# Minimum magnitude response to a fixed input

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**Abstract**

In this paper we investigate the *exact* solution, minimizing the $l_\infty$ norm of the regulated output for a fixed input in SISO discrete-time feedback control systems. This is achieved by allowing non-zero steady state value and parametrizing the output to have a rational transfer function with chosen poles on the stability boundary. Alongside these $l_\infty$-optimal solutions, we obtain solutions in $l_1$ with the same order transfer functions and arbitrarily close $l_\infty$ norms.

## 1 Introduction

The problem of minimizing the peak magnitude of the tracking error in a feedback control system driven by a fixed command input is an important practical one and is not solved using $l_2$ approaches. The problem was posed in [1] as an interpolation constrained $l_\infty$ minimization and solved using results on minimum norm duality and linear programming, using an approach related to that proposed [2] for $l_1$-optimal control. It was found in [1] that, in general, the optimal solution to the $l_\infty$ problem was not in $l_1$. In order to obtain a response having a steady state value of zero, [1] proposed a truncation-based solution approach which gave a deadbeat (FIR) system of sufficient order to achieve performance within any specified tolerance of the optimum. The main focus of the present paper is to show that if we drop the requirement that the regulated output be zero in the steady state, then in many cases, the optimal solutions have rational transfer functions and can be found using linear programs.

As an alternative to the use of linear programs to obtain numerical FIR solutions, closed-form solutions for some simple truncated $l_\infty$ minimization problems with only two interpolation constraints are in [3]. Limiting values of performance as the truncation length tended to infinity were readily obtained from these closed-form solutions. Another approach for obtaining FIR solutions is [6], where improved numerical performance is claimed through the use of a Chebyshev approximation formulation.

[1]Work supported by the Australian Research Council.

Now, with regard to the implementation of controllers, although a sequence of deadbeat solutions could be used for obtaining bounds on achievable performance, they have the undesirable feature that high order closed-loop systems and thus high order controllers may be required if performance approaching achievable bounds is needed. The possibility of using other than deadbeat systems has been raised in [4, 5], however, as indicated in [4], the selection of the closed-loop poles was an open problem. Results on the design of fixed order controllers to minimize an upper bound on the $l_\infty$ norm are in [7]. Dual formulations for more general performance criteria are obtained in [8].

Aspects of the continuous-time version of the problem are in [9, 10, 11, 12]. In [9, 10, 11] the solution to the $\mathcal{L}_\infty$ norm optimization, the so-called $\mathcal{L}_\infty$-optimal solution, was considered separately from its projection onto $\mathcal{L}_1$. An intermediate step in [10] used the discrete-time counterpart, namely the $l_\infty$-optimal solution, and an example with two real interpolation constraints was given.

Here we develop a partial solution to the problem of obtaining exact or $l_\infty$-optimal solutions for problems with any number of interpolation points, real or in complex conjugate pairs. The approach used is a development of [5] which has also been extended [13] to obtain stable rational $l_1$-suboptimal solutions. The $l_\infty$-optimal solutions are obtained by dropping the requirement that the regulated output be zero at steady state. We show that in many cases, these solutions have rational transfer functions and can be determined using linear programs. These solutions are useful for several reasons. Firstly, they give exact bounds on the limits of performance achievable by stable discrete-time closed-loop systems. Secondly, as we show, they provide a method for designing stabilizing controllers which give better $l_\infty$ performance than the best deadbeat controller of given order, for sufficiently high orders. Thirdly, they arise in the procedure in [10] for obtaining rational continuous-time designs. Fourth, these solutions provide a partial answer to the problem posed in [4] of how to perform $l_\infty$ minimization over the closed-loop pole locations. Finally, they are of theoretical interest, since they are the solutions being approximated by FIR approaches.

