

# Gap Metrics, Representations, and Nonlinear Robust Stability\*

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## Abstract

Various alternative definitions for the nonlinear  $L_2$ - and  $\nu$ -gap metrics are studied. The concept of  $\beta$ -conjugacy and multiplicative homogeneity are introduced to relate the metrics to each other and to compare the stability margins of nonlinear feedback loops expressed in terms of the norms of complementary parallel projections. Left and right representations for the graph of a nonlinear system are studied. A new definition of “normalized” is introduced for left representations. Formulas for the gap metrics as the norm of the product of left and right representations are derived.

**Keywords.** Robust control, nonlinear systems, gap metric, graph representations.

## 1 Introduction

This paper is concerned with the approach to robust stability of nonlinear systems using gap metrics following the work of [1], [4], [13]. The paper considers several related issues surrounding the following basic robustness theorem: feedback stability is preserved if gap perturbations do not exceed the inverse of the norm of a nonlinear parallel projection operator associated with the feedback loop. A circle of ideas concerned with the definition and computation

of gap metrics, graph representations, and controller synthesis to achieve norm bounds on the parallel projection operators, will be explored.

The paper compares and relates several contrasting definitions of the gap and  $\nu$ -gap which generalize definitions of the  $L_2$ - and  $\nu$ -gap from the linear case (Section 5). Various versions of the main robustness theorem will be given involving the two complementary parallel projections of a feedback loop. In particular, the “ $\delta$ -type” (resp. “ $\rho$ -type”) gap metrics are needed for results involving the parallel projection onto the plant (resp. controller) graph. The  $\delta$ -type and  $\rho$ -type gaps are shown to be equal subject to a certain conjugacy transformation on one of the systems (Lemma 5.1). Under similar conditions, the norms of the two complementary parallel projections are shown to be equal (Theorem 3.1).

The connection between gap metrics and representations of the graph are investigated. Operators whose image (resp. kernel) generates the graph are termed right (resp. left) representations of the system. These operators, called graph symbols, generalize the usual notions of right and left coprime factorizations. As usual, we call a right representation normalized if the symbol is inner. We introduce a new definition of “normalized” for left representations which requires that the “amplification” or “gain” of any  $L_2$ -function by the symbol is equal to the minimal distance to the graph (Section 4.2).

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This allows formulas to be derived (Theorem 5.4), involving norms of products of the left and right symbols, for the various  $\rho$ -type gap metrics. This also allows versions of the robustness theorem to be obtained involving  $\infty$ -norm errors between the graph symbols to account for system uncertainty (Theorem 6.2). For full details, proofs and other related results the reader is referred to the full paper [7].

## 2 Preliminaries

In the definitions to follow  $n$  denotes the generic dimension of the range of the signals. Write  $L_2 = L_2(-\infty, \infty) = \{w : (-\infty, \infty) \rightarrow \mathbf{R}^n \mid \|w\|_2 < \infty\}$  for the Lebesgue space of signals on the doubly infinite time axis, where  $\|w\|_2 = \|w\|_{L_2(-\infty, \infty)} \triangleq \left(\int_{-\infty}^{\infty} |w(s)|^2 ds\right)^{1/2}$ . The signals in the following space are zero before a finite time and square integrable on each finite interval:

$$L_{2,ce}(-\infty, \infty) = \left\{ w : (-\infty, \infty) \rightarrow \mathbf{R}^n \mid \begin{array}{l} \mathbf{T}_T w = 0 \text{ some } T \in (-\infty, \infty) \\ \text{and } \|\mathbf{T}_T w\|_2 < \infty \forall T \geq 0 \end{array} \right\},$$

where

$$(\mathbf{T}_T w)(s) = \begin{cases} w(s) & s \leq T \\ 0 & s > T. \end{cases}$$

The space of signals defined for positive times and square integrable on each finite interval is  $L_{2,e} = L_{2,e}[0, \infty) = \{w : [0, \infty) \rightarrow \mathbf{R}^n \mid \|\mathbf{T}_T w\|_2 < \infty \forall T \geq 0\}$ . We can regard  $L_{2,e}$  as a subset of  $L_{2,ce}$  by defining elements of  $L_{2,e}$  to be 0 before time 0. For  $w \in L_{2,ce}$  we write  $\|w\|_T = \|\mathbf{T}_T w\|_2$ .

