

PWM-Type Discrete VSS Controller for On/Off Actuator Systems

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Abstract

In this paper, a regulation problem for systems whose control input signal is restricted to On/Off pulses with large sampling periods is considered. A control law is designed for a class of single input systems with full state information and does not require the system to be stable. First, a sliding sector for discrete-time system is defined based on a norm of the system state where the norm decreases inside the sliding sector. Then the PWM-type discrete-time VSS control law is designed to move the system state from the outside to the inside of the sliding sector. Simulation show the effectiveness of the proposed controller.

1 Introduction

In spite of the difficulties in analyzing the system caused by its nonlinearity, PWM(Pulse-Width Modulated) systems has been widely used in industry for many years. Some application examples are control of D.C motors, attitude control of a satellite, signal processing, power control systems, and so on. There are, roughly speaking, two advantages using PWM systems. First, since PWM signal is an on/off signal, the actuator part can be made simple which implies high reliability and low cost. There are indeed some cases that an actuator can only accept on/off signals, where some way of designing on/off signals, like PWM controller, is essential. The second reason is that it can use maximum power for control. Therefore, it has a low sensitivity to noise. Also, generally speaking, compared to the PAM(Pulse-Amplitude Modulated) signals, there is low energy loss. (For instance, to drive a DC motor, using PWM signal makes the heat-loss low.)

In the early 60's, many researchers have reported the analysis of PWM feedback systems [1] [2] [3] [4]. The stability condition of the closed-loop system required the plant to have all its poles in the left half of the s-plane, except possibly one pole at the origin, which is

a very strict condition and not realistic.

The question is then how to design a controller for PWM systems. In industries, the principle of equivalent areas [5], is the commonly used and may be the only practical way to find an equivalent PWM controller with fast sampling rate from an existing continuous-time or PAM controllers. The principle of equivalent areas can be briefly stated as follows: Two input signals to a linear continuous-time system are dynamically equivalent, if their integrals over a corresponding equal time interval are equal, and the time interval is suitably small. The basic assumption of a "small" sampling time interval in the principle limits the potential applications of the existing PWM controllers to systems which require long sampling periods [6]. In the attitude control of satellites for example, longer sampling periods are often desirable for the limited propellant consumption.

For this reason, a controller for PWM system not relying on approximation such as linearization nor redesigns is desirable for these applications, and is the primary motivation of our work. We have proposed an online continuation method controller in [7], [9] which is a control strategy for PWM systems using numerically calculated solution as the control input. In this paper, the PWM-type discrete-time VSS (Variable Structure System) control law is proposed. In section 3, the PR-sliding sector [14] is defined. In section 4, our proposed controller is derived. By applying our controller, the system state will move from the outside to the inside of the PR-sliding sector. Inside the PR-sliding sector, it is designed so as the P-norm of the system state will decrease. In section 5, we demonstrate the efficiency of our controller by showing computer simulations of a 4-th order system and is compared to the classical controller based on the principle of equivalent area.

2 Preliminary

We consider continuous time dynamic systems whose state-space equation is given by

$$\dot{x}(t) = Ax(t) + bu(t) \quad , \quad x \in \mathbf{R}^n \quad , \quad u \in \mathbf{R} \quad (1)$$

where x is state vector, u is input vector, and A, b are matrices of appropriate dimensions. It is assumed that the plant is linear, time invariant and inputs are constrained to PWM-type, e.g. ,

$$u(t) = \begin{cases} u_A \text{ or } -u_A & \text{for } t \in [kT, kT + w_k T) \\ 0 & \text{for } t \in [kT + w_k T, (k+1)T) \end{cases} \quad (2)$$

where $u_A (> 0)$ is the amplitude of input pulse, T is the sampling period, w is the duty ratio of the input pulse, and the subscript “ k ” means that it is the k th sampling. So w_k , for example, is the duty ratio at the k th sampling, which means that the input pulse width is $w_k T$. The discrete time state space representation can then be written in the following form:

$$x_{k+1} = \Phi x_k + g(w_k, d) \quad (3)$$

where

$$\begin{aligned} \Phi &:= e^{AT} \\ g(w_k, d) &:= \text{sgn}(d) \int_0^{w_k T} e^{A(T-\tau)} d\tau b u_A \end{aligned} \quad (4)$$

and $0 \leq w_k < 1$. So, the control parameters are $\text{sgn}(d)$ (direction of input pulse i.e. $+u_A$ or $-u_A$) and the duty ratio w_k .