## 2 Preliminaries and Problem Setup

The $z$-transform, $\hat{h}$, of a sequence, $h = \{h_k\}_{k=0}^{\infty}$, is defined by $\hat{h} = \sum_{k=0}^{\infty} h_k z^k$. With this definition, a stable transfer function has all of its poles at values of $z : |z| > 1$ . The symbol $z$ also denotes the unit delay. The space of sequences $h$ for which $\sum_{k=0}^{\infty} |h_k| < \infty$ is $l_1$, and is equipped with norm given by $\|h\|_1 = \sum_{k=0}^{\infty} |h_k|$. The space of sequences $h$ where $\sup_k |h_k| < \infty$ is $l_\infty$, and is equipped with norm given by $\|h\|_\infty = \sup_k |h_k|$.

Given a scalar rational plant $\hat{P}(z)$ and command $w \in l_\infty$, we wish to design a closed-loop regulated output $\phi = w - y$, where $y$ is the plant output, in such a way that $\|\phi\|_\infty$ is minimized. To preclude hidden unstable modes in the closed-loop system, $\hat{\phi}(z)$ must satisfy a finite set of interpolation constraints imposed by the unstable plant poles and the nonminimum phase zeros of the plant and of $\hat{w}(z)$.

We assume the plant has no poles or zeros at $z : |z| = 1$. Likewise the command $\hat{w}(z)$ has no zeros at $z : |z| = 1$. We denote the real poles and zeros to be interpolated as $a_1, \ldots, a_{n_{re}}$ and one element from each complex conjugate pair as $a_{1+n_{re}}, \ldots, a_{n_{cx}+n_{re}}$. We define $r_j$, $\theta_j$ from $a_{j+n_{re}} = r_j(\cos\theta_j + i\sin\theta_j)$, $j = 1, \ldots, n_{cx}$, where $i = \sqrt{-1}$. Each complex conjugate pair of points to be interpolated gives rise to a pair of real constraints so the total number of constraints $n$ is given by $n = n_{re} + 2n_{cx}$. The $n$ interpolation constraints are then

$$\sum_{k=0}^{\infty} a_j^k \phi_k = b_j; \qquad j = 1, 2, \ldots, n_{re}, \tag{1}$$

$$\sum_{k=0}^{\infty} r_j^k \cos(k\theta_j)\phi_k = \Re b_{j+n_{re}}; \qquad j = 1, 2, \ldots, n_{cx}, \tag{2}$$

$$\sum_{k=1}^{\infty} r_j^k \sin(k\theta_j)\phi_k = \Im b_{j+n_{re}}; \qquad j = 1, 2, \ldots, n_{cx}, \tag{3}$$

where $\Re$ denotes real part and $\Im$ denotes imaginary part. The $b_j$ arise as follows: firstly if $a_j$ is a zero of $\hat{P}(z)$ or of $\hat{w}(z)$, then $b_j = \hat{w}(a_j)$; secondly, in a one-parameter configuration only, if $a_j$ is a pole of $\hat{P}(z)$, then $b_j = 0$. We write (1, 2, 3) in abbreviated form as $\hat{\phi}(a) = b$ where

$$b = (b_1, \ldots, b_{n_{re}}, \Re b_{n_{re}+1}, \ldots, \Re b_{n_{re}+n_{cx}}, \Im b_{n_{re}+1}, \ldots, \Im b_{n_{re}+n_{cx}}),$$

denoting a column vector.

The scalar $l_\infty$ minimization problem and its dual can be written [1]:

**Theorem 1.**

$$\nu := \min_{\substack{\phi \in l_\infty \\ \hat{\phi}(a) = b}} \|\phi\|_\infty = \max_{\substack{\alpha \in \mathcal{R}^n \\ \|\phi^*\|_1 \leq 1}} \alpha^T b. \tag{4}$$

where $\phi^* = \{\phi_k^*\}_{k=0}^{\infty}$ with

$$\begin{aligned} \phi_k^* &= \sum_{j=1}^{n_{re}} \alpha_j a_j^k + \sum_{j=1}^{n_{cx}} \alpha_{j+n_{re}} r_j^k \cos k\theta_j \\ &\quad + \sum_{j=1}^{n_{cx}} \alpha_{j+n_{re}+n_{cx}} r_j^k \sin k\theta_j; \quad k = 0, 1, \ldots \end{aligned} \tag{5}$$

Any optimal $\phi$ and corresponding optimal $\phi^*$ satisfy the following alignment condition:

$$\phi_k^* \neq 0 \;\Rightarrow\; \phi_k = \nu \,\mathrm{sgn}\phi_k^*, \tag{6}$$

$$\phi_k^* = 0 \;\Rightarrow\; |\phi_k| \leq \nu. \tag{7}$$

■

This theorem shows the relation between two optimization problems, namely a primal minimization and a dual maximization. Define $\phi_{opt}$ to be the sequence $\phi$ which solves this problem and define $\nu := \|\phi_{opt}\|_\infty$. In general $\phi_{opt} \notin l_1$.