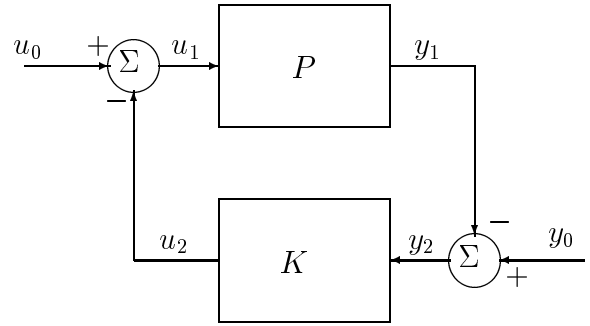
The signal spaces  $\mathcal{U}$ ,  $\mathcal{Y}$ ,  $\mathcal{W}$ , etc, will be  $L_{2,ce}$  spaces of suitable range dimension. The plant  $P$  and controller  $K$  will be operators defined on these spaces. We will also consider restrictions to  $L_{2,e}$  and  $L_2$ .

We write

$$\begin{aligned} \|P\|_\infty &= \limsup_{T \rightarrow \infty} \sup_{u \in L_{2,ce}(-\infty, \infty), \|u\|_T \neq 0} \frac{\|Pu\|_T}{\|u\|_T} \\ &= \sup_{u \in L_2[0, \infty), u \neq 0} \frac{\|Pu\|_2}{\|u\|_2} \end{aligned}$$

for the induced norm for a causal, time invariant operator  $P : L_{2,ce} \rightarrow L_{2,ce}$ ; this is often called the  $H_\infty$  norm of  $P$  even if  $P$  is nonlinear.

The feedback configuration  $[P, K]$  shown in Figure 2.1 consists of a *plant*  $P : \mathcal{U} \rightarrow \mathcal{Y}$  and a *controller*  $K : \mathcal{Y} \rightarrow \mathcal{U}$ , both causal, time-invariant maps defined on  $L_{2,ce}$  signal spaces  $\mathcal{U}$ ,  $\mathcal{Y}$  and which satisfy  $P0 = 0$  and  $K0 = 0$ .



**Figure 2.1:** Feedback configuration.

In Figure 2.1,  $u_i \in \mathcal{U}$  and  $y_i \in \mathcal{Y}$ , ( $i = 0, 1, 2$ ) and we write  $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$ . The system  $[P, K]$  is assumed to well-posed. Namely, for any  $w = (u_0, y_0) \in \mathcal{W}$  there exist unique  $u_1, u_2 \in \mathcal{U}$ ,  $y_1, y_2 \in \mathcal{Y}$  such that the following closed loop equations hold:  $u_0 = u_1 + u_2$ ,  $y_0 = y_1 + y_2$ ,  $y_1 = Pu_1$ ,  $u_2 = Ky_2$ , and furthermore the map

$$\begin{aligned} H_{P,K} : \mathcal{W} &\rightarrow \mathcal{W} \times \mathcal{W}, \\ &: \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \left( \begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right) \end{aligned} \quad (2.1)$$

is causal. Well-posedness will always be assumed for both nominal and perturbed systems.

It is convenient to consider *graphs* of operators. The graph of the plant  $P$  is

$$\mathcal{G}_P = \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} : u \in \mathcal{U}, Pu \in \mathcal{Y} \right\} \subset \mathcal{W},$$

and the graph of the controller  $K$  is

$$\mathcal{G}_K = \left\{ \begin{pmatrix} Ky \\ y \end{pmatrix} : y \in \mathcal{Y}, Ky \in \mathcal{U} \right\} \subset \mathcal{W}.$$

We often write  $\mathcal{M} = \mathcal{G}_P$ ,  $\mathcal{N} = \mathcal{G}_K$ . Of central importance to robustness of the feedback system  $[P, K]$  are the *parallel projection* operators

$$\Pi_{\mathcal{M}||\mathcal{N}} = \Pi_1 H_{P,K}, \quad \Pi_{\mathcal{N}||\mathcal{M}} = \Pi_2 H_{P,K},$$

[4]. Here,  $\Pi_i : \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{W}$  denote the natural projections ( $i = 1, 2$ ). The operators  $\Pi_{\mathcal{M}||\mathcal{N}}$ ,  $\Pi_{\mathcal{N}||\mathcal{M}}$  both enjoy the defining property:  $\Pi(\Pi w_1 + (I - \Pi)w_2) = \Pi w_1$  ( $I$  denotes the identity operator) for any  $w_1, w_2 \in \mathcal{W}$ . Also,  $H_{P,K} = (\Pi_{\mathcal{M}||\mathcal{N}}, \Pi_{\mathcal{N}||\mathcal{M}})$ , and  $\Pi_{\mathcal{M}||\mathcal{N}} + \Pi_{\mathcal{N}||\mathcal{M}} = I$ . Consequently, stability of the feedback system  $[P, K]$ , i.e. the finiteness of  $\|H_{P,K}\|_\infty$ , is equivalent to the stability of either parallel projection.