One way to design the PWM controller is to determine the duty ratio w_k so that

$$\int_{kT}^{(k+1)T} u(t)_{(Oth-hold)} dt = \int_{kT}^{(k+1)T} u(t)_{(PWM)} dt \quad (6)$$

becomes the same as to the linear discretized system case assuming the 0th hold operator. This conversion has been widely used in industries for many years. In 1960, R. E. Andeen proved in his paper “The principle of equivalent areas” [5] the validity of this conversion under the assumption that the sampling period is suitably small. Our main motivation is to design a controller for PWM systems whose sampling period is not so small.

3 PR-sliding sector

Definition 1 The P-Norm $\|\cdot\|_p$ of the system state is defined as

$$\|x_k\|_p = (x_k^T P x_k)^{\frac{1}{2}}, \quad x_k \in \mathbf{R}^n \quad (7)$$

where $P \in \mathbf{R}^{n \times n}$ is a positive definite symmetric matrix.

The square of the P-norm is denoted as

$$L_k = \|x_k\|_p^2 = x_k^T P x_k > 0, \quad \forall x_k \in \mathbf{R}^n, x_k \neq 0 \quad (8)$$

If the autonomous system of (1) is stable then there exist symmetric positive definite matrices P such that

$$L_{k+1} - L_k = x_k^T (\Phi^T P \Phi - P) x_k < 0, \quad \forall x_k \in \mathbf{R}^n \quad (9)$$

while there is no control action. However, for an unstable system, the inequality (9) does not hold. It is possible that $L_{k+1} - L_k \geq 0$ for some $x_k \in \mathbf{R}^n$, and $L_{k+1} - L_k < 0$ for some other $x_k \in \mathbf{R}^n$. The latter ones form a special subset in which the P-norm $\|\cdot\|_p$ decreases. Accordingly we define the special subset called the “PR-sliding sector” as follows.

Definition 2 The PR-sliding sector is a subset of \mathbf{R}^n defined by

$$S = \{x_k | x_k^T (\Phi^T P \Phi - P) x_k \leq -x_k^T R x_k\} \quad (10)$$

where $P \in \mathbf{R}^{n \times n}$ and $R \in \mathbf{R}^{n \times n}$ are positive definite symmetric matrices respectively.

These definitions were originally stated in [14]. Note that in the original paper, the PR-sliding sector was defined by using a positive semi-definite symmetric matrix \mathbf{R} . In our paper, the definition of PR-sliding sector is modified.

Inside the PR-sliding sector, the P-norm $\|\cdot\|_p$ of the plant (1) without any control action decreases because

$$L_{k+1} - L_k \leq -x_k^T R x_k < 0, \quad \forall x_k \neq 0 \in S \quad (11)$$

There are special cases when sliding sector becomes an equilibrium point or the whole state-space. If we admit these cases, the existence of such a sliding sector for the discrete-time plant (3) is shown (see Theorem 1 of [14]).

There is a more simple form of expressing PR-sliding sectors. The simplified form is shown in the following proposition.

Proposition 1 (Proposition 3 of [14]) For discrete plant (3), the PR-sliding sector defined by (10) exists for any positive definite symmetric matrix P and any positive semi-definite symmetric matrix R . And it can be rewritten as

$$S = \{x_k | s_k^2 \leq \delta_k^2\} \quad (12)$$

where

$$s_k^2 = x_k^T P_1 x_k \geq 0 \quad (13)$$

$$\delta_k^2 = x_k^T P_2 x_k \geq 0 \quad (14)$$

P_1 and P_2 are $n \times n$ positive semi-definite symmetric matrices.

Next, we will show an example to illustrate when the PR-sliding sector becomes the special cases and also show how P_1 and P_2 of proposition 1 is related to the P and R of the definition.