In [1], a suboptimal solution (i.e. $\phi : \|\phi\|_\infty > \nu$), $\phi \in l_1$ was obtained by truncating the set of dual constraints (5) so that $\phi^* = \{\phi_i^*\}_{i=0}^{i=N}$ and $\|\phi^*\|_1 = \sum_{i=0}^{N} |\phi_i^*|$. Implicit is this approach is that the transfer function of $\phi$ has the parametrization,

$$\hat{\phi}(z) = \sum_{i=0}^{N} \phi_i z^i. \tag{8}$$

The value of the primal and dual optimization criterion for such a truncated problem is denoted $\nu_N$. It is shown in [1] that $\nu_N \geq \nu_{N+1}$ and, further, that $lim_{N\to\infty}\nu_N = \nu$. This allows the value of $\nu$ to be found to within an arbitrarily small specified tolerance by solving a sufficiently large linear program.

As they stand, both the primal minimization and dual maximization problems in Theorem 1 are infinite linear programs. Call a sequence $\phi$ which solves this problem, $\phi_{opt}$, so that $\nu = \|\phi_{opt}\|_\infty$. Our goal is to calculate $\nu$ and $\phi_{opt}$ exactly by solving finite linear programs.

Note a routine modification, removing $\phi_0$ from the vector of coefficients of $\phi$ being minimized, can be incorporated for plants with delay, in order to prevent $\phi_0$ from determining $\nu$.

## 3 Rational $l_\infty$-optimal solutions

As pointed out in [1], the optimal solution for most problems of interest has infinitely many $\phi_i$ taking the maximum magnitude. We will make use of the fact that in many cases, these sequences have *rational* transfer functions.

Now, from the structure of the dual, and the alignment condition (6,7) between primal and dual solutions, it is clear that the slowest decaying mode in the dual constraints plays a role in determining the sign properties of the tail of the primal solution. We distinguish this mode:

**Definition D1.** Define $a_s$ as the $a_j$ of largest magnitude such that $\alpha_j \neq 0$ in a solution to the dual maximization problem of Theorem 1.

We will assume that $|a_s| > 0$, thereby excluding trivial plants. One of the difficulties encountered in this work is that we do not know a-priori which out of the $a_i$'s is $a_s$, since we cannot say which if any $\alpha_j$ will be zero valued in the solution to the dual problem in Theorem 1. For many problems, $a_s$ is the $a_i$ of largest magnitude, but, in general, we cannot know if this is so in advance.

We require the following restrictions.

**Assumption A1.** All real $a_j$ and all complex conjugate pairs are distinct in magnitude. Also $\theta_j = 2\pi p_j/q_j$ for $j = 1, \ldots, n_{cx}$ where $p_j, q_j$ are integers with $0 < p_j < q_j$ for $j = 1, \ldots, n_{cx}$.

The following key result is a consequence of Theorem 1 under Assumption A1.

**Theorem 2.** If A1 holds, then $\hat{\phi}_{opt}(z)$ is rational with $z$-transform as follows. Firstly, if $a_s > 0$, then for some $m \geq 0$,

$$\hat{\phi}_{opt}(z) = \sum_{k=0}^{m-1} \phi_k z^k + \frac{\phi_m z^m}{1 - z}. \tag{9}$$

Secondly, if $a_s < 0$, then for some $m \geq 0$,

$$\hat{\phi}_{opt}(z) = \sum_{k=0}^{m-1} \phi_k z^k + \frac{\phi_m z^m}{1 + z}. \tag{10}$$

