A New Design of Reduced-order Controllers for Singular $\mathcal{H}_\infty$ Control Problems Based on ARE Approach

Xin Xin

Department of Communication Engineering
Faculty of Computer Science and System Engineering
Okayama Prefectural University
111 Kuboki, Soja, Okayama 719-1197, JAPAN
Email: xxin@c.oka-pu.ac.jp

Abstract

The reduced-order controllers for singular $\mathcal{H}_\infty$ control problems with infinite zeros are studied via ARE (Algebraic Riccati Equation) approach in this paper. First, we analyze the relation between the eigenstructure corresponding to the infinite zeros and the parameterization of $\mathcal{H}_\infty$ controllers given in the descriptor form. Next, we propose a method to choose appropriately the free parameter in the parameterization for constructing the $\mathcal{H}_\infty$ controllers whose orders equal to the size of a reduced-order ARE involved in solvability conditions of the singular $\mathcal{H}_\infty$ control problems. The parameterization of the reduced-order $\mathcal{H}_\infty$ controllers is also discussed.

1 Introduction

Consider a generalized plant $G(s)$ with the following stabilizable and detectable realization

$$\begin{aligned}\begin{bmatrix} z \\ y \end{bmatrix} &= G(s)\begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix} \\ &= \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]\begin{bmatrix} w \\ u \end{bmatrix}, \end{aligned} \tag{1}$$

where $z \in R^m$, $y \in R^q$, $w \in R^r$ and $u \in R^p$ are the controlled error, the observation output, the exogenous input and the control input, respectively, and $A \in \mathcal{R}^{n\times n}$. The $\mathcal{H}_\infty$ control problem is to find a proper control law $u = K(s)y$ which internally stabilizes the closed-loop system and satisfies $||\Phi||_\infty < 1$, where Φ is the closed-loop transfer function from w to z. If both $G_{12}(s)$ and $G_{21}(s)$ have no zeros on the extended $j\omega$-axis including the infinity, it is well known that the $\mathcal{H}_\infty$ control problem can be solved via the solutions of two AREs. If the above assumption about the zeros does not hold, the corresponding $\mathcal{H}_\infty$ control problem is usually termed as *singular* one. The research on the singular $\mathcal{H}_\infty$ control problem has attracted considerable interests [2], [7], [10], [9], [4], [6], [17].

For a general $\mathcal{H}_\infty$ control problem, though it is difficult to derive an explicit form of controllers with order less than plant order n, it is possible for the singular case when D_{12} or D_{21} is not full rank. In what follows, we give a brief review of the existed results for designing reduced-order controllers for singular $\mathcal{H}_\infty$ control problems. The results can be divided into two categories: ARE approach based one and LMI approach based one. It is regarded that both ARE and LMI approaches have their merit and each yields a different kind of insight into the $\mathcal{H}_\infty$ control problems.

As to the ARE approach, it needs the assumption that $G_{12}(s)$ and $G_{21}(s)$ have no finite $j\omega$-axis zeros. If one of $G_{12}(s)$ and $G_{21}(s)$ has only infinite zeros with order not greater than one, [8] proposed a method to construct reduced-order controllers based on a reduced-order ARE. [2] obtained the same result for the above problem based on generalized eigenvalue problem approach. Moreover, by using perturbation method, [2] extended the above result to the case where the above constraint on the order of infinite zeros was not satisfied. Based on QMI approach (where a QMI can be solved via reduced-order an ARE), [11] constructed a reduced-order controller using a lower-order observer for the singular $\mathcal{H}_\infty$ control problem in which both $G_{12}(s)$ and $G_{21}(s)$ can be allowed to have arbitrary infinite zeros. Under the assumption that D_{21} is full column rank, a reduced-order controller was constructed based on ARE approach in [12]. As to the LMI approach, [14] proposed a method of designing reduced-order controllers for both continuous and discrete-time singular $\mathcal{H}_\infty$ control problems without any assumptions on infinite or finite zeros.

Through different approaches under different assumptions, the known result about the reduced-order controller for singular $\mathcal{H}_\infty$ control problems is: If the $\mathcal{H}_\infty$ control problem is solvable, then the controller order n_c satisfies $n_c \le n^*$, where

$$\begin{aligned} n^* = \min(n - \operatorname{rank}\begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} + \operatorname{rank} D_{12}, \\ n - \operatorname{rank}\begin{bmatrix} C_2 & D_{21} \end{bmatrix} + \operatorname{rank} D_{21}). \end{aligned} \tag{2}$$

As to the bound given by (2), it implies that the reduced-order controller can be achieved by using the state variables which can be directly measured (after an appropriate state transformation). It is generally difficult to derive a controller with order $n_c < n^*$.