We note that the parallel projections can be viewed in terms of generalized plants

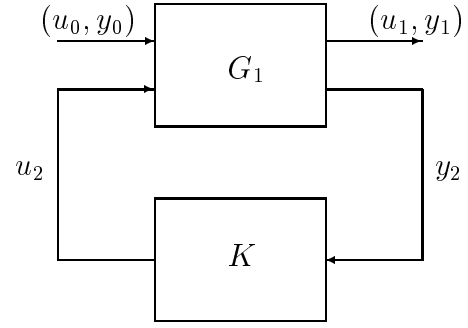
$$G_1 : \begin{pmatrix} u_0 \\ y_0 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ y_1 \\ y_2 \end{pmatrix} \quad \text{and} \quad G_2 : \begin{pmatrix} u_0 \\ y_0 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ y_2 \\ y_2 \end{pmatrix}$$

with feedback  $K$  so that  $\|\Pi_{\mathcal{M}||\mathcal{N}}\|_\infty = \|(G_1, K)\|_\infty$ , where  $(G_1, K)$  is the operator mapping  $(u_0, y_0)$  to  $(u_1, y_1)$  in Figure 2.2. In a similar manner,  $\|\Pi_{\mathcal{N}||\mathcal{M}}\|_\infty = \|(G_2, K)\|_\infty$ . Robust stabilization is then equivalent to either of these two standard  $H^\infty$  control problems: find controller(s) to minimize either of these quantities. These synthesis problems are solved in [7] using the methods in [6].

### 3 Conjugate Norm Equivalence

For linear systems  $\|\Pi_{\mathcal{M}||\mathcal{N}}\|_\infty = \|\Pi_{\mathcal{N}||\mathcal{M}}\|_\infty$ , however in the general nonlinear case it is known that  $\|\Pi_{\mathcal{M}||\mathcal{N}}\|_\infty \neq \|\Pi_{\mathcal{N}||\mathcal{M}}\|_\infty$  (see [2] for an example where both  $\mathcal{M}$  and  $\mathcal{N}$  are piecewise linear static functions).

For  $\gamma > 1$  let  $\beta = \sqrt{1 - \gamma^{-2}} < 1$ . We say that controllers  $K_1, K_2$  are  *$\beta$ -conjugate* if  $K_1 =$



**Figure 2.2:** The parallel projection  $\Pi_{\mathcal{M}||\mathcal{N}}$  and generalized plant  $G_1$ .

$\beta^2 K_2 \beta^{-2}$ . An operator  $P$  is (*positively, multiplicatively*) *homogeneous* if  $\alpha P = P\alpha$  for all real  $\alpha > 0$ . Linear systems, of course, enjoy these properties.

**Theorem 3.1** *Any plant  $P$  enjoys conjugate norm equivalence:*

$$\|\Pi_{\mathcal{M}||\mathcal{N}_1}\|_\infty < \gamma \quad \text{iff} \quad \|\Pi_{\mathcal{N}_2||\mathcal{M}}\|_\infty < \gamma \quad (3.1)$$

whenever  $K_1, K_2$  are  $\beta$ -conjugate, where  $\mathcal{M} = \mathcal{G}_P$ ,  $\mathcal{N}_i = \mathcal{G}_{K_i}$ , ( $i = 1, 2$ ). Furthermore,

$$\inf_{K_1} \|\Pi_{\mathcal{M}||\mathcal{N}_1}\|_\infty = \inf_{K_2} \|\Pi_{\mathcal{N}_2||\mathcal{M}}\|_\infty, \quad (3.2)$$

and if  $P$  and/or  $K$  is homogeneous, then

$$\|\Pi_{\mathcal{M}||\mathcal{N}}\|_\infty = \|\Pi_{\mathcal{N}||\mathcal{M}}\|_\infty. \quad (3.3)$$

## 4 Plant Representations

In the literature, there are a number of generalizations of right and left coprime factorizations for an operator  $P$ , see, e.g. [9], [10]. For our purposes it is convenient to adopt definitions which capture the essential energy balances, for both finite and infinite time horizons, and of course the graph  $\mathcal{G}_P$  of the operator  $P : u \mapsto y$ . The notion of “coprimeness” of a right (resp. left) factorization is usually translated into the context of graph representations in the form of a left (resp. right) invertibility condition on the graph symbol. Since we will not need such a notion explicitly in what follows, we will not formalize this concept here.