Finally, if $\Im a_s \neq 0$, then for some $m \geq 0$,

$$\hat{\phi}_{opt}(z) = \sum_{k=0}^{m-1} \phi_k z^k + \frac{\sum_{k=1}^{q} c_k z^{m+k-1}}{1 - z^q}, \tag{11}$$

where $q = \operatorname{lcm}(q_j)$ if there are no $a_j < 0$ or $q = \operatorname{lcm}(2, q_j)$ if some $a_j < 0$. ∎

We introduce two linear programming problems: $P1$, developed from [5]; and $P2$.

$$P1: \qquad \nu_1 = \min_{\{\phi_0,\ldots,\phi_m\}} \max\{|\phi_0|, \ldots, |\phi_m|\}$$

subject to

$$\sum_{k=0}^{m-1} \phi_k a_j^k + \phi_m \frac{a_j^m}{1 - \lambda a_j} = b_j, \quad j = 1, 2, \ldots, n_{re}$$

$$\sum_{k=0}^{m-1} \phi_k r_j^k \cos(k\theta_j) + \phi_m r_j^m \frac{\cos m\theta_j - \lambda r_j \cos(m-1)\theta_j}{1 - 2\lambda r_j \cos(\theta_j) + \lambda^2 r_j^2} = \Re b_{j+n_{re}}, \quad j = 1, 2, \ldots, n_{cx}$$

$$\sum_{k=1}^{m-1} \phi_k r_j^k \sin(k\theta_j) + \phi_m r_j^m \frac{\sin m\theta_j - \lambda r_j \sin(m-1)\theta_j}{1 - 2\lambda r_j \cos(\theta_j) + \lambda^2 r_j^2} = \Im b_{j+n_{re}}, \qquad j = 1, 2, \ldots, n_{cx}.$$

$P2$ :

$$\nu_2 = \min_{\substack{\{\phi_0,\ldots,\phi_{m-1},\\ c_1,\ldots,c_q\}}} \max\{|\phi_0|, \ldots, |\phi_{m-1}|, |c_1|, \ldots, |c_q|\}$$

subject to

$$\sum_{k=0}^{m-1} \phi_k a_j^k + a_j^m \frac{\sum_{k=1}^{q} c_k a_j^{k-1}}{1 - \lambda a_j^q} = b_j, \quad j = 1, 2, \ldots, n_{re}$$

$$\sum_{k=0}^{m-1} \phi_k r_j^k \cos(k\theta_j) + \sum_{k=1}^{q} c_k r_j^{m+k-1} \frac{\cos(m+k-1)\theta_j}{1 - \lambda r_j^q} = \Re b_{j+n_{re}}, \qquad j = 1, 2, \ldots, n_{cx}$$

$$\sum_{k=1}^{m-1} \phi_k r_j^k \sin(k\theta_j) + \sum_{k=1}^{q} c_k r_j^{m+k-1} \frac{\sin(m+k-1)\theta_j}{1 - \lambda r_j^q} = \Im b_{j+n_{re}}, \qquad j = 1, 2, \ldots, n_{cx}.$$

Solving $P1$ minimizes $\|\phi\|_\infty$ for $\hat{\phi}(z)$ having signal parametrization,

$$\hat{\phi}(z) = \sum_{k=0}^{m-1} \phi_k z^k + \frac{\phi_m z^m}{1 - \lambda z}, \qquad \lambda \in [-1, 1], \tag{12}$$

giving $\|\phi\|_\infty = \nu_1$. Solving $P2$ minimizes $\|\phi\|_\infty$ for $\hat{\phi}(z)$ with parametrization,

$$\hat{\phi}(z) = \sum_{k=0}^{m-1} \phi_k z^k + \frac{\sum_{k=1}^{q} c_k z^{m+k-1}}{1 - \lambda z^q}, \qquad \lambda \in [-1, 1], \tag{13}$$

giving $\|\phi\|_\infty = \nu_2$. This leads directly to the following result.