From [17], for singular $\mathcal{H}_\infty$ control problems, we know that solvability conditions can be expressed in terms of the stabilizing solutions of two reduced-order AREs with the same forms as those in [3]. Also, the parameterization of all suitable controllers is given in terms of a linear fractional transformation involving a certain fixed transfer function matrix and together with a stable transfer function matrix with gain less than 1 which is free apart from satisfying certain interpolation conditions. Since the $\mathcal{H}_\infty$ controllers are not unique, it is possible to obtain a reduced-order controller by appropriately choosing the free parameter in the parameterization of $\mathcal{H}_\infty$ controllers. However, it is still not clear how to choose such free parameter analytically for general cases.

Note that the dimensions of the reduced-order AREs in [17] are usually smaller than n^*. We will study how to design reduced-order controllers with order less than n^* based on ARE approach under the following assumptions which have been adopted in [2] and [8].

A1 (A, B_2, C_2) is stabilizable and detectable.
A2 $\mathrm{rank} D_{12} < p, \quad \mathrm{rank} D_{21} = q.$
A3 $P_{12}(s)$ and $P_{21}(s)$ have no $j\omega$-axis invariant zeros.

First, we analyze the relation between the eigenstructure corresponding to the infinite zeros and the parameterization of $\mathcal{H}_\infty$ controllers given in the descriptor form in [17]. Next, we propose a method to choose appropriately the free parameter in the parameterization of controllers for constructing the controllers whose orders equal to the size of the reduced-order ARE involved in solvability conditions of $\mathcal{H}_\infty$ control problems.

Notations: The open left complex plane and open complex plane are denoted by $\mathcal{C}_-$ and $\mathcal{C}$, respectively. The set of all $p \times q$ constant real matrices is denoted by $\mathcal{R}^{p\times q}$. I_r denotes the identity matrix of size $r \times r$. $\mathcal{RH}_\infty^{p\times q}$ denotes the set of all $p\times q$ rational stable proper matrices, and $\mathcal{BH}_\infty^{p\times q}$ denotes the subset of $\mathcal{RH}_\infty^{p\times q}$ with $\mathcal{H}_\infty$-norm less than 1. $\rho(X)$ is the maximum eigenvalue of X. $\mathrm{Im}\, A$ and $\mathrm{Ker}\, A$ denote the image space and null space of a matrix A, respectively.

$$C(sE-A)^{-1}B + D := \left[\begin{array}{c|c} -sE+A & B \\ \hline C & D \end{array}\right], \quad E \neq I.$$

If $E = I$, the term $-sE + A$ in the above notation is replaced by A. For invertible matrices M and N, the identity

$$\left[\begin{array}{c|c} -sE+A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} -sMEN+MAN & MB \\ \hline CN & D \end{array}\right] \tag{3}$$

is termed as restricted equivalent transformation under (M, N). If $N = M^{-1}$, the corresponding equivalent transformation is called as similarity transformation under M.

2 Preliminaries

2.1 Infinite eigenstructure

Denote the system matrix pencil of $G_{12}(s)$ with its realization induced from (1) as $-sP_E + P_A$, where

$$P_E := \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad P_A := \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix}. \tag{4}$$

According to assumption **A3**, the above pencil has full normal column rank.

Based on Kronecker theory [5], since $\dim(\mathrm{Ker}\, P_E) = p$, there exist linear independent infinite eigenvectors and extended infinite eigenvectors such that

$$P_E v_j^1 = 0, \quad j = 1, \cdots, p, \tag{5}$$

$$P_E v_j^{k+1} = P_A v_j^k, \quad k = 1, \cdots, k_j - 1, \tag{6}$$

respectively, where $v_j^{k_j}$ is the last (highest) one of each infinite eigenvector chain, satisfying $P_A v_j^{k_j} \notin \mathrm{Im}\, P_E$. Now construct

$$V_\infty := \begin{bmatrix} V_r & V_h \end{bmatrix}, \tag{7}$$

where $V_h \in \mathcal{R}^{(n+p)\times p}$ contains all the *last (highest)* infinite eigenvectors and $V_r \in \mathcal{R}^{(n+p)\times n_r}$ contains all the remaining eigenvectors (in any order), where

$$n_r := n_\infty - p, \quad n_\infty := \sum_{j=1}^{p} k_j. \tag{8}$$

Owing to (5) and (6), $k_j \geq 1$ $(j = 1, \cdots, p)$. Also, if $\mathrm{rank}\, D_{12} = p$, then $k_j = 1$ $(j = 1, \cdots, p)$ and $n_r = 0$. Furthermore, it is easy to see that

$$n_r \geq p - \mathrm{rank}\, D_{12}, \tag{9}$$

where the equality holds if and only if $k_j \leq 2$ $(j = 1, \cdots, p)$, i.e., the highest order of the infinite zero of $G_{12}(s)$ is one owing to [1].