Example First, we define a matrix Δ as follows.

$$\Delta := \Phi^T P \Phi - P + R \quad (15)$$

Then the PR-sliding sector defined by equation (10) is determined by

$$\text{PR-sliding sector} : \{x_k | x_k^T \Delta x_k \leq 0\} \quad (16)$$

By considering the most simple first-order discrete system

$$x_{k+1} = ax_k + bu_k, \quad (17)$$

Δ of this system is calculated.

$$\Delta = Pa^2 - P + R = P(a^2 - 1) + R \quad (18)$$

Considering the equation (16), if $\Delta \leq 0$, then the PR-sliding sector is the whole state-space. Also, if $\Delta > 0$, then the sliding sector is reduced to the equilibrium point. Obviously, since $P > 0$ and $R > 0$, if $|a| > 1$, that is when the original plant is unstable, then the PR-sliding sector is always reduced to the origin. This shows that P_1 and P_2 of proposition 1 is determined from the eigenvalues of Δ .

4 PWM-type VSS controller

Inside the PR-sliding sector defined in the previous section, the P-norm decreases. Thus, if there exists a VSS control law which enables the system state move from outside to the inside of the PR-sliding sector in finite steps, and to stay inside it forever, then the VSS control law results in a stable system. In lemma 5 below, we will show the PWM-type VSS control law which ensures such movement of the system state.

Before showing the PWM-type VSS control law, another lemma is given as a preliminary.

Lemma 1 *The mapping $g : w_k \rightarrow g(w_k, + \text{ or } -)$ given in system equation (3) (5) is a continuous mapping. Therefore, for any constant matrix $C \in \mathbf{R}^{1 \times n}$, and any scalar β which satisfies*

$$|\beta| \leq \max_{w_k \in [0,1]} |Cg(w_k, \cdot)| =: |Cg(w_k^*, \cdot)|, \quad (19)$$

there exist a direction d and a pulse duration w_k such that

$$\beta = Cg(w_k, d) \quad (20)$$

holds.

Proof The continuous character of $g(\cdot)$ can be easily shown for g can be differentiable by w_k as many times as one wants. (So actually it is smooth).

$$\begin{aligned} \frac{dg(w_k, +)}{dw_k} &= \frac{d}{dw_k} \int_0^{w_k T} e^{A(T-\tau)} d\tau b u_A \\ &= T e^{AT(1-w_k)} b u_A \end{aligned} \quad (21)$$

$$g''(w_k, +) = -AT^2 e^{AT(1-w_k)} b u_A \quad (22)$$

$$g^{(n)}(w_k, +) = (-1)^{n-1} A^{n-1} T^n e^{AT(1-w_k)} b u_A \quad (23)$$

Also, notice that

$$0 = Cg(0, \cdot) \leq |\beta| \leq |Cg(w_k^*, \cdot)| \quad (24)$$

$$g(w_k, -d) = -g(w_k, d) \quad (25)$$

From the continuous character of $g(w_k, \cdot)$ and the existence of w_k^* , the existence of w_k is assured. \square

Now we are ready to state our proposed controller.

Lemma 2 PWM-type VSS controller *For the discrete-time plant (3)(5), the following PWM-type discrete-time VSS control law enables the system state to move from the outside to the inside of the PR-sliding sector in a finite steps and enable the P-norm to decrease inside the PR-sliding sector.*

- if $x_k \in S$

$$u(t) = 0 \quad \text{for } kT < t \leq (k+1)T \quad (26)$$

- if $x_k \notin S$

$$u(t) = \begin{cases} \text{sgn}(d)u_A & \text{while } h(t) < 0 \\ 0 & \text{after } h(t) = 0 \end{cases} \quad (27)$$

for $kT < t \leq (k+1)T$

where

$$d = -\text{sgn} \left(\frac{C\Phi x_k + K \text{sgn}(s_k \delta_k) \delta_k}{Cg(w_k^*, +)} \right) \quad (28)$$

$$h(t) = \left\{ Cg \left(\frac{t - kT}{T}, d \right) \right\}^2 - \{C\Phi x_k + K \text{sgn}(s_k \delta_k) \delta_k\}^2 \quad (29)$$

and the coefficient K satisfies $0 \leq K < 1$, S is the PR-sliding sector defined by the following.

$$S = \{x_k | s_k^2 \leq \delta_k^2\} \quad (30)$$

$$s_k := Cx_k \quad (31)$$

$$\delta_k := \sqrt{x_k^T Q x_k} \quad (32)$$

We assume u_A is large enough so that for any given state x_k of our interest,

$$\beta := | -C\Phi x_k - K \text{sgn}(s_k \delta_k) \delta_k | \leq |Cg(w_k^*, \cdot)| \quad (33)$$

holds.