**Theorem 3.** The $l_\infty$ norm of $\hat{\phi}(z)$ with the structure (9) is minimized by solving $P1(\lambda = 1)$. Similarly the $l_\infty$ norm of $\hat{\phi}(z)$ of (10) is minimized by solving $P1(\lambda = -1)$. The $l_\infty$ norm of $\hat{\phi}(z)$ of (11) is minimized by solving $P2(\lambda = 1)$. ∎

Taken together, Theorems 2 and 3 tell us that if A1 holds, then $\hat{\phi}_{opt}(z)$ can be calculated by solving a finite number of linear programs.

## 4 Obtaining $l_\infty$-optimal solutions

Use of Theorem 3 to find $\hat{\phi}_{opt}(z)$ for a given plant and command, requires selection of $m$ and a signal parametrization out of (9), (10), (11). We proceed by firstly, selecting a "candidate" $m$ and a structure; secondly, solving the corresponding LP from Theorem 3 to obtain a primal feasible candidate solution, which we will call $\hat{\phi}_c(z)$ and, finally, testing if this solution corresponds with a solution to the untruncated dual problem in Theorem 1. This test would seem to require the solution of an infinite LP. We propose two finite approaches, both based on the following result.

**Theorem 4.** Candidate primal feasible solution $\phi_c = \{(\phi_c)_k\}_{k=0}^{\infty}$ is an optimal solution if and only if there exists a nonzero $\phi^* \in l_1$ of the form (5) aligned with $\phi_c \in l_\infty$.

**Proof.** The dual maximization problem from Theorem 1 can be written:

$$\nu = \max_{\substack{\alpha\in\mathcal{R}^n \\ \|\phi^*\|_1\leq 1}} \phi_c \cdot \phi^* \tag{14}$$

where $\phi_c$ is any primal feasible solution. If there exists a $\phi^* \neq 0$ of the form (5) that is aligned with $\phi_c$, then $\nu = \|\phi_c\|_\infty$. If there is no such $\phi^*$, then $\nu < \|\phi_c\|_\infty$. ■

Notice that, from the primal side, the alignment conditions are:

$$\begin{aligned} \phi_k = \nu_c \quad &\Rightarrow \quad \phi_k^* \geq 0, & (15)\\ \phi_k = -\nu_c \quad &\Rightarrow \quad \phi_k^* \leq 0, & (16)\\ |\phi_k| < \nu_c \quad &\Rightarrow \quad \phi_k^* = 0. & (17)\end{aligned}$$

### 4.1 Method 1

Direct application of Theorem 4 allows us to test if $\hat{\phi}_c(z) = \hat{\phi}_{opt}(z)$ through constructing a function with required sign properties from a given finite set of stable exponential functions. For problems where $n$ is small, this construction can be carried out almost by inspection, as illustrated next.

**4.1.1 Example:** If $\hat{w}(z) = 28/57 - 20z/57$ and $\hat{P}(z) = (z+1/2)(z-9/10)$, then $\phi^*$ takes the form,

$$\phi_k^* = \alpha_1(-1/2)^k + \alpha_2(9/10)^k, \qquad k = 0,1,2\ldots.$$

Solving $P1(\lambda = 1)$ with $m = 1$ gives $\hat{\phi}_c(z) = 0.649123 - .0526316z/(1-z)$. With this solution, alignment of $\phi^*$ with $\phi_c$ would require $\phi_0^* > 0$ and $\phi_k^* = 0, \forall k = 1,2,\ldots$, which cannot be met for any $\alpha_1, \alpha_2$.

Solving $P1(\lambda = -1)$ with $m = 1$ gives $\hat{\phi}_c(z) = 1/(3(1+z))$ for which alignment is satisfied using $\alpha_1 > 0, \alpha_2 = 0$. Thus $\hat{\phi}_{opt}(z) = 1/(3(1+z))$.

### 4.2 Method 2

This is a more systematic way of constructing $\phi^*$ for verifying the optimality of a candidate $\hat{\phi}_c(z)$. It is based around solving the corresponding finite dimensional dual problem to $P1(\lambda = 1)$, $P1(\lambda = -1)$, $P2(\lambda = 1)$; we name these $D1(1), D1(-1), D2(1)$ respectively.

