Communication-Limited Stabilization of Linear Systems

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Abstract

This paper investigates the problem of stabilizing a linear, discrete-time plant using a digital link with a finite data rate. The plant model is infinite-dimensional and time-varying, with a real-valued output which is zero at negative times and distributed according to a probability density p at time zero. Finite and infinite horizon costs in terms of the m-th output moment are defined and the equations of the optimal, finite horizon *coder-controller* derived. Asymptotic quantization theory is then used to obtain the solution as the horizon tends to infinity, without needing to explicitly solve the finite horizon problem. It is shown that this limiting coder-controller is optimal with respect to the infinite horizon cost, provided that p satisfies certain technical conditions. This immediately leads to a necessary and sufficient condition for the existence of a coder-controller that takes the m-th output moment to zero asymptotically with time. If the open-loop plant is finite-dimensional and time-invariant, this condition simplifies to an inequality involving the data rate and the open-loop pole with greatest magnitude. Analagous results automatically hold for the related problem of state estimation with a finite data rate.

Keywords: stabilization, communication, quantization.

1 Introduction

In recent years, a number of researchers have begun to investigate the effects of finite communication rates in control problems [1, 9, 17, 15]. Traditionally the communications channel between the plant and controller goes unmodelled, for it is simply assumed that the outputs of the former are available to the latter completely and with infinite precision. Clearly, this is untrue if the link between them can only carry a limited number of digital symbols per unit time. In addition to introducing both delay and quantization, the finite data rate raises the question of how to choose the bits of information that would be most useful for control.

The focus in this area so far has generally been on *memoryless* coding, in which the output is quantized without reference to its past. In [4] it was shown that if the states of a discrete-time, linear, time-invariant (LTI) system are passed through a fixed, memoryless quantizer then it is no longer controllable in the asymptotic sense. The communication delays in this model were explicitly included in [9] and sufficient conditions given for the state to eventually remain within a given bound. Wong and Brockett proved that if the initial condition of a continuous-time LTI system is within a given bounded set then memoryless coding *and* control suffice to bound the state, given certain conditions [17]. Investigating discrete-time, Gaussian, LTI systems, Borkar and Mitter derived a separation principle and a memoryless, certainty equivalent controller, under the constraint that the coder was a memoryless quantizer acting on the *innovations* of a Kalman filter [1].

The only case in which coders with memory have been dealt with in the context of communication-limited control appears to be in [15], in which a noisy, analog channel was considered. In this paper, we assume a noiseless, digital channel and permit the coder to have potentially unlimited memory. The motivation for this comes from the the closely related field of communication-limited state estimation, in which recursive coding schemes have been studied extensively [3, 16, 12, 13]. Moreover, this divorces the effects of the structural constraint imposed by memoryless coding and/or control from those of the communication constraint proper, thereby allowing the latter to be analyzed more clearly.

We consider a general linear, time-varying and infinite-dimensional plant in discrete-time, with a directly observed, real-valued output which is zero before time zero and governed by a probability distribution function P at time zero. We show that the optimal *coder-controller* under a certain finite horizon, mean m-th power cost is given by a causally reformulated optimal quantizer for the initial output. The insights gained from the finite horizon analysis are then combined with asymptotic quantization theory [10, 5, 2, 8] to derive the limiting scheme as the horizon tends to infinity, *without having to solve the finite horizon problem first.* It is shown that this limiting coder-controller is optimal in an infinite horizon sense, provided that the initial condition probability density p is continuous and satisfies certain technical conditions. This immediately leads to a necessary and sufficient condition for there to exist a coder-controller with data rate R that asymptotically stabilizes the plant, in the sense that the m-th output moment is taken to zero with time. For plants with finite-dimensional and time-invariant open-loop dynamics, this condition simplifies to

$$R > \log_2 |\lambda|, \tag{1}$$

where λ is the open-loop pole with largest magnitude. It is then observed that analagous results automatically hold for the problem of state estimation with a finite data rate.