From (7), the complete infinite eigenstructure of $-sP_E + P_A$ is defined by

$$(-sP_E + P_A)V_\infty = P_A V_\infty(-sN + I_{n_\infty}), \tag{10}$$

where $N \in R^{n_\infty \times n_\infty}$ is a nilpotent matrix. From (6), $P_A V_r \in \mathrm{Im}\, P_E$ holds which leads to $\begin{bmatrix} C_1 & D_{12} \end{bmatrix} V_r = 0$; hence we can decompose $P_A V_\infty$ as

$$P_A V_\infty = \left[\begin{array}{cc} A & B_2 \\ \hdashline C_1 & D_{12} \end{array}\right] \left[\begin{array}{c:c} V_r & V_h \end{array}\right] =: \begin{bmatrix} T & \hat{B}_2 \\ 0 & \hat{D}_{12} \end{bmatrix}, \tag{11}$$

which yields

$$T := \begin{bmatrix} A & B_2 \end{bmatrix} V_r, \quad \hat{B}_2 := \begin{bmatrix} A & B_2 \end{bmatrix} V_h, \tag{12}$$

$$\hat{D}_{12} := \begin{bmatrix} C_1 & D_{12} \end{bmatrix} V_h. \tag{13}$$

It follows from Lemma C.2 in [1] that $\hat{D}_{12}$ has full column rank.

2.2 Stable eigenstructures

From the realization of $G_{12}(s)$ induced from (1), a realization of $G_{12}^T(-s)G_{12}(s)$ is readily found, with its system matrix

$$W_{12}(s) := \begin{bmatrix} -sI+A & 0 & B_2 \\ -C_1^TC_1 & -sI-A^T & -C_1^TD_{12} \\ D_{12}^TC_1 & B_2^T & D_{12}^TD_{12} \end{bmatrix}. \tag{14}$$

Let real matrices $\begin{bmatrix} U_1^T & U_2^T & U_3^T \end{bmatrix}^T$ span the stable eigenspaces of $W_{12}(s)$, respectively (corresponding to eigenvalues in $\mathcal{C}_-$). There exists stable Λ_{12} such that

$$W_{12}(s)\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ 0 \end{bmatrix}(-sI+\Lambda_{12}). \tag{15}$$

Under assumption **A3**, from [15],

$$S := \begin{bmatrix} U_1 & T \end{bmatrix} \tag{16}$$

is square and nonsingular.

2.3 Solution to $\mathcal{H}_\infty$ control problems

From [17], we recall the following result.

LEMMA 1 *Under assumptions* **A1** $-$ **A3***, the* $\mathcal{H}_\infty$ *control problem for the plant* $G(s)$ *in (1) is solvable if and only if the following statements hold.*

(i) The following ARE has a stabilizing solution $X_r \geq 0$,

$$(A_r-\hat{B}_{r2}E_{12}^{-1}\hat{D}_{12}^TC_{r1})^TX_r+X_r(A_r-\hat{B}_{r2}E_{12}^{-1}\hat{D}_{12}^TC_{r1})+$$

$$X_r(B_{r1}B_{r1}^T-\hat{B}_{r2}E_{12}^{-1}\hat{B}_{r2}^T)X_r+C_{r1}^T(I-\hat{D}_{12}E_{12}^{-1}\hat{D}_{12}^T)C_{r1}=0, \tag{17}$$

where $E_{12} := \hat{D}_{12}^T\hat{D}_{12}$, *and*

$$A_r := L_1AU_1, \quad B_{r1} := L_1B_1, \tag{18}$$

$$\hat{B}_{r2} := L_1\hat{B}_2, \quad C_{r1} := C_1U_1, \tag{19}$$

$$\begin{bmatrix} L_1^T & L_2^T \end{bmatrix}^T := S^{-1} = \begin{bmatrix} U_1 & T \end{bmatrix}^{-1}, \tag{20}$$

where T, $\hat{B}_2$ *are defined in (12),* $\hat{D}_{12}$ *and* U_1 *are defined in (13) and (15), respectively.*

(ii) The following ARE has a stabilizing solution $Y \geq 0$,

$$Y(A-B_1D_{21}^TE_{21}^{-1}C_2)^T+(A-B_1D_{21}^TE_{21}^{-1}C_2)Y$$

$$+Y(C_1^TC_1-C_2^TE_{21}^{-1}C_2)Y+B_1(I-D_{21}^TE_{21}^{-1}D_{21})B_1^T=0, \tag{21}$$

where $E_{21} := D_{21}D_{21}^T$.