We give details for the case where $\phi_c$ has the form of (9). The other cases (10,11) are similar. The following is the dual [5] to $P1(\lambda = 1)$, namely $D1(1)$,

$$D1(1): \qquad \nu_c = \max_{\substack{\alpha\in\mathcal{R}^n \\ \|\phi^{*m}\|_1\leq 1}} \alpha^T b \tag{18}$$

where $\phi^{*m} = \{\phi_k^{*m}\}_{k=0}^{k=m}$ with $\phi_k^{*m} = \phi_k^*$ of (5) for $k = 0,\ldots,m-1$ and

$$\begin{aligned}\phi_m^{*m} \quad = \quad & \sum_{j=1}^{n_{re}} \alpha_j \frac{a_j^m}{1-a_j} \\ & + \sum_{j=1}^{n_{cx}} \alpha_{j+n_{re}} r_j^m \frac{\cos m\theta_j - r_j\cos(m-1)\theta_j}{1 - 2r_j\cos(\theta_j) + r_j^2} \\ & + \sum_{j=1}^{n_{cx}} \alpha_{j+n_{re}+n_{cx}} r_j^m \frac{\sin m\theta_j - r_j\sin(m-1)\theta_j}{1 - 2r_j\cos(\theta_j) + r_j^2}.\end{aligned}$$

This dual maximization can be solved to find the value of $\alpha$, which is then used to reconstruct $\phi^*$ using (5). A condition (exact if the dual solution is unique; sufficient if is not unique) for $\hat{\phi}_c(z) = \hat{\phi}_{opt}(z)$ is that $\phi^* \in l_1$ is aligned with $\phi_c \in l_\infty$. If the signs do not match, then a summation (infinite) may be carried out to calculate $\|\phi^*\|_1$ in order to bound the gap: if $\|\phi^*\|_1 = 1 + \delta > 1$ then

$$\frac{\nu_c}{1+\delta} \leq \nu \leq \nu_c. \tag{19}$$

It is clearly desirable that the finite dual out of $D1(1), D1(-1), D2(1)$ being tested has a unique solution, since this allows determination that $\hat{\phi}_c(z) = \hat{\phi}_{opt}(z)$.

**4.2.1 Unique solution to finite dual:** The following condition is sufficient for $D1(1), D1(-1)$ to have a unique solution. Corresponding conditions for $D2(1)$ are similar.

**C1**: Candidate solution to $P1(\lambda = \pm 1)$ has at least $n-1$ out of $\{\phi_0, \ldots, \phi_m\}$ satisfying $|\phi_k| < \nu_c$ and of the corresponding $\phi_k^{*m}$, exactly $n-1$ are linearly independent of each other and of $b$. ■

If C1 holds, an LP is not needed to calculate $\alpha$: taking $n-1$ independent equations $\phi_k^* = 0$ together with $\alpha^T b = \nu_c$ gives $n$ linear equations which can be solved for $\alpha$.

**4.2.2 Nonunique solution to finite dual:** The main drawback of Method 2 is that in some cases where the solution to $D1(1)$, $D1(-1)$, or $D2(1)$ is nonunique, verification that $\hat{\phi}_c(z) = \hat{\phi}_{opt}(z)$ cannot be achieved by using a LP. This is because the solution $\alpha$ obtained, while optimizing the finite dual, causes the untruncated dual in (4) to be infeasible.

Cases of nonunique solution to the finite dimensional dual problems can be exactly identified, but it is not clear how to proceed other than by re-solving the primal and finite dimensional dual for larger values of $m$ to reduce the indicated interval (19) bounding $\nu$. This allows the value of $\nu$ to be found arbitrarily closely, but our aim was to find it exactly. Nonetheless, even in such cases, if $\hat{\phi}_c(z) = \hat{\phi}_{opt}(z)$, the method offers advantages in two respects over existing approaches to find progressively tighter upper and lower bounds using FIR $\hat{\phi}(z)$'s. Firstly, the order of $\hat{\phi}_c(z)$ will not increase as the value of $m$ is increased, and secondly, $\|\phi_c\|_\infty = \nu$.