The function $h(t)$ can be considered as a reference signal function which determines the timing of the control pulse.

Proof Sketch of the proof is as follows. Outside the PR-sliding sector, i.e., $x_k \notin S$, from equation (27),(28) and (29), the duration w_k and direction d is designed so as

$$Cg(w_k, d) = -C\Phi x_k - K \text{sgn}(s_k \delta_k) \delta_k \quad (34)$$

holds. By applying the VSS controller, the following equation satisfies.

$$\begin{aligned} Cx_{k+1} &= C\Phi x_k + Cg(w_k, d) \\ &= -K \text{sgn}(s_k \delta_k) \delta_k \end{aligned} \quad (35)$$

So, the generalized variable s_{k+1} satisfies

$$s_{k+1}^2 = K^2 \delta_k^2 < K^2 s_k^2 \quad (36)$$

which implies that the $|s_k|$ will decrease until the system state moves into the inside of the PR-sliding sector. If the coefficient K is chosen to be small enough (or 0), then the system state will move into the PR-sliding sector in finite steps (or one step).

Inside the PR-sliding sector, the VSS control law is

$$w_k = 0. \quad (37)$$

Thus the P-norm will decrease inside the PR-sliding sector according to the design of the PR-sliding sector.

□

5 Simulation

We consider the plant of the form (1) and assume that it is discretized with a 0th hold operator.

$$x_{k+1} = \Phi x_k + \Gamma u_k \quad (38)$$

A performance index is chosen as

$$J = \sum_{k=0}^{\infty} (x_k^T Q x_k + u_k^2) \quad (39)$$

where $Q \in \mathbf{R}^{n \times n}$ is a positive definite symmetric matrix. The optimal control minimizing the performance index for the plant (38) exists and has the form

$$u_k = -F x_k. \quad (40)$$

The gain matrix F is defined as

$$F = (1 + \Gamma^T P \Gamma)^{-1} \Gamma^T P \Phi \quad (41)$$

where $P \in \mathbf{R}^{n \times n}$ is a positive definite symmetric matrix which satisfies the following discrete time Riccati equation:

$$P = Q + \Phi^T P \Phi - \Phi^T P \Gamma (1 + \Gamma^T P \Gamma)^{-1} \Gamma^T P \Phi. \quad (42)$$

In the simulation, the proposed PWM-type discrete time VSS controller is compared with the traditional redesigned controller.

The PR-sliding sector for the proposed PWM-type discrete VSS controller is designed as follows:

$$S = \{x_k | s_k^2 \leq \delta_k^2\} \quad (43)$$

where

$$s_k = Cx_k, \quad C = \sqrt{1 + \Gamma^T P \Gamma} F \quad (44)$$

$$\delta_k = \sqrt{x_k^T Q x_k} \quad (45)$$

For the traditional redesigned controller the duration w_k and the direction d was given as below:

$$w_k = \frac{|u_k|}{u_A} \quad (46)$$

$$d = u_k \quad (47)$$

where $u_k = -F x_k$. Note that this controller is based on the principle of equivalent area.

The plant we have used is a model of a rotational inverted pendulum whose parameters are given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 6.5531 & -0.9386 & -0.1444 \\ 0 & 81.0655 & -1.3382 & -1.7863 \end{bmatrix}, \quad (48)$$

$$b = [0 \quad 0 \quad 30.4747 \quad 43.4473]^T. \quad (49)$$

The weight matrix Q we have used is

$$Q = \text{diag}\{100000, 10000, 2000, 1000\}. \quad (50)$$

The only case for the traditional redesigned controller to succeed in stabilizing the plant was case 1 ($u_A = 100, T = 0.01$), where as our proposed PWM-type VSS controller succeeded to stabilize the plant in each cases. This comes from the fact that the principle of equivalent area is valid only when the sampling period is sufficiently small. Since the nonlinearity of PWM systems differs from what range of duration w_k we use, generally it is profitable to use an actuator which allows large u_A . From the simulation, we conclude that our proposed controller is efficient against systems with not only long sampling interval T , but also with a small input pulse amplitude u_A .