(iii) $\rho(XY) < 1$, *where*

$$X := S^{-T}diag\{X_r, 0\}S^{-1} = L_1^TX_rL_1. \tag{22}$$

Then all $\mathcal{H}_\infty$ *controllers are given by*

$$K(s) = HM(\Pi(s), Q(s))$$

$$:= (\Pi_{11}(s)Q(s)+\Pi_{12}(s))(\Pi_{21}(s)Q(s)+\Pi_{22}(s))^{-1}, \tag{23}$$

where

$$\Pi(s) = \begin{bmatrix} \Pi_{11}(s) & \Pi_{12}(s) \\ \Pi_{21}(s) & \Pi_{22}(s) \end{bmatrix}$$

$$:= \left[\begin{array}{cc|cc} -sI+A_n & B_{n2} & 0 & B_{n1} \\ C_{n1} & N_{12} & -I_p & 0 \\ \hline 0 & I_p & 0 & 0 \\ C_{n2} & 0 & 0 & E_{21}^{1/2} \end{array}\right], \tag{24}$$

where

$$A_n := A+B_1B_1^TX+ZYF_\infty^TF_\infty, \tag{25}$$

$$B_{n1} := -ZL_\infty, \quad B_{n2} := B_2-ZYF_\infty^TN_{12}, \tag{26}$$

$$C_{n1} := -F_\infty, \quad C_{n2} := C_2+D_{21}B_1^TX, \tag{27}$$

$$F_\infty := -E_{12}^{-1/2}(\hat{B}_2^TX+\hat{D}_{12}^TC_1), \tag{28}$$

$$L_\infty := -(YC_2^T+B_1D_{21}^T)E_{21}^{-1/2}, \tag{29}$$

$$Z = (I-YX)^{-1}, \tag{30}$$

$$N_{12} := E_{12}^{-1/2}\hat{D}_{12}^TD_{12}, \tag{31}$$

and $Q(s) \in \mathcal{BH}_\infty^{p\times q}$ *such that* $K(s)$ *in (23) is proper.*

3 Reduced-order $\mathcal{H}_\infty$ Controllers

If D_{12} is singular, so is N_{12} in (31). Then according to Lemma 1, $K(s)$ is non-proper for general $Q(s)$. [17] gives explicit interpolation conditions with respect to infinite zeros to construct proper controllers; however, how to find a reduced-order controller is not discussed there.

Note that ARE (17) is of size $n-n_r$ where n_r is defined in (8). Under assumptions **A2** and **A3**,

$$n-n_r \leq n-p+\operatorname{rank} D_{12} = n^* \tag{32}$$

holds owing to (9). The strict inequality in (32) holds if $k_j > 2$ holds for at least one $j \in \{1, \cdots, p\}$. For the nonsingular $\mathcal{H}_\infty$ control problem, the orders of both "the central $\mathcal{H}_\infty$ controller" and the ARE is n [3]. Therefore, is there a reduced-order $\mathcal{H}_\infty$ controller with order $n-n_r$ for arbitrary infinite zeros? We will attempt to answer this question in what follows.

To this end, with quantities as defined in Section 2, we first need the following result of the infinite eigenstructure of $G_{12}(s)$ from Lemma 3 in [15]: There exists a matrix $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ with $A_{11} \in R^{n_r\times n_r}$ and $A_{22} \in R^{p\times p}$ such that

$$\begin{bmatrix} -sI+A & B_2 \\ C_1 & D_{12} \end{bmatrix}\begin{bmatrix} T & 0 \\ 0 & I_p \end{bmatrix}$$

$$= \begin{bmatrix} T & \hat{B}_2 \\ 0 & \hat{D}_{12} \end{bmatrix}\begin{bmatrix} -sI+A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{33}$$

$$\left| \begin{array}{cc} -sI + A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right| \neq 0, \quad \forall s \in \mathcal{C}. \tag{34}$$