2 Formulation of Control Problem

Consider the infinite-dimensional, time-varying, ARMA plant model

$$x_{k+1} = \sum_{j=0}^{\infty} a_{k,j} x_{k-j} + b_{k,j} u_{k-j}, \quad k = 0, 1, 2, \ldots, \tag{2}$$

where $x_k, u_k \in \mathbf{R}$ are the output and control respectively at time k, with $x_k = u_k = 0$ when $k < 0$, and $a_{k,j}, b_{k,j} \in \mathbf{R}$, $j, k \geq 0$, are known parameters. We assume that $b_{k,0} \neq 0$, $\forall k \geq 0$, so that the control always affects the output at the next time instant. In addition, we further assume that x_0 is a realization of a random variable X_0 on the probability space $(\mathbf{R}, \mathcal{L}, P)$, where $\mathcal{L}$ is the σ-algebra of Lebesgue sets on the real line and P is a known probability measure such that $\mathrm{E}|X_0|^m < \infty$ for some $m > 0$.

Suppose a coder observes the outputs and then sends real-time data to a distant controller over a digital channel that can carry only one symbol s_k from an alphabet $\mathbf{Z}_M \triangleq [0, 1, \ldots, M-1]$ during each sampling interval. The corresponding *data rate* is $R \triangleq \log_2 M$ bits per interval. Neglecting the propagation delay and transmission errors, the finite data rate implies that each symbol takes one sampling interval to reach the other end of the channel. Hence at time k the controller has $s_0, \ldots, s_{k-1}$ available and generates

$$u_k = \upsilon_k(\tilde{s}_{k-1}), \quad \forall k \geq 0, \tag{3}$$

where the notation $\tilde{y}_k$ denotes a sequence $\{y_j\}_{j=0}^{k}$ and $\upsilon_k : \mathbf{Z}_M^k \to \mathbf{R}$ is the controller function at time k.

If no restrictions save causality are placed on the structure of the coder, each symbol s_k which it transmits may be a function of the sequences of past and present outputs $\tilde{x}_k$ and past symbols $\tilde{s}_{k-1}$. However, (2) and (3) imply that x_k in its turn is completely determined by the initial condition and $\tilde{s}_{k-2}$, so s_k is consequently a function of x_0 and $\tilde{s}_{k-1}$,

$$s_k = \gamma_k(x_0, \tilde{s}_{k-1}), \quad \forall k \geq 0, \tag{4}$$

where $\gamma_k : \mathbf{R} \times \mathbf{Z}_M^k \to \mathbf{Z}_M$ is the coder function at time k.

Note that there is no explicit communication constraint between the controller and actuator. This is obviously a reasonable assumption if they are colocated, but even otherwise the formulation above would be unchanged, since the location of the controller is purely nominal. The symbols that would be transmitted by it over an additional link to the actuator would have to be translated once again into control signals, making intermediate calculations redundant. The number R should thus be regarded as the *overall* rate of the complete link from sensor to actuator. In addition, we remark that there are slightly different ways in which the digital link can be defined; see [17, 1] for details.

We call the pair of sequences $(\gamma, \upsilon) \triangleq (\{\gamma_k\}_{k\geq 0}, \{\upsilon_k\}_{k\geq 0})$ a *coder-controller* and the objective is to find one that minimizes an infinite horizon cost of the form

$$J_m \triangleq \limsup_{k\to\infty} \rho_k^{-1} \mathrm{E}|X_k|^m, \tag{5}$$

where X_k is the random variable corresponding to the output x_k. This compares the asymptotic behaviour of the m-th output moment against some positive sequence $\{\rho_k\}_{k\geq 0}$ which serves as a rough benchmark of the desired output behaviour. Although it does not attach a cost to the magnitudes of the controls or intermediate states, it does succeed in capturing the asymptotic stochastic behaviour of the closed-loop system. However, before addressing this objective, we first investigate the finite horizon cost

$$J_{m,N} \triangleq \mathrm{E}|X_{N+1}|^m. \tag{6}$$

The insights gained then lead to an optimal solution of the infinite horizon problem.

