# A Study of Tracking-Differentiator


Xuejun Wang*, Stephen Yau*, Jie Huang†

*: Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago, Chicago, IL 60607
Email: yau@uic.edu

†: Department of Mechanical and Automation Engineering
The Chinese University of Hong Kong, Hong Kong
Email: jhuang@mae.cuhk.edu.hk



### Abstract

A second-order dynamics, Tracking-Differentiator, is given to generate smooth approximation of the incoming measurement and its derivertive, which can be used as the desired trajectory for the control system.


## 1 Introduction

In tracking control problems, the smooth assumption is often made for the desired trajectory of the control system[2]. However, in many tracking problems, the smooth assumption is not satisfied. Generally, a second-order linear system like

$$\ddot{y}_d + k_1\dot{y}_d + k_2 y_d = k_2 y_a(t) \tag{1}$$

will be used to generate desired position and velocity for the tracking system. $k_1, k_2$ needs to be chosen such that $y_d(t)$ is fast enough to closely approximate $y_a(t)$. A famous conclusion about this second-order linear system is the contradiction between fast response and over shot. That is, a fast response always results in an over shot. In practical problem , a compromise must be made. This will dramatically affect the efficiency of tracking system. We have the fast response and over shot dilemma for this reference model.

Han proposed a parameterized second-order nonlinear dynamics, called Tracking-Differentiator(TD) in [3] to solve the above dilemmas. Han uses the following criteria to describe the accuracy of smooth approximation:

$$\lim_{R\to\infty}\int_0^T |y_d(t) - y_a(t)|dt = 0 \tag{2}$$

where $T > 0$ is a finite number, $R > 0$ is a parameter.

In [3], a sufficient condition is given for a second-order system to be a TD. In this paper, we will examine some properties of TD. Finally, a simulation result will show the advantage of nonlinear TD over linear TD.

## 2 Main theorem and other results

The following theorem is presented by Han in [3].

**Theorem 1** *(The Basic Convergence Theorem of Tracking Differentiator) If the*

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = f(z_1, z_2) \end{cases} \tag{3}$$

*is globally asymptotically stable at origin, then for any bounded integrable function $v(t)$ and $T > 0$, the solution of the following system:*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = R^2 f(x_1 - v(t), \frac{x_2}{R}) \end{cases} \tag{4}$$

*satisfies*

$$\lim_{R\to\infty}\int_0^T |x_1(t) - v(t)|dt = 0 \tag{5}$$

This theorem provides a way to construct a TD from any globally asymptotically stable system. $R$ can be used to control the accuracy of the tracking signal. If $f(x_1, x_2)$ is linear, we have much stronger result:

**Theorem 2** *(Linear Tracking Differentiator) If the*

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = ax_1 + bx_2 \end{cases} \tag{6}$$

*is asymptotically stable at origin, then for any bounded integrable function $v(t)$ and $T > 0$, the solution of the following system:*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = R^2[a(x_1 - v(t)) + b\frac{x_2}{R}] \end{cases} \tag{7}$$

*satisfies*

$$\lim_{R\to\infty} x_1(t) = v(t), \forall t \in [0, T] \tag{8}$$

The proof of this theorem is relatively long, and will appear elsewhere. Notice there is a big difference between

$x_1(t)$ and $v(t)$. $x_1(t)$ is smooth and has derivertive while $v(t)$ may even not continuous. In the following, we will study how parameter $R$ affects the response time of the TD. First we need to introduce two definitions about convergence speed of stable systems.

We will use the definition of Globly Finite Time Stable given in [1]. If the system is asymptotically stable, we give the following definition of $\epsilon$-setting time:

**Definition 1** *($\epsilon$-setting time) Consider the system*

$$\dot{x} = f(x), f(0) = 0, x \in R^n, x(0) = x_0$$

*which is globally asymptotically stable at the origin. For any given $\epsilon > 0$, Let*

$$T(x_0, \epsilon) = min\{T : \|x(t)\| < \epsilon, \forall t > T\}$$

*then $T(x_0, \epsilon)$ is the $\epsilon$-setting time of the system.*

The following two theorems show the two most important properties of $\epsilon$-setting time.

**Theorem 3** *Let*

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = f(z_1, z_2) \end{cases} \tag{9}$$

*be globally asymptotically stable at origin with $\epsilon$-setting time $T(z_0, \epsilon)$. Let $c$ be any constant. The system*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = R^2 f(x_1 - c, \frac{x_2}{R}) \end{cases} \tag{10}$$

*is globally asymptotically stable at (c, 0) with setting time less than $\frac{T(x_0,\epsilon)}{R}$, where $x_0 = (x_1(0), x_2(0))$.*

This theorem is also true for setting time of globally finite time stable systems. It is stated as following:

**Theorem 4** *Let*

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = f(z_1, z_2) \end{cases} \tag{11}$$

*be globally finite time stable at origin with setting time $T(z_0)$. Let $c$ be any constant. The system*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = R^2 f(x_1 - c, \frac{x_2}{R}) \end{cases} \tag{12}$$

*is globally finite time stable at (c, 0) with setting time equal to $\frac{T(x_0)}{R}$, where $x_0 = (x_1(0), x_2(0))$.*

These two theorems show that the parameter $R > 1$ in system (12) will reduce the $\epsilon$-setting time. A big $R$ will decrease the system response time.

For the linear system, we have more specific result.

**Theorem 5** *If the following linear system*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = ax_1 + bx_2 \end{cases} \tag{13}$$

*is stable with two eigenvalues $\lambda_1, \lambda_2$. Then the following system*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = R^2[ax_1 + b\frac{x_2}{R}) \end{cases} \tag{14}$$

*is also stable, and has eigenvalues as $R\lambda_1, R\lambda_2$*

This can be easily shown by noticing that eigenvalues of (14) are given by

$$\begin{aligned} \overline{\lambda}_{1,2} &= \frac{bR \pm \sqrt{b^2R^2 + 4aR^2}}{2} \\ &= \frac{bR \pm R\sqrt{b^2 + 4a}}{2} \\ &= R\lambda_{1,2} \end{aligned}$$

Since the real part of $\lambda_1, \lambda_2$ must be negative, this result shows that a bigger $R$ will make the real part of the eigenvalue more smaller (tends to negative). This will definitely increase the system speed. While in the case that the eigenvalues have imaginary part, the absolute value of imaginary part will also increase as $R$ increases. This big absolute value of imaginary part causes the overshoot in the system response. So for linear system, one has to balance the speed and overshoot. That is why we need to look for nonlinear system for quick response and small overshoot.

## 3 Conclusion

This paper shows the nature of parameter $R$ in Tracking-Differentiator. TD provides an easy method to smooth the noised signal.

## References


[1] S.P. Bhat and D.S. Bernstein *Continuous finite stabilization of the translational and rotational double integrators*, IEEE Trans. Automat. Contr., vol 43, no. 5, pp. 678-682, 1998

[2] R.M. Hirschorn, J. Davis, *Global output tracking for nonlinear system*, SIAM J. Contol Optm. 26 (1988) pp. 1321-1330

[3] Jingqing Han, Wei Wang, *Nonlinear Tracking-Differentiator*, System Science and Mathematics, 1994, 14(2), pp. 177-183

[4] V.T.Haimo, *Finite time controller*, SIAM J. Contr. Opti. vol. 24 no. 4, pp. 760-770, 1986

[5] E.P.Ryan, *Singular optimal controls for second-order saturating system*, Int. J. Contr., Vol. 36, no. 4, pp. 549-564, 1979