Next, for analyzing (24), we give other expressions of X in (22) and F_∞ in (28) in terms of the generalized eigenvalue problems. To begin with, we define

$$W_{12b}(s) := \left[\begin{array}{ccc} -sI + A & B_1 B_1^T & B_2 \\ -C_1^T C_1 & -sI - A^T & -C_1^T D_{12} \\ D_{12}^T C_1 & B_2^T & D_{12}^T D_{12} \end{array} \right], \tag{35}$$

which is the Hamiltonian matrices associated with a Riccati equation for the standard H^∞ control problem. Let real matrices $\left[\begin{array}{ccc} P_{12}^T & \Phi_{12}^T & U_{12}^T \end{array} \right]^T$ span the stable eigenspaces of $W_{12b}(s)$ (corresponding to eigenvalues in $\mathcal{C}_-$). Then there exists real stable Λ such that

$$W_{12b}(s) \left[\begin{array}{c} P_{12} \\ \Phi_{12} \\ U_{12} \end{array} \right] = \left[\begin{array}{c} P_{12} \\ \Phi_{12} \\ 0 \end{array} \right] (-sI + \Lambda). \tag{36}$$

Now we are ready to give the following result which can be proved by using [13] and [16].

LEMMA 2 *Suppose that the $\mathcal{H}_\infty$ control problem for plant $G(s)$ with assumption $\mathbf{A1} - \mathbf{A3}$ is solvable. Then*

$$P := \left[\begin{array}{cc} P_{12} & T \end{array} \right] \tag{37}$$

is nonsingular, and X in (22) and F_∞ in (28) can be expressed as

$$X = \left[\begin{array}{cc} \Phi_{12} & 0 \end{array} \right] \left[\begin{array}{cc} P_{12} & T \end{array} \right]^{-1}, \tag{38}$$

$$F_\infty = E_{12}^{1/2} \left[\begin{array}{cc} A_{22} U_{12} & -A_{21} \end{array} \right] \left[\begin{array}{cc} P_{12} & T \end{array} \right]^{-1}, \tag{39}$$

respectively, where P_{12} and Φ_{12} are defined in (36), and T is given in (12), A_{12} and A_{22} are given in (33).

We will analyze (24) for constructing reduced-order controllers in what follows. Firstly, by performing the following equivalent transformation on (24)

$$\text{Row 1} = \text{Row 1} + Z Y F_\infty^T \times \text{Row 2},$$

we obtain

$$\Pi(s) = \left[\begin{array}{cc|cc} -sI + A + B_1 B_1^T X & B_2 & -ZYF_\infty & B_{n1} \\ -F_\infty & N_{12} & -I_p & 0 \\ \hline 0 & I_p & 0 & 0 \\ C_{n2} & 0 & 0 & E_{21}^{1/2} \end{array} \right]. \tag{40}$$

Under the similar transformation under $\left[\begin{array}{cc} P & 0 \\ 0 & I_p \end{array} \right]$ on (40), we have

$$\Pi(s) = \left[\begin{array}{ccc|cc} -sI + \Lambda - V_1 B_2 U_{12} & V_1 A T & V_1 B_2 & -V_1 Z Y F_\infty^T & V_1 B_{n1} \\ -V_2 B_2 U_{12} & -sI + V_2 A T & V_2 B_2 & -V_2 Z Y F_\infty^T & V_2 B_{n1} \\ -E_{12}^{1/2} A_{22} & E_{12}^{1/2} A_{21} & N_{12} & -I_p & 0 \\ \hline U_{12} & 0 & I_m & 0 & 0 \\ C_2 P_{12} + D_{21} B_1^T \Phi_{12} & C_2 T & 0 & 0 & E_{21}^{1/2} \end{array} \right], \tag{41}$$

where

$$\left[\begin{array}{c} V_1 \\ V_2 \end{array} \right] := P^{-1}. \tag{42}$$

Since $V_1 T = 0$ and $V_2 T = I$, then from (33), we obtain

$$V_1 A T = V_1 \hat{B}_2 A_{21}, \quad V_1 B_2 = V_1 \hat{B}_2 A_{22},$$

$$V_2 A T = V_2 \hat{B}_2 A_{21} + A_{11}, \quad V_2 B_2 = V_2 \hat{B}_2 A_{22} + A_{12}.$$

It follows from (33) and (31) that $D_{12} = \hat{D}_{12} A_{22}$ and $N_{12} = E_{12}^{1/2} A_{22}$ hold.