3 Finite Horizon Cost

If the initial condition were known with complete accuracy by the controller, it could easily use the plant equations to generate controls that yield $x_1 = x_2 = \cdots = 0$. However, in the presence of the data rate constraint this is evidently impossible. One way around this is for the coder to transmit a progressively more accurate estimate of x_0 to the controller, by using the bits available at each time instant to quantize it recursively. The controller can then use these estimates to generate the controls. It is shown in this section that for linear plants of the form (2), the optimal finite horizon scheme has exactly this structure. More precisely, the core of the $J_{m,N}$-optimal coder-controller is shown to be an optimal, M^N-level quantizer for X_0 that has been reformulated to operate in a sequential manner.

Observe that by using downward induction on k, the plant output can be expressed in terms of the initial condition and past controls,

$$x_{k+1} = \alpha_k x_0 + \sum_{j=0}^{k} \beta_{k,j} u_{k-j}, \quad \forall k \geq 0, \tag{7}$$

where $\alpha_{-1} \triangleq 1$ and α_k, $\beta_{k,j}$ are given recursively by

$$\begin{aligned} \alpha_{k+1} &\triangleq \sum_{j=0}^{k+1} a_{k+1,j}\alpha_{k-j}, \\ \beta_{k+1,j} &\triangleq b_{k+1,j} + \sum_{i=0}^{j-1} a_{k+1,i}\beta_{k-i,j-i-1}, \end{aligned} \tag{8}$$

$\forall k \geq 0, j \in [0,\ldots,k+1]$. Notice that the problem becomes trivial if $\alpha_k = 0$ for some $k \in [0,\ldots,N]$, since from (7) and (2) controls $u_k,\ldots,u_N$ can then be found which yield $x_{k+1} = \cdots = x_{N+1} = 0$. As such, we focus on the nontrivial case when $\alpha_k \neq 0$, $\forall k \in [0,\ldots,N]$.

The next step is to change variables. Define the mappings $\eta_0, \eta_1, \ldots, \eta_N$ by

$$\eta_k(\tilde{t}_{k-1}) \triangleq -\alpha_k^{-1} \sum_{j=0}^{k} \beta_{k,j} v_{k-j}(\tilde{t}_{k-j-1}), \quad \forall \tilde{t}_{k-1} \in \mathbf{Z}_M^k, \tag{9}$$

so that $-\alpha_k \eta_k(\tilde{s}_{k-1})$ represents the cumulative effect of past controls on the output at time $k+1$. As $\beta_{k,0} = b_{k,0}$, which by hypothesis is nonzero, the matrix of coefficients $\{-\alpha_k^{-1}\beta_{k,j}\}_{k,j}$ is triangular with nonzero diagonal elements. Hence the equation above can easily be inverted to express the sequence $\tilde{v}_N$ of controller functions in terms of $\tilde{\eta}_N$ via

$$v_k(\tilde{t}_{k-1}) \triangleq \sum_{j=0}^{k} \tau_{k,j}\eta_{k-j}(\tilde{t}_{k-j-1}), \quad \forall \tilde{t}_{k-1} \in \mathbf{Z}_M^k, \tag{10}$$

where $\tau_{k,j}$ is given by the recursion

$$\tau_{k,0} \triangleq -\beta_{k,0}^{-1}\alpha_k, \quad \tau_{k,j} \triangleq -\beta_{k,0}^{-1} \sum_{i=1}^{j} \beta_{k,i}\tau_{k-i,j-i}, \tag{11}$$

$\forall k \in [0,\ldots,N]$ and $j \in [0,\ldots,k]$. This one-to-one correspondence between $\tilde{\eta}_N$ and $\tilde{v}_N$ implies that it is perfectly equivalent to optimize $J_{m,N}$ with respect to either. Substituting (7) and (9) into (6), we obtain

$$J_{m,N} = |\alpha_N|^m \mathrm{E}|X_0 - \eta_N(\tilde{S}_{N-1})|^m, \tag{12}$$

where $\tilde{S}_{N-1}$ is the random variable corresponding to the symbol sequence $\tilde{s}_{N-1}$. As $\eta_N(\tilde{s}_{N-1})$ is a function of x_0 with up to M^N distinct values, one for each possible symbol sequence, the function R_N defined by $R_N(x_0) \triangleq \eta_N(\tilde{s}_{N-1})$ can be regarded as an M^N-level quantizer for x_0. The R.H.S. of (12) is then simply its mean m-th power error (MmPE), scaled by $|\alpha_N|^m$. As such, if we can find a pair (γ^*, η^*) that achieves the minimum MmPE for an M^N-level quantizer for X_0, it will automatically be $J_{m,N}$-optimal. We show next how to construct such a pair.

