

$$0 \le \gamma(e^{(i)})/e^{(i)} \le \beta, \qquad (12)$$

$\forall\ e^{(i)} \in$ R with $e^{(i)} \neq 0$. Let G be a causal, stable, time-invariant map $R^q \to R^q$, having finite gain and suppose that G has a z-transform $\hat{G}(e^{j\theta})$, which is bounded on the unit circle. Suppose that for some $q \geq 0$, there is a $\delta > 0$ such that

$$< (G + q\nabla G + I\beta^{-1})e, e > \ge \delta < e, e >, \qquad (13)$$

$\forall\ e^{(i)} \in \mathrm{R}^n$. Under these conditions then $u \in \ell_2 \Rightarrow e, \gamma(e) \in \ell_2$.

Proof: From (11) we have $\nabla u = \nabla e + \nabla G\gamma(e)$ and

$$u + q\nabla u = (G + q\nabla G + I\beta^{-1})\gamma(e) + e - \beta^{-1}\gamma(e) + q\nabla e. \qquad (14)$$

Then

$$< u + q\nabla u, \gamma(e) > = < e - \beta^{-1}\gamma(e), \gamma(e) > + < q\nabla e, \gamma(e) > + < (G + q\nabla G + \beta^{-1})\gamma(e), \gamma(e) >. \qquad (15)$$

Consider the first term on the right hand side of (15), since the $\gamma(e)$ consists of noninteracting nonlinearities, then

$< \gamma(e), e > = \sum_{i=1}^{n} < \gamma(e^{(i)}), e^{(i)} >$ and

$$< e - \gamma(e)\beta^{-1}, \gamma(e) > = \sum_{i=1}^{n} < e^{(i)} - \gamma(e^{(i)})\beta^{-1}, \gamma(e^{(i)}) > \ge 0.$$

Using the approach of [8] together with the sector conditions on the identical noninteracting nonlinearities (12), it can be shown that expanding out the second term of (15), yields $< \nabla e, \gamma(e) > \ge 0$. Using $\| \nabla u \| \le 2 \| u \|$ (p. 192 of [8]), the Schwartz inequality and the triangle inequality, it can be shown that

$$< (u + q\nabla u, \gamma(e) > \le (1 + 2q)\| u \|. \qquad (16)$$

It follows from (13), (15) and (16) that $\| \gamma(e) \|^2 \le (1 + 2q)\delta^{-1} \| u \|$, hence $\gamma(e) \in \ell_2$. We also have $G\gamma(e) \in \ell_2$ since the gain of G is finite.

If the linear part is stable and bounded on the unit circle then the condition (13) becomes

$$\lambda_{\min}\{[I + q(I - z^{-1}I)][\hat{G}(e^{j\theta}) + \hat{G}(e^{j\theta})^*] + \beta^{-1}\} \ge \delta, \qquad (17)$$

for $z = e^{j\theta}$, $0 \le \theta \le \pi$ (see pp. 175, 194 of [8]).

V. APPLICATION TO TRACKING SUPERIMPOSED SIGNALS

In this section we set out a parallel demodulator for the case where multiple signals are present within the observed data. A demodulator for tracking two or more superimposed frequency modulated signals follows by constructing an EKF for an augmented state space system that includes the multiple signal components [2]. Since the multiple component problem is a simple generalisation of the two component problem, the treatment is restricted to the latter.

Consider two superimposed frequency or phase modulated signals modelled by (1) where $x_k = [a_k^{(1)}, \omega_k^{(1)}, \phi_k^{(1)}, a_k^{(2)}, \omega_k^{(2)}, \phi_k^{(2)}]^T$, $A=diag[A^{(1)}, A^{(2)}]$, in which $A^{(i)} = \begin{bmatrix} \mu_a & 0 & 0 \\ 0 & \mu_b & 0 \\ 0 & 1 & 0.9999 \end{bmatrix}$. The states $a_k^{(i)}$, $\omega_k^{(i)}$ and $\phi_k^{(i)} \in R$ represent the instantaneous amplitude, frequency and phase components. Let