Secondly, based on the above relations and by performing the restricted equivalent transformation in (3) under

$$\left(\left[\begin{array}{ccc} I & 0 & -V_1 \hat{B}_2 E_{12}^{-1/2} \\ 0 & I & -V_2 \hat{B}_2 E_{12}^{-1/2} \\ 0 & 0 & E_{12}^{-1/2} \end{array} \right], \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & I & 0 \\ U_{12} & 0 & I \end{array} \right] \right)$$

on (41), we obtain

$$\Pi(s) = \left[\begin{array}{ccc|cc} -sI + \Lambda & 0 & 0 & V_1 Z R E_{12}^{-1/2} & V_1 B_{n1} \\ 0 & -sI + A_{11} & A_{12} & V_2 Z R E_{12}^{-1/2} & V_2 B_{n1} \\ 0 & A_{21} & A_{22} & -E_{12}^{-1/2} & 0 \\ \hline U_{12} & 0 & I_m & 0 & 0 \\ C_2 P_{12} + D_{21} B_1^T \Phi_{12} & C_2 T & 0 & 0 & E_{21}^{1/2} \end{array} \right], \tag{43}$$

where

$$R := \hat{B}_2 + Y C_1^T \hat{D}_{12}. \tag{44}$$

Denote

$$\left[\begin{array}{cc} N_1 & N_3 \\ N_2 & N_4 \end{array} \right] := \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right]^{-1}, \tag{45}$$

$$\left[\begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array} \right] := \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right]^{-1} \left[\begin{array}{cc} V_2 Z R E_{12}^{-1/2} & V_2 B_{n1} \\ -E_{12}^{-1/2} & 0 \end{array} \right] \tag{46}$$

$$N_\infty = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right]^{-1} \left[\begin{array}{cc} I_{n_r} & 0 \\ 0 & 0 \end{array} \right]. \tag{47}$$

Calculating (43) yields

$$\Pi(s) = \Pi_r(s) + \left[\begin{array}{cc} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{array} \right] \tag{48}$$

where

$$\begin{aligned} \Pi_r(s) &= \left[\begin{array}{cc} \Pi_{r11}(s) & \Pi_{r12}(s) \\ \Pi_{r21}(s) & \Pi_{r22}(s) \end{array} \right] \\ &= \left[\begin{array}{c|cc} \Lambda & V_1 Z R E_{12}^{-1/2} & V_1 B_{n1} \\ \hline U_{12} & -M_{21} & -M_{22} \\ C_2 P_{12} + D_{21} B_1^T \Phi_{12} & 0 & E_{21}^{1/2} \end{array} \right], \end{aligned} \tag{49}$$

$$\begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix} = \begin{bmatrix} sN_2 \\ C_2T \end{bmatrix} (sN_1 - I)^{-1} \begin{bmatrix} M_{11} & M_{12} \end{bmatrix}. \tag{50}$$

Thirdly, we analyze (50). To this end, we need the following result.

LEMMA 3 *Let*

$$k^* = \max k_j, \quad j = 1, \cdots, p, \tag{51}$$

with k_j being defined in (6). Then

$$(i) \quad N_1^{k^*} = 0, \quad N_2 N_1^{k^*-1} = 0,$$

$$(ii) \quad TN_1^{k^*-1} = 0, \quad B_2 N_2 N_1^{k^*-2} = TN_1^{k^*-2}.$$

Proof From the definition of k^*, for N_∞ in (47), we know

$$N_\infty^{k^*} = \left(\begin{bmatrix} N_1 & 0 \\ N_2 & 0 \end{bmatrix} \right)^{k^*} = 0. \tag{52}$$

Therefore, (i) holds. From (33), we obtain $ATN_1 + B_2N_2 = T$, together with (i), we know that (ii) holds. □

Remark From [1], the highest order of the infinite zero of $G_{12}(s)$ is $k^* - 1$.

From (46) and (29),

$$M_{12} = -N_1 V_2 Z (YC_2^T + B_1 D_{21}^T) E_{21}^{-1/2} \tag{53}$$

holds which contains the factor N_1. Using Lemma 3, we obtain the four elements of $\Pi(s)$ in (48) as

$$\Pi_{11}(s) = \Pi_{r11}(s) - (sN_2 + s^2 N_2 N_1 + \cdots + s^{k^*-1} N_2 N_1^{k^*-2}) M_{11}, \tag{54}$$

$$\Pi_{12}(s) = \Pi_{r12}(s) - (sN_2 + s^2 N_2 N_1 + \cdots + s^{k^*-2} N_2 N_1^{k^*-3}) M_{12}, \tag{55}$$

$$\Pi_{21}(s) = \Pi_{r21}(s) - (C_2T + sC_2TN_1 + \cdots + s^{k^*-2} C_2 T N_1^{k^*-2}) M_{11}, \tag{56}$$

$$\Pi_{22}(s) = \Pi_{r22}(s) - (C_2T + sC_2TN_1 + \cdots + s^{k^*-3} C_2 T N_1^{k^*-3}) M_{12}, \tag{57}$$

where Π_{rij} $(i, j = 1, 2)$ are defined as in (49).

We are ready to give the following result.