Denote the points of the MmPE-optimal, M^N-level quantizer for X_0 by $q_N(0), q_N(1), \ldots, q_N(M^N - 1)$, ranked from least to greatest, and set

$$\eta_N^*(\tilde{t}_{N-1}) \triangleq q_N\left(\sum_{k=0}^{N-1} t_k M^{N-1-k}\right), \quad \forall \tilde{t}_{N-1} \in \mathbf{Z}_M^N, \tag{13}$$

i.e. $\eta_N^*(t_0,\ldots,t_{N-1})$ is the $(1 + \sum_{k=0}^{N-1} t_k M^{N-1-k})$-th quantizer point. By the *nearest neighbour rule* [6], the optimal quantizer selects the point closest to x_0, so choose the symbol sequence by

$$\tilde{s}_{N-1} = \arg \min_{\tilde{t}_{N-1} \in \mathbf{Z}_M^N} |x_0 - \eta_N^*(\tilde{t}_{N-1})|^m, \tag{14}$$

breaking possible ties by selecting the greater point. As this equation can be expressed recursively, $\forall k \in [0,\ldots,N-1]$, as

$$\begin{aligned} s_k = \gamma_k^*(x_0, \tilde{s}_{k-1}) \triangleq \arg \min_{t_k \in \mathbf{Z}_M} \Big\{ &\min_{t_{k+1},\ldots,t_{N-1} \in \mathbf{Z}_M} \\ &|x_0 - \eta_N^*(\tilde{s}_{k-1}, t_k, t_{k+1}, \ldots, t_{N-1})|^m \Big\}, \end{aligned} \tag{15}$$

it is realizable within our framework. Substituting (14) into (12) and denoting the optimal cost by $J_{m,N}^*$, we obtain

$$\begin{aligned} |\alpha_N|^{-m} J_{m,N}^* &= \int \min_{\tilde{t}_{N-1}} |x_0 - \eta_N^*(\tilde{t}_{N-1})|^m dP(x_0) \\ &= \int \min_{j \in \mathbf{Z}_{M^N}} |x_0 - q_N(j)|^m dP(x_0). \end{aligned} \tag{16}$$

As the last integral is simply the expression for the MmPE of the optimal M^N-level quantizer for X_0 [6], the pair (γ^*, η^*) is $J_{m,N}$-optimal.

We now recast the coder equation (15) in a somewhat simpler form. First extend q_N over the real interval $[-1, M^N]$, so that it remains increasing and is furthermore continuous, and for convenience set $q_N(-1) \triangleq -\infty$, $q_N(M^N) \triangleq \infty$. Next define

$$e_N(z) \triangleq \left(q_N(M^N z - 1) + q_N(M^N z)\right)/2, \tag{17}$$

$$\zeta_k \triangleq \sum_{j=0}^{k} s_j M^{-j-1}, \tag{18}$$

$\forall z \in [0,1]$ and $k \geq 0$. Hence $e_N(\zeta_{N-1})$ is halfway between the neighbouring quantizer points $q_N(M^N\zeta_{N-1} - 1)$ and $q_N(M^N\zeta_{N-1})$. From (14), the sequence $\tilde{s}_{N-1}$ is transmitted iff the quantizer point closest to x_0 is $q_N(M^N\zeta_{N-1})$, equivalent to x_0 lying inside the interval $\left[e_N(\zeta_{N-1}), e_N(\zeta_{N-1} + M^{-N})\right)$. As e_N is increasing and continuous, x_0 lies in this interval iff $e_N^{-1}(x_0) \in [\zeta_{N-1}, \zeta_{N-1} + M^{-N})$. Referring to (18), this in turn is equivalent to $\tilde{s}_{N-1}$ being the first N digits of the M-ary representation for $e_N^{-1}(x_0)$. That is, the optimal coder simply applies a transformation e_N^{-1} to the initial condition and then transmits the first N digits of its M-ary expansion. This and the previous results are encapsulated below:

Coder 1 *First, transform the initial condition x_0 of the plant (2) to obtain $\zeta \triangleq c_N(x_0) \triangleq e_N^{-1}(x_0)$, where e_N is given by (17). Then at time k, transmit the $(k+1)$-th most significant digit in the M-ary representation of ζ as the symbol s_k.*

Controller 1 *Upon receiving the symbol s_{k-1} at time k, calculate the number ζ_k using (18). Set*

$$\begin{aligned} \eta_N^*(\tilde{s}_{N-1}) &\triangleq q_N(M^N\zeta_{N-1}), \\ \eta_k^*(\tilde{s}_{k-1}) &\triangleq c_N^{-1}(\zeta_{k-1} + 0.5/M^k),\ 0 \leq k \leq N-1, \end{aligned} \tag{19}$$

where $q_N(0) < q_N(1) < \cdots < q_N(M^N - 1)$ are the points of the MmPE-optimal, M^N-level quantizer for X_0, and use (10) to calculate the control $v_k(\tilde{s}_{k-1})$.

We make several comments here. Firstly, the optimal coder-controller is basically a *compander*, i.e. it consists of a *compressor* c_N which maps $x_0 \in \mathbf{R}$ to $\zeta \in [0,1]$, followed by a uniform, M^N-level quantizer which maps this to ζ_{N-1} and then an *expander* $q_N(M^N\cdot)$ which transforms ζ_{N-1} into an estimate of x_0 [6]. Secondly, the mappings $\eta_0^*, \ldots, \eta_{N-1}^*$ are actually completely arbitrary, since they do not affect the integrand in (16). However the choice above is intuitively appealing, as $\zeta_k + \frac{1}{2M^{k+1}}$ is the midpoint of the interval of length M^{-k-1} which the controller knows that ζ lies in, from the sequence $\tilde{s}_k$. Furthermore, this makes the infinite horizon analysis of the next section somewhat easier. Thirdly, although the optimal quantizer may be unique, η_N^* can be defined in as many different ways as there are one-to-one maps from the integers $\mathbf{Z}_{M^N}$ to the M-ary sequences $\mathbf{Z}_M^N$. The choice of mapping taken here, as implied in equation (13), is one of the more tractable ones. Finally, explicit expressions for the optimal coder-controller are generally impossible to derive, since q_N can normally only be obtained numerically for a given p and N [10, 6]. One of the few exceptions is when X_0 is *Laplacian* and $m = 1$, for which case a closed form solution parametrized by N and the mean and variance of X_0 has been obtained [11].

4 Infinite Horizon Cost

In the previous section, we observed that $J_{m,N}$-optimal coder-controllers are usually impossible to derive in closed form. However, we demonstrate here that when $N \to \infty$ the limiting coder-controller can be obtained directly, *without explicitly solving the finite horizon problem.* We then prove that this limiting scheme is in fact optimal with respect to an infinite horizon cost of the form (5), under certain mild conditions on the probability density p governing the initial output x_0.

The key is the classic result that as the number of MmPE-optimal quantizer points approaches infinity, their normalized density per unit x_0 approaches

$$\nu(x_0) \triangleq \left(\int p(y)^{1/(m+1)}\, dy \right)^{-1} p(x_0)^{1/(m+1)},\ \forall x_0 \in \mathbf{R}, \tag{20}$$

under certain technical conditions on p [5, 2]. As $q_N(M^N\zeta_{N-1})$ is the $(M^N\zeta_{N-1} + 1)$-th quantizer point by (13), the nearest neighbour rule implies that there are $M^N\zeta_{N-1} + O(1)$ points less than or equal to x_0. Observing that ζ_{N-1}, being a sum of exponentially decaying terms, must converge to a number $\bar{\zeta} \in [0,1]$ as $N \to \infty$, we may define the infinite horizon compressor

$$c(x_0) \triangleq \bar{\zeta} = \lim_{N\to\infty} \frac{M^N\zeta_{N-1} + O(1)}{M^N} = \int_{y \leq x_0} \nu(y)\, dy. \tag{21}$$