The situation of nonunique solution to the dual occurs precisely when the dual cost hyperplane $\nu_c = \alpha^T b$ lies parallel to a face of the dual feasible set $\|\phi^*\|_1 = 1$. Given a candidate solution $\phi_c$, the corresponding signs of the $\phi_k^*$ are found from (15, 16, 17). The constraint $\|\phi^*\|_1 = 1$ can then be written in terms of $\alpha$ in the form $\alpha^T \gamma = 1$ where $\gamma \in \mathcal{R}^n$. Then if $\gamma = sb$, where $s$ is any nonzero scalar, the dual has a nonunique solution.

Nonuniqueness can occur in problems where one or more of the primal interpolation constraints is redundant in the sense that its removal does not affect the solution. However it is not clear that removal of such redundant constraints will be effective in all cases. Examples without redundant constraints can be found where $\alpha$ is not unique.

## 5 Properties of $l_\infty$-optimal solutions

**Solution order**: The order of transfer function $\hat{\phi}_{opt}(z)$ is unbounded with respect to $n$, since both $m$ and $q$ can be arbitrarily large. A case with $n = 2$ and unbounded $m$ can occur for the plant $\hat{P}(z) = (z - \beta)/(z - \gamma)$ with $1 > \gamma > \beta > 0$, tracking $\hat{w}(z) = 1/(1 - z)$ in a one-parameter configuration. Results from [3] give

$$m = 1 + \left\lfloor \frac{\log\left(\frac{1}{2}\right)}{\log \gamma} \right\rfloor \xrightarrow{\gamma \uparrow 1} \infty,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Solutions with unbounded $m$ can also arise with small magnitude $a_j$.

**Uniqueness**: The solutions are not always unique as demonstrated in Example 7.3.

**Effect of** $w$: In a two-parameter system, $w$ is a primal feasible solution to (4) and can be regarded as $\phi_c$ in Theorem 4. This gives the following result.

**Corollary 5**: In a two-parameter system, $\nu \leq \|w\|_\infty$. Furthermore $\nu = \|w\|_\infty$ and $\phi_{opt} = w$ if only if there is a nonzero $\phi^* \in l_1$ aligned with $w$.

For instance if, in a two-parameter system, $\hat{w}(z) = 1/(1-z)$ and $\hat{P}(z)$ has a zero at $z \in (0, 1)$, then $\phi_{opt} = w$. This is the case in the Example in Section 7.4. On the other hand, if $\hat{w}(z)$ is FIR, and $\hat{P}(z)$ has no delay and at least one nonminimum phase zero, in a two-parameter configuration, then $\nu < \|w\|_\infty$. The Example in Section 4.1.1 illustrates such a case.

## 6 Obtaining $l_\infty$-suboptimal solutions in $l_1$

Practical considerations of zero steady state error and closed-loop stability motivate the use of solutions in $l_1$, which are necessarily suboptimal with respect to $l_\infty$-norm minimization.

Given a candidate primal feasible solution $\hat{\phi}_c(z)$ of the form of one of (9,10,11), with $\|\phi_c\|_\infty = \nu_c$, a solution in $l_1$ having $l_\infty$ norm arbitrarily close to $\nu_c$ is found by solving $P1(\lambda = 1 - \epsilon)$, $P1(\lambda = -1 + \epsilon)$, or $P2(\lambda = 1 - \epsilon)$ respectively using sufficiently small $\epsilon > 0$.

By arguments based on LP sensitivity, $\lim_{\epsilon \downarrow 0} \|\phi\|_\infty = \nu_c$.

## 7 Examples

### 7.1 Optimal solution with a positive pole

We consider the example from [6] where the plant

$$\hat{P}(z) = \frac{z(z - 0.5)}{(z - 0.1)(1 - 0.5z)} \qquad (20)$$

is to track a unit step, $\hat{w}(z) = 1/(1 - z)$ in a one-parameter configuration. Here,

$$\begin{aligned} \nu \quad &= \quad \min_{\phi \in l_\infty} \|\phi\|_\infty \\ \text{subject to} \quad & \quad \hat{\phi}(0) = 1, \hat{\phi}(0.5) = 2, \hat{\phi}(0.1) = 0. \end{aligned}$$