$$\begin{bmatrix} y_k^{(1)} = \sum_{i=1}^{2} a_k^{(i)} \cos\phi_k^{(i)} + v_k^{(1)} \\ y_k^{(2)} = \sum_{i=1}^{2} a_k^{(i)} \sin\phi_k^{(i)} + v_k^{(2)} \end{bmatrix} \tag{18}$$

denote the complex, baseband observations where $y_k^{(i)}, v_k^{(i)} \in R$. Expanding the prediction error to terms linear in the estimation error yields $C_k = [C_k^{(1)}, C_k^{(2)}]$ where

$$C_k^{(i)} = \begin{bmatrix} \cos\hat{\phi}_k^{(i)} & 0 & -\hat{a}_k^{(i)} \sin\hat{\phi}_k^{(i)} \\ \sin\hat{\phi}_k^{(i)} & 0 & \hat{a}_k^{(i)} \cos\hat{\phi}_k^{(i)} \end{bmatrix}. \tag{19}$$

This form suggests $D_k = \begin{bmatrix} D_k^{(1)} \\ D_k^{(2)} \end{bmatrix}$ where

$$D_k^{(i)} = \begin{bmatrix} \cos\hat{\phi}_k^{(i)} & \sin\hat{\phi}_k^{(i)} \\ \dfrac{\sin\hat{\phi}_k^{(i)}}{\hat{a}_k^{(i)}} & \dfrac{\cos\hat{\phi}_k^{(i)}}{\hat{a}_k^{(i)}} \end{bmatrix}. \tag{20}$$

VI. EXAMPLE

We consider the problem of demodulating two superimposed FM signals. Two 8 kHz speech samples (i.e, “Matlab is number one” and “Number one is Matlab”), centred at ± 0.25 rads/sec, were used to synthesize two superimposed, unity amplitude, frequency modulated signals. An EKF was constructed using the model (1) and (17),with $\mu_\omega^{(i)}=0.9$ and $(\sigma_\omega^{(i)})^2= 0.02$. A fake ARE filter (3) was constructed using (5), (8), (19) and (20). It was found that a suitable parameter choice for an arbitrary solution to (10) is $\Sigma = diag\,[\Sigma^{(1)}, \Sigma^{(2)}]$ where $\Sigma^{(i)} = \begin{bmatrix} 0.001 & 0.08 \\ 0.08 & 0.7 \end{bmatrix}$. Neglecting observation noise, a suitable approximation of (6) in the form (11), is

$$\begin{bmatrix} \tilde{\omega}_{k+1}^{(1)} \\ \tilde{\phi}_{k+1}^{(1)} \\ \tilde{\omega}_{k+1}^{(2)} \\ \tilde{\phi}_{k+1}^{(2)} \end{bmatrix} = (A - K_k \overline{C}) \begin{bmatrix} \tilde{\omega}_{k}^{(1)} \\ \tilde{\phi}_{k}^{(1)} \\ \tilde{\omega}_{k}^{(2)} \\ \tilde{\phi}_{k}^{(2)} \end{bmatrix} - K_k \begin{bmatrix} \sin\tilde{\phi}_k^{(1)} - \tilde{\phi}_k^{(1)} \\ \sin\tilde{\phi}_k^{(2)} - \tilde{\phi}_k^{(2)} \end{bmatrix}, \tag{20}$$

where $\overline{C} = \begin{bmatrix} 0100 \\ 0001 \end{bmatrix}$. It follows that the linear part of (20) may be written as $G(z) = \overline{C}(zI - (A - K_k\overline{C}))^{-1} K_k$. From Section V, for the stability of (20), *G(z)* must be stable and a $\delta > 0$ must be found satisfying (17) for a $q > 0$. In this example $\beta = 1.2$ and with $q = 0.001$, a $\delta = 0.82$ was found for stable *G(z)*, obtained by redefining $K_k = (0.96)^i K_k$, where at each time *k*, the *i* is a sufficiently large integer value such that $|\lambda_{\max}\{A - K_k\overline{C}\}| < 1$.