By analogy with Coder-Controller 1, the following scheme as the horizon N becomes unbounded is suggested:

Coder 2 *First, transform the initial condition x_0 of the plant (2) to yield $\bar{\zeta} \triangleq c(x_0)$, where c is given by (21). Then at time k transmit the $(k+1)$-th most significant digit in the M-ary representation of $\bar{\zeta}$ as the symbol s_k.*

Controller 2 *Upon receiving the symbol s_{k-1} at time k, calculate*

$$\eta_k(\tilde{s}_{k-1}) = c^{-1}(\zeta_{k-1} + 0.5/M^k), \tag{22}$$

where ζ_{k-1} is defined by (18), and use (10) to generate the control signal $v_k(\tilde{s}_{k-1})$.

For instance, for *Laplacian* X_0 with mean μ and mean absolute deviation ϵ, it can be shown that

$$c(x_0) = 0.5 + \operatorname{sign}(x_0 - \mu)(1 - \mathrm{e}^{-|x_0-\mu|/((m+1)\epsilon)})/2,$$

and for Gaussian X_0 with mean μ and standard deviation ϵ, we have

$$c(x_0) = F\left((x_0 - \mu)/(\epsilon\sqrt{m+1})\right),$$

where F is the unit normal distribution function.

Next we fix the weights ρ_k, $k \geq 0$, in the infinite horizon cost (5). Observe that as $N \to \infty$,

$$\begin{aligned} & \min_{\gamma,v} M^{mN} |\alpha_N|^{-m} \mathrm{E}|X_{N+1}|^m \\ &= \min_{\gamma,\eta} M^{mN} \mathrm{E}|X_0 - \eta_N(\tilde{S}_{N-1})|^m \\ &\to (m+1)^{-1} 2^{-m} \|p\|_{1/(m+1)}, \end{aligned} \quad (23)$$

where the limit is a well-known result of asymptotic quantization theory [2] and $\|p\|_r \triangleq \left(\int p(x_0)^r \, dx_0\right)^{1/r}$. This is nearly what we want, except that in (5) the minimization is to be performed after the limit is taken. This suggests that an appropriate choice of weighting sequence is

$$\rho_k = |\alpha_{k-1}|^m / M^{m(k-1)}, \quad \forall k \geq 0. \quad (24)$$

The main result of this section can now be stated.

Theorem 1 *Let the initial output x_0 of the system (2) be governed by a continuous probability density function p which decreases with $|x_0|$ for sufficiently large $|x_0|$ and satisfies $\mathrm{E}|X_0|^{m+n} < \infty$, for some $m, n > 0$. Suppose further that*

$$\begin{aligned} pc^{-1}(2z-1) &\leq Apc^{-1}(z), \quad \forall z \in [1-\delta, 1], \\ pc^{-1}(2z) &\leq Apc^{-1}(z), \quad \forall z \in [0, \delta], \end{aligned}$$

for some $A, \delta > 0$, where c is given by (21). Then Coder-Controller 2 is J_m-optimal and achieves

$$\begin{aligned} J_m^* &= \min_{\gamma,v} \lim_{k\to\infty} |\alpha_{k-1}|^{-m} M^{m(k-1)} \mathrm{E}|X_k|^m \\ &= \frac{1}{(m+1)2^m} \left(\int p(x_0)^{1/(m+1)} \, dx_0 \right)^{m+1} \end{aligned} \quad (25)$$

where α_k, $k \geq 0$, are given by (8). Furthermore, for a given coding alphabet size M, a coder-controller that takes $\mathrm{E}|X_k|^m \to 0$ exists if and only if

$$\alpha_k / M^k \to 0, \quad \text{as } k \to \infty. \quad (26)$$

The proof of this theorem may be found in [14] and depends on a result in [8] on optimal scalar companding. Condition (26) compares the the accuracy of the initial condition estimate, proportional to M^k, with the dynamical coefficient α_k which propagates the uncertainty in x_0, and states that the plant can be stabilized if and only if the former increases more rapidly than the latter. Note that it is trivially satisfied by asymptotically stable plants, for which $\alpha_k \to 0$. When the open-loop plant is d-dimensional, linear and time-invariant (LTI), the first equation in (8) becomes