6 Conclusion

Discrete-time VSS control law for PWM systems has been proposed. The proposed controller was applied to a 4th order plant, and computer simulation showed that our proposed method was efficient against systems with long sampling period T , and also with small input pulse amplitude u_A . The PWM-type VSS control law given in Lemma 5 ensures the system state to move outside to the inside of the PR-sliding sector, and the decreasing of the P-norm inside the PR-sliding sector is also ensured. But for the discrete time system, the system state, generally speaking, cannot stay inside the PR-sliding sector forever. In some step, the system state will move out the PR-sliding sector. In this case, the VSS control law should enable the system state to move back inside the PR-sliding sector, again. While the system state moves into the PR-sliding sector, the P-norm may increase or decrease. The example we showed in section 3 suggest that by taking large enough R , it is always possible to reduce the PR-sliding sector to the equilibrium point. In this case, we do not have to bother about the above argument. To be rigorous, it should be taken into consideration whether the P-norm decrease or increase while the system state moves from the outside to inside of the PR-sliding sector with the proposed VSS controller, which is our future work.

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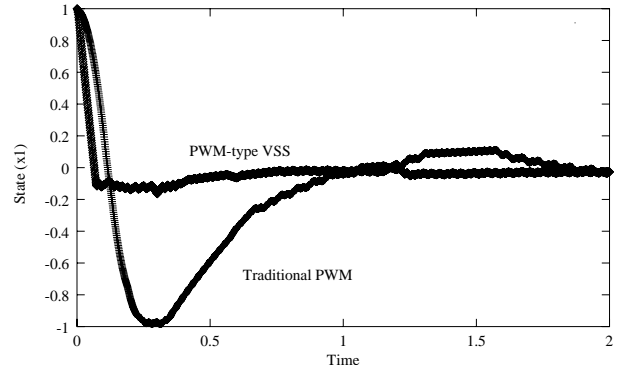


Fig.1a x_1 vs time

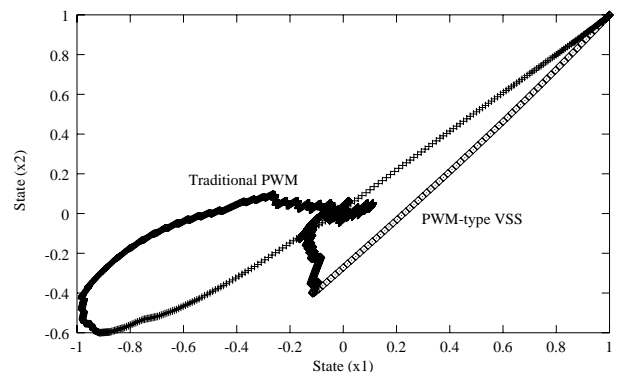


Fig.1b Phase plane(x_1 vs x_2)

Figure 1: Case 1. Sampling interval $T = 0.01$, pulse amplitude $u_A = 100$.