Simulations were conducted with 100 realisations of additive Gaussian white measurement noise, from 0 dB SNR to 30 dB SNR in 6 dB steps. A histogram of mean square error (MSE) exhibited by the two demodulators is shown in Fig. 2. In contrast with [2], here the outliers in the frequency message estimates were not precluded in the calculation of MSE. It can be seen that the EKF performance degrades with increasing SNR. The presence of co-channel signals causes outliers in the frequency estimates. The locally stable fake ARE filter is seen to provide some robustness to outliers; in particular at 30 dB SNR, the reduction in MSE approaches 20 dB.

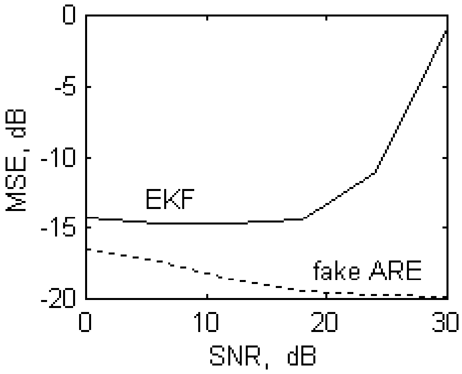

Fig. 2. Performance of cochannel demodulators.

Two mechanisms have been observed for occurrence of outliers or faults within co-channel demodulators. Firstly errors can occur in the state attribution, i.e., there is correct tracking of some component message segments but the tracks are inconsistently associated with the individual signals. This is illustrated by the example frequency estimate tracks shown in Figure 3. Secondly, the phase unwrapping can be erroneous so that the frequency tracks bear no resemblance to the underlying messages. These faults can occur without any significant deterioration in the error residual. An insight into co-channel fault behaviour follows from an observability perspective. (Observability refers to whether or not the states can be uniquely reconstructed from the measurements.) It is conjectured that the phase ambiguities appear because the locally linearised system loses observability. The co-channel demodulators have been observed to be increasingly fault prone at higher SNR. This arises because lower SNR designs possess narrower bandwidths and so are less sensitive to nearby frequency components. Figure 3 illustrates the trade-off between stability and robustness:

the fake ARE approach provides improved stability whereas the EKF exhibits superior fidelity.

There are many possible techniques for mitigating cochannel demodulation faults. An iterative method, which we have used to advantage in [10], involves introducing saturating nonlinearities into the component speech models, which compress the dynamic range of the component signal estimates.

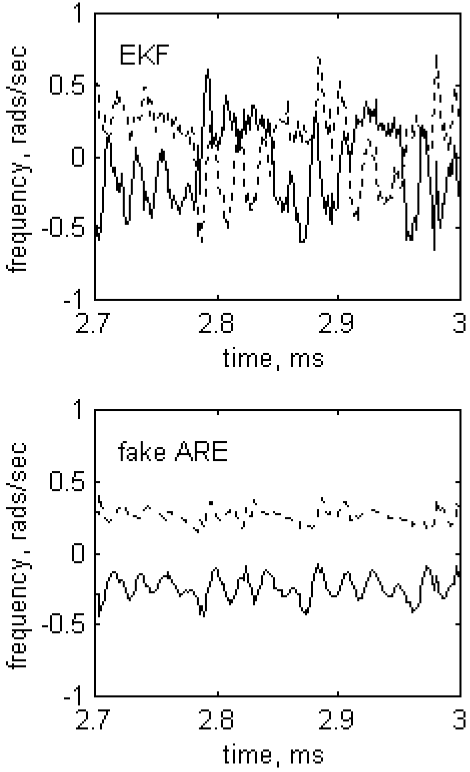

Fig. 3. Example EKF and fake ARE frequency estimate tracks.

VII. CONCLUSION

The application of the fake ARE technique has been applied in the development of an adaptive nonlinear filter for tracking multiple signals. A passivity framework has been used to arrive at conditions for local error stability. The results of simulation studies for the problem of demodulating two superimposed FM signals have been presented in which the fake ARE approach demonstrates improved stability at the cost of fidelity.

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