$$\alpha_{k+1} = \sum_{j=0}^{d-1} a_j \alpha_{k-j},$$

where $\alpha_{-1} = 1$ and $\alpha_{-2} = \ldots = \alpha_{-d} = 0$. The solution to this is of the form

$$\alpha_k = \sum_{j=0}^{d-1} h_j(k) \theta_j^k,$$

where $\theta_0, \ldots, \theta_{d-1}$ are the poles of the system and $h_0, \ldots, h_{d-1}$ are polynomial functions. Hence if λ is the pole with largest magnitude then $\alpha_k \sim \lambda^k$ for large k, to within a polynomial factor in k. Substituting this into (26), we see that the plant is asymptotically stabilizable in m-th moment iff $M > |\lambda|$. As the data rate $R = \log_2 M$, we obtain the condition $R > \log_2 |\lambda|$. This makes precise the notion that the more unstable a system is, the higher the data rate needed to stabilize it. This result is summarized below:

Corollary 1 *Consider a linear plant with time-invariant, d-dimensional open-loop dynamics*

$$x_{k+1} = \sum_{j=0}^{d-1} a_j x_{k-j} + \sum_{j=0}^{\infty} b_{k,j} u_{k-j}, \quad \forall k \geq 0,$$

where $x_k, u_k \in \mathbf{R}$ are the output and control respectively at time k, $a_j, b_{k,j} \in \mathbf{R}$ with $b_{k,0} \neq 0$, $\forall j, k \geq 0$, and $x_k = u_k = 0$ when $k < 0$. If the probability density function governing x_0 satisfies the conditions in Theorem 1, then a coder-controller with data rate R which takes $\mathrm{E}|X_k|^m \to 0$ exists if and only if

$$R > \log_2 |\lambda|,$$

where λ is the open-loop pole with largest magnitude.

The technical conditions on p in Theorem 1 limit the speed of decay of $pc^{-1}(z)$ as z approaches 0 and 1. They can be shown to be satisfied by any p such that $p(y) \sim |y|^v \exp(-B|y|^w)$ for large $|y|$ and some $B, w > 0$, $v \in \mathbf{R}$, which includes densities such as the Gaussian and Laplacian. We conjecture that the infimum of J_m assuming only that p is Lebesgue-integrable is also given by (23), since it should be possible to construct compressors c_i, $i \geq 0$, which satisfy the conditions of [8] and approach c in an appropriate integral sense as $i \to \infty$. However, it is much more difficult to prove that Coder-Controller 2 actually achieves this lower bound for any such general p, despite being the limiting form of the optimal finite horizon scheme.

We remark that the results above automatically apply to the problem of output estimation under a data rate constraint [16, 12]. The only differences in the formulation are that the controls in the system equation (2) are set to zero, the controller is replaced by an estimator

$$\hat{x}_k = \delta_k(\tilde{s}_{k-1}), \quad \forall k \geq 0, \quad (27)$$

and the objective is to find a *coder-estimator* $(\gamma, \delta) \triangleq (\{\gamma_k\}_{k\geq 0}, \{\delta_k\}_{k\geq 0})$ that minimizes the distortion

$$D_m \triangleq \limsup_{k\to\infty} \rho_k^{-1} \mathrm{E}|X_k - \hat{X}_k|^m. \tag{28}$$

The optimal coder-estimator is then the same as Coder-Controller 2, except that $\eta_k(\tilde{s}_{k-1})$ is used to generate an estimate $\delta_k(\tilde{s}_{k-1}) = \alpha_{k-1}\eta_k(\tilde{s}_{k-1})$ rather than a control.

Finally, note that the results of this section indicate that *one-bit mean coding schemes* [16, 7] are suboptimal with respect to the infinite horizon, mean-square-error cost J_2. Such schemes have intuitive appeal, as they proceed by simply partitioning a coding interval I containing x_0 at the conditional mean $\mathrm{E}\{X_0 | X_0 \in I\}$ to form two new candidate intervals. However, it is easy to show from the discussion at the beginning of this section that the intervals formed by Coder 2 always contain equal proportions of optimal, infinite-level quantizer points. For a one-bit scheme, this means that each coding interval $[a, b)$ should be divided at the point u such that

$$\int_a^u \nu(x_0)dx_0 = \int_u^b \nu(x_0)dx_0,$$

which in general does not coincide with $\mathrm{E}\{X_0 | X_0 \in [a, b)\}$. Hence, although the conditional mean is the mean-square-error-optimal reconstruction point, it is not the J_2-optimal partition point.