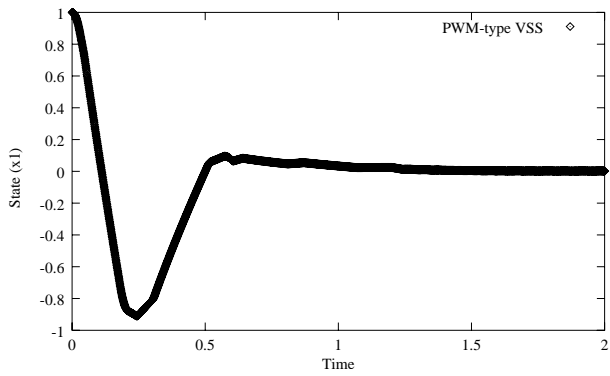


Fig.2a x_1 vs time

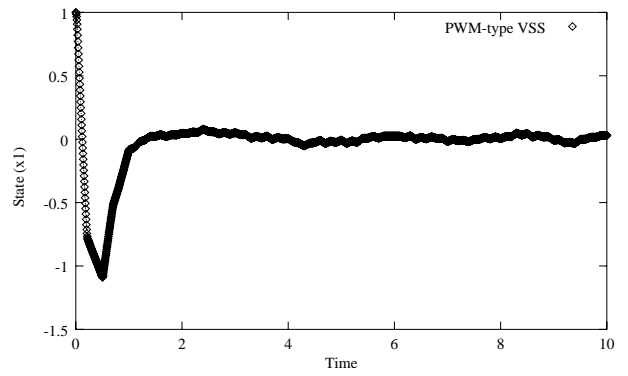


Fig.3a x_1 vs time

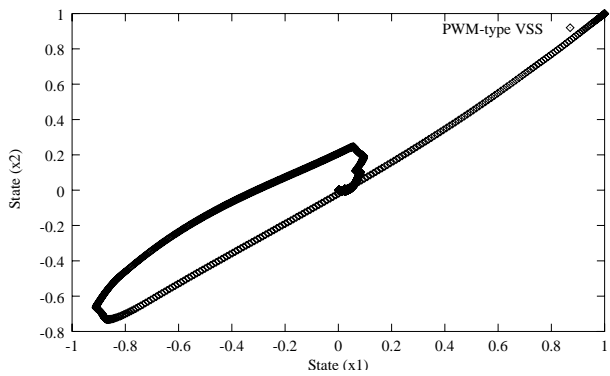


Fig.2b Phase plane(x_1 vs x_2)

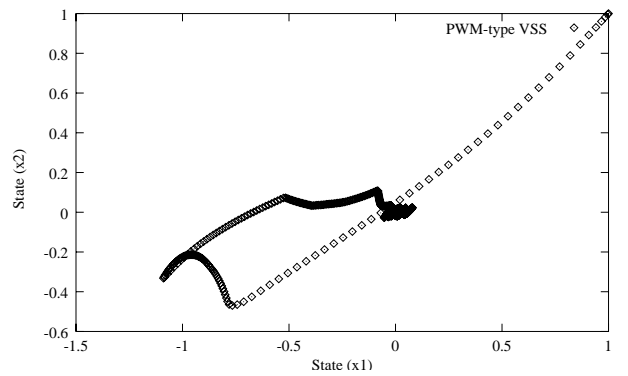


Fig.3b Phase plane(x_1 vs x_2)

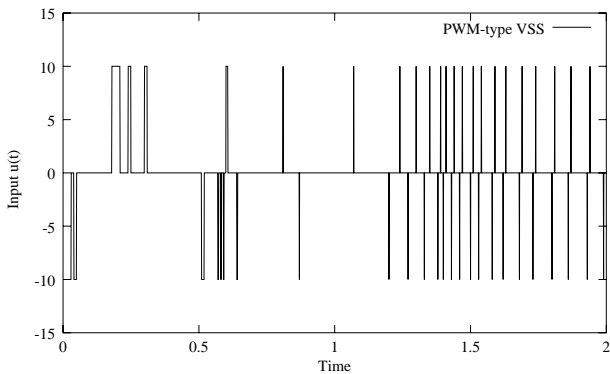


Fig.2c Input vs time

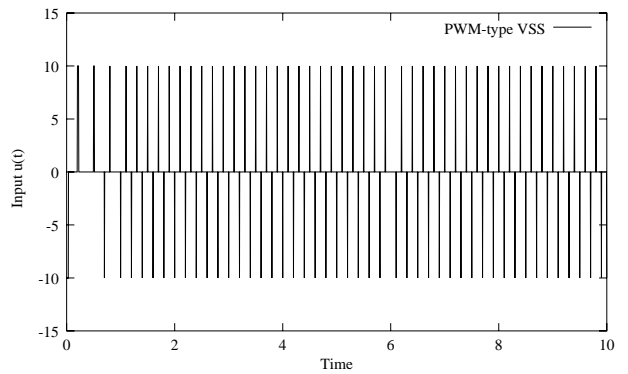


Fig.3c Input vs time

Figure 2: Case 2. Sampling interval $T = 0.01$, pulse amplitude $u_A = 10$. The traditional redesigned controller failed to stabilize the plant.

Figure 3: Case 3. Sampling interval $T = 0.1$, pulse amplitude $u_A = 10$. The traditional redesigned controller failed to stabilize the plant.