Since all $a_j \geq 0$, we know a-priori that $a_s > 0$ and consequently from Theorems 2 and 3, that $\hat{\phi}_{opt}(z)$ is found by solving $P1(\lambda = 1)$. The value of $m$ is not known a-priori. We find

$$\hat{\phi}_{opt}(z) = 1 - 11.5z + \frac{13.5z^2}{1 - z}, \qquad (21)$$

with $\nu = 13.5$. This value is noted in [6]. Solving $P1(\lambda = 0.999)$ with $m = 2$ gives,

$$\hat{\phi}(z) = 1 - 11.5015z + \frac{13.515z^2}{1 - 0.999z},$$

with $\|\phi\|_\infty = 13.515$. An FIR solution needs to be of order 11 to achieve this value of $\|\phi\|_\infty$.

### 7.2 Optimal solution with a negative pole

Here plant $\hat{P}(z) = z(z+.7)(z-.5)/(z-.6)$ is to track $\hat{w}(z) = 1/(1-z)$ in a one-parameter configuration. Solving $P1(\lambda = -1)$ gives:

$$\begin{aligned}\hat{\phi}_{opt}(z) &= 1 + 5.52196z + 2.84656z^2 \\ &\quad -9.89088\left(\sum_{k=3}^{16} z^k + \frac{z^{17}}{1+z}\right),\end{aligned}$$

with $\nu = 9.89088$. Notice that a third order suboptimal solution,

$$\hat{\phi}(z) = 1 + 5.50557z + 2.88428z^2 - \frac{9.89574z^3}{1-z},$$

is nearly as good with $\|\phi\|_\infty = 9.89574$ and with probably a more desirable closed-loop pole.

### 7.3 Optimal solution with complex poles

This example has $n_{cx} = 1$ with $a_1 = 1/2 + i/2, b_1 = 1 + i/2$. Here $a_s$ is complex and $q = 8$. Solving $P2(\lambda = 1)$ gives a solution; in fact the solution is nonunique:

$$\begin{aligned}\hat{\phi}_{opt}(z) &= \frac{9/20(1+z+z^2) + \gamma z^3}{1-z^8} \\ &\quad + \frac{9/20(-z^4 - z^5 - z^6) + z^7(4\gamma + 3/2)}{1-z^8}\end{aligned}$$

with $\nu = 9/20$ and $\gamma \in [-9/20, -21/80]$. Although two dual variables take value zero, they are linearly dependent, so C1 holds, and the solution to $D2(1)$ is unique.

### 7.4 Problem where dual solution is nonunique

Here $\hat{P}(z) = (z+.9)(z-0.5)$ and $\hat{w}(z) = 1/(1-z)$. Solving $P1(\lambda = 1)$ with any $m > 0$ gives $\hat{\phi}_c(z) = 1/(1-z)$ with $\phi_c = \{1, 1, \ldots\}$ and $\nu_c = 1$. For this plant and command, $\phi^*$ takes the form,

$$\phi_k^* = \alpha_1(-.9)^k + \alpha_2(.5)^k, \qquad k = 0, 1, 2 \ldots.$$

Vectors $\phi_c$ and $\phi^*$ are aligned when $\alpha_1 = 0, \alpha_2 > 0$. By Theorem 4, $\hat{\phi}_{opt}(z) = 1/(1-z)$. Method 2, however, fails to identify that $\hat{\phi}_c(z) = \hat{\phi}_{opt}(z)$, since LP solutions to $D1(1)$ do not set $\alpha_1 = 0$. Removing the redundant constraint, $\hat{\phi}(-0.9) = \hat{w}(-0.9)$, removes the difficulty.

## 8 Conclusion

We have presented results on minimizing the $l_\infty$ norm of signals in feedback control systems. These solutions are useful for characterizing achievable performance, and in some cases they provide a basis for designing low order systems. They can be obtained by solving a sequence of finite linear programs, although we do not give a-priori bounds for the size of these programs. The main limitation of the work is that, in some cases, there are difficulties in verifying that one has found the optimal solution. Nonetheless the approach can give designs with advantages over FIR solutions and the work contributes towards answering the question posed in [4] of how to select closed-loop poles to give a good $l_\infty$ response.

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