THEOREM 1 *Suppose that the $\mathcal{H}_\infty$ control problem for (1) is solvable under assumptions $\mathbf{A1} - \mathbf{A3}$. If there exists $Q(s) \in \mathcal{BH}_\infty^{p \times q}$ and a proper $U(s)$ of size $n_r \times q$ such that*

$$M_{11} Q(s) + M_{12} = N_1^{k^*-1} U(s) \tag{58}$$

holds, then

$$K(s) = HM(\Pi_r(s); Q(s)) \tag{59}$$

are proper, where $\Pi_r(s)$ is defined as in (49).

Proof From Lemma 3 and (58), we obtain

$$\begin{bmatrix} sN_2 \\ C_2T \end{bmatrix} (sN_1 - I)^{-1} (M_{11} Q(s) + M_{12}) = \begin{bmatrix} sN_2 \\ C_2T \end{bmatrix} (sN_1 - I)^{-1} N_1^{k^*-1} U(s) = 0.$$

This completes the proof of Theorem 1. □

We will discuss the condition on $Q(s)$ in (58) further in what follows. We assume $Q(s) = N_{12} Q_1(s)$. In this case, using the facts $ZT = T$, $V_2 T = I$ and $N_1 A_{12} + N_3 A_{22} = 0$, and from (44), we have

$$M_{11} = N_1 V_2 Z (B_2 + YC_1^T D_{12}). \tag{60}$$

Then (58) is reduced to

$$N_1 V_2 Z [(B_2 + YC_1^T D_{12}) Q_1(s) - (YC_2^T + B_1 D_{21}^T) E_{21}^{-1/2}] = N_1^{k^*-1} U(s). \tag{61}$$

Therefore, we have

THEOREM 2 *Suppose that the $\mathcal{H}_\infty$ control problem for (1) is solvable under assumptions $\mathbf{A1} - \mathbf{A3}$. If there exist $Q_1(s)$ and $U(s)$ satisfying (61) and $N_{12} Q_1(s) \in \mathcal{BH}_\infty^{p \times q}$, then reduced-order $\mathcal{H}_\infty$ controllers are*

$$K(s) = HM(\Pi_r(s); N_{12} Q_1(s)). \tag{62}$$

As an example, we will use Theorems 1 and 2 to analyze a class of singular $\mathcal{H}_\infty$ control problems when $G_{12}(s)$ has only the infinite zeros with order not greater than one.

THEOREM 3 *Suppose that the $\mathcal{H}_\infty$ control problem for (1) is solvable under assumptions $\mathbf{A1} - \mathbf{A3}$. Also suppose $G_{12}(s)$ has only the infinite zeros with order not greater than one, or equivalently, k^* in (51) satisfies $k^* = 2$. Then take $Q(s) = N_{12} Q_1(s) \in \mathcal{BH}_\infty^{p \times q}$, the $\mathcal{H}_\infty$ control problem has the reduced-order controllers in the form of (62).*

Proof Since $k^* = 2$, take $Q_1(s) = N_{12} Q_1(s) \in \mathcal{BH}_\infty^{p \times q}$ and

$$U(s) = V_2 Z [(B_2 + YC_1^T D_{12}) Q_1(s) - (YC_2^T + B_1 D_{21}^T) E_{21}^{-1/2}],$$

then (61) holds. This completes the proof of Theorem 3. □

Theorem 3 gives a family of reduced-order $\mathcal{H}_\infty$ controllers based on (61). We will construct another family of reduced-order $\mathcal{H}_\infty$ controllers by exploiting (54)–(57).

THEOREM 4 *Suppose that the $\mathcal{H}_\infty$ control problem for (1) is solvable under assumptions* **A1**$-$**A3**. *Suppose $G_{12}(s)$ has only the infinite zeros with order not greater than one, or equivalently, k^* in (51) satisfies $k^* = 2$. Then take $Q(s) = Q_1(s)/(s+a)$, where $a > 0$ and $Q_1(s)$ is stable such that $Q(s) \in \mathcal{BH}_\infty^{p\times q}$, the reduced-order $\mathcal{H}_\infty$ controllers are*

$$K(s) = HM(\Pi_r(s); Q(s)). \tag{63}$$

Proof Since $k^* = 2$, from Lemma 3, $N_1^2 = 0$ and $N_2N_1 = 0$. Thus, from (53), $N_1M_{12} = 0$. Therefore, from (54)–(57), we obtain

$$\Pi_{11}(s) = U_{12}(sI-\Lambda)^{-1}V_1ZRE_{12}^{-1/2} - sN_2M_{11} - M_{21},$$

$$\Pi_{12}(\infty) = -M_{22},$$

$$\Pi_{21}(\infty) = -C_2TM_{11},$$

$$\Pi_{22}(\infty) = E_{21}^{1/2}.$$

Thus, Theorem 4 holds. □

As to $k^* > 2$, we have to find $Q(s) \in \mathcal{BH}_\infty$ and $U(s)$ such that (58) or (61) holds. In comparison with the results of [2] and [8], the above results have illustrated further the structure of singular $\mathcal{H}_\infty$ control systems.