5 Conclusion

In this paper, a communication-limited control problem was formulated for linear plants with initial output probability density p. It was shown that the optimal finite horizon coder-controller is an optimal quantizer for the initial condition that has been formulated to operate sequentially. Asymptotic quantization theory was then used to directly obtain the limiting scheme as the horizon approaches infinity. Under certain technical conditions on the initial condition probability density, this scheme is optimal in an infinite horizon sense and yields a necessary and sufficient condition, in terms of the coding alphabet size and dynamical constants, for a given plant to be asymptotically stabilizable in m-th output moment. For the case of linear, time-invariant plants this condition simplifies to an inequality involving the data rate and the open-loop pole with largest magnitude. Further work is currently being undertaken on extending the results presented here to stochastic and state-space systems.

References

[1] V. S. Borkar and S. K. Mitter. LQG control with communication constraints. In A. Paulraj, V. Roychowdhury, and C. D. Schaper, editors, *Communications, Computation, Control and Signal Processing*, pages 365–73. Dordrecht, Boston, 1997.

[2] J. A. Bucklew and G. L. Wise. Multidimensional asymptotic quantization with rth power distortion measures. *IEEE Trans. Info. Theory*, 28:239–47, 1982.

[3] D. F. Delchamps. Extracting state information from a quantized output record. *Systems and Control Letters*, 13:365–72, 1989.

[4] D. F. Delchamps. Stabilizing a linear system with quantized state feedback. *IEEE Trans. Autom. Contr.*, 35:916–24, 1990.

[5] A. Gersho. Asymptotically optimal block quantization. *IEEE Trans. Info. Theory*, 25:373–80, 1979.

[6] A. Gersho and R. M. Gray. *Vector Quantization and Signal Compression*. Kluwer, Boston, 1993.

[7] X. Li and W. S. Wong. State estimation with communication constraints. *Systems and Control Letters*, 28:49–54, 1996.

[8] T. Linder. On asymptotically optimal companding quantization. *Problems of Control and Information Theory*, 20:475–84, 1991.

[9] X. Liu and W. S. Wong. Controllability of linear feedback control systems with communication constraints. In *Proc. 36th IEEE Conf. Decision and Control*, pages 60–5, 1997.

[10] S. P. Lloyd. Least squares quantization in PCM. *IEEE Trans. Info. Theory*, 28:129–37, 1982. Reproduction of an unpublished Bell Laboratories manuscript, 1957.

[11] G. N. Nair. *State Estimation Under Information Constraints*. PhD thesis, University of Melbourne, 1999.

[12] G. N. Nair and R. J. Evans. State estimation via a capacity-limited communication channel. In *Proc. 36th IEEE Conf. Decision and Control*, pages 866–71, 1997.

[13] G. N. Nair and R. J. Evans. A finite-dimensional coder-estimator for rate-constrained state estimation. In *Proc. 14th IFAC World Congress 1999*, volume I, pages 19–24, 1999.

[14] G. N. Nair and R. J. Evans. Stabilization with data-rate-limited feedback: tightest attainable bounds. *Systems and Control Letters*, 41(1):49–56, 2000.

[15] S. Tatikonda, A. Sahai, and S. K. Mitter. Control of LQG systems under communication constraints. In *Proc. 37th IEEE Conf. Decision and Control*, pages 1165–70, 1998.

[16] W. S. Wong and R. W. Brockett. Systems with finite communication bandwidth constraints I. *IEEE Trans. Autom. Contr.*, 42:1294–9, 1997.

[17] W. S. Wong and R. W. Brockett. Systems with finite communication bandwidth constraints II: stabilization with limited information feedback. *IEEE Trans. Autom. Contr.*, 44:1049–53, 1999.