4 Conclusions

The parameterization of reduced-order controllers for singular $\mathcal{H}_\infty$ control problems with infinite zeros has been studied by the ARE approach in this paper. By analyzing the $\mathcal{H}_\infty$ controllers in the descriptor from, we have shown the effect of the eigenstructure corresponding to the infinite zeros on all admissible $\mathcal{H}_\infty$ controllers. When the generalized plant satisfies certain assumptions, several results have been provided to choose the free parameter in the parametrization of controllers appropriately for constructing reduced-order the controllers.

References

[1] Copeland, B. R. and M. G. Safonov, 'Zero cancelling compensation for singular control problems and their application to the inner-outer factorization problem', *International Journal of Robust and Nonlinear Control*, **2**, 139-164, (1992).

[2] Copeland, B. R., and M. G. Safonov, 'A generalized eigenproblem solution for singular H^2 and $\mathcal{H}_\infty$ problems', Advances in Robust Control Systems Techniques & Applications (Ed: C.T. Leondes), Academic Press, New York (1992).

[3] Doyle, J. C., K. Glover, P. P. Khargonekar and B. A. Francis, 'State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems', *IEEE Transactions on Automatic Control*, **34**, 831–847 (1989).

[4] Gahinet, P. and P. Apkarian, 'A linear matrix inequality approach to H^∞ control', *International Journal of Robust and Nonlinear Control*, **4**, 421-448, (1994).

[5] Gantmacher, F. R., *Matrix Theory*, Chelsea, New York, 1959.

[6] Iwasaki, T. and R. E. Skelton, 'All controllers for the general H^∞ control problem: LMI existence conditions and state space formulas', *Automatica*, **30**, 1307-1317 (1994).

[7] Hara, S., T. Sugie and R. Kondo, '$\mathcal{H}_\infty$ control problem with $j\omega$-axis zeros', *Automatica*, **28**, 55-70 (1992).

[8] Kimura, H., Y. Lu, and R. J. Kawatani, 'On the structure of $\mathcal{H}_\infty$ control systems and related extensions', *IEEE Transactions on Automatic Control*, **36**, 653-667 (1991).

[9] Scherer, C., '$\mathcal{H}_\infty$-optimization without assumptions on finite or infinite zeros', *SIAM Journal on Control Optimization*, **30**, 143-166 (1992).

[10] Stoorvogel, A. A., 'The singular $\mathcal{H}_\infty$ control with dynamic measurement feedback', *SIAM Journal on Control Optimization*, **29**, 160-184 (1991).

[11] Stoorvogel, A. A., A. Saberi and B. M. Chen, 'A reduced order observer based controller design for $\mathcal{H}_\infty$-optimization', *IEEE Transactions on Automatic Control*, **39**, 355-360 (1994).

[12] Watanabe, T., K. Yasuda and R. Yokoyama, 'Reduced order solution of non-standard $\mathcal{H}_\infty$ control problem', *Transactions of Society of Instrument and Control Engineering*, **32**, 16-25 (1996).

[13] Xin, X. and H. Kimura, 'Singular (J, J')-lossless factorization for strictly proper function', *International Journal of Control*, **59**, 1383-1400 (1994).

[14] Xin, X., Guo, L. and C. B. Feng, Reduced-order controllers for continuous and discrete time singular H_∞ control problems based on LMI, *Automatica*, **32**, 581-585 (1996).

[15] Xin, X. and T. Mita, Inner-outer factorization for non-square proper functions with infinite and finite $j\omega$-axis zeros, *International Journal of Control*, **71**, 145-161 (1998).

[16] Xin, X., T. Mita and H. Kimura, (J, J')-lossless factorization approach to H_∞ control problems with infinite and finite $j\omega$-axis zeros, *Proceedings of 38th IEEE Conference on Decision and Control*, 3132–3137, Phoenix (1999).

[17] Xin, X., B. D. O. Anderson and T. Mita, Complete solution of the 4-block H_∞ control problem with infinite and finite $j\omega$-axis zeros, *International Journal of Robust and Nonlinear Control*, **10**, 59-81 (2000).

[18] Zhou, K. M., J. C. Doyle, and K. Glover, *Robust and Optimal Control*, New Jersey: Prentice Hall (1995).