## 4.1 Right Representations

Roughly speaking, we wish to generalize the idea that  $P = NM^{-1}$  has a normalized coprime factorization, with  $R = \begin{bmatrix} M \\ N \end{bmatrix}$  defining a range representation of the graph of  $P$ :  $\{w \in \mathcal{W} \mid w = R\psi, \text{ some } \psi \in \mathcal{U}\} = \mathcal{G}_P$ , and satisfying an inner property.

A causal operator

$$\mathbf{R}_{eP} : \mathcal{U} \rightarrow \mathcal{W}$$

is called a *right representation* of  $P$  provided it maps onto the graph:

$$\begin{aligned} \text{range } \mathbf{R}_{eP} &= \mathcal{G}_P \\ \text{range } \mathbf{R}_{eP}|_{L_2} &= \mathcal{G}_P \cap L_2 \\ \text{range } \mathbf{R}_{eP}|_{L_2[0,\infty)} &= \mathcal{G}_P \cap L_2[0,\infty) \end{aligned} \quad (4.1)$$

(i)  $\mathbf{R}_{eP}$  is *contractive* if:

$$\|\mathbf{R}_{eP}\psi\|_T \leq \|\psi\|_T \quad \forall T \in (-\infty, \infty), \psi \in \mathcal{U}, \quad (4.2)$$

(ii)  $\mathbf{R}_{eP}$  is *inner* (or *normalized*) if:

$$\|\mathbf{R}_{eP}\psi\|_2 = \|\psi\|_2 \quad \forall \psi \in \mathcal{U} \cap L_2. \quad (4.3)$$

## 4.2 Left Representations

We next generalize the idea that  $P = \tilde{M}^{-1}\tilde{N}$  has a normalized left coprime factorization, with  $L = [-\tilde{N}, \tilde{M}]$  defining a kernel representation of  $P$ :  $\{w \in \mathcal{W} \mid Lw = 0\} = \mathcal{G}_P$ , and satisfying a coininner property.

A causal operator

$$\mathbf{L}_{eP} : \mathcal{W} \rightarrow \mathcal{Y}$$

is called a *left representation* of  $P$  provided its kernel is the graph:

$$\ker \mathbf{L}_{eP} = \mathcal{G}_P. \quad (4.4)$$

(i)  $\mathbf{L}_{eP}$  is *contractive* if:

$$\|\mathbf{L}_{eP}w\|_T \leq \|w\|_T \quad \forall T \in (-\infty, \infty), w \in \mathcal{W}. \quad (4.5)$$

(ii)  $\mathbf{L}_{eP}$  is *coininner* if:

$$\|\mathbf{L}_{eP}w\|_T = \inf \{\|w - \tilde{w}\|_T \mid \tilde{w} \in \mathcal{G}_P\} \quad (4.6)$$

for all  $T \in (-\infty, \infty)$  and all  $w \in \mathcal{W}$ ,  $T \in (-\infty, \infty)$ , and

$$\|\mathbf{L}_{eP}w\|_2 = \inf \{\|w - \tilde{w}\|_2 \mid \tilde{w} \in \mathcal{G}_P \cap L_2\} \quad (4.7)$$

for all  $w \in L_2$ .

Representations satisfying the above properties can be constructed for input-affine systems subject to appropriate regularity assumptions; this is done in [7].

## 5 Gap Metrics

Gap metrics are used to provide a measure of the stability margin for a closed loop system. If  $[P_0, K]$  is stable, then one wants  $[P_1, K]$  stable for  $P_1$  “sufficiently close” to  $P_0$ . Here, closeness is relative to closed-loop behaviour; a basic idea of the gap metric is to provide an open-loop measure for this type of closeness [15].

### 5.1 Definitions and Basic Properties

There are a number of definitions for gap metrics in the literature. We now give the definitions of some of these together with some new ones.

We write  $\mathcal{G}_0 = \mathcal{G}_{P_0}$ ,  $\mathcal{G}_1 = \mathcal{G}_{P_1}$ , for two plants  $P_0$  and  $P_1$ . Define the  $\delta$ -type “gap metrics” following [4], and [13]

$$\begin{aligned} \vec{\delta}_g(P_0, P_1) &= \limsup_{T \rightarrow \infty} \sup_{x_1 \in \mathcal{G}_1 \cap L_2, ce(-\infty, \infty), x_1 \neq 0} \\ &\quad \inf_{x_0 \in \mathcal{G}_0 \cap L_2, ce(-\infty, \infty), x_0 \neq 0} \frac{\|x_1 - x_0\|_T}{\|x_0\|_T} \\ \vec{\delta}_{L_2}(P_0, P_1) &= \sup_{x_1 \in \mathcal{G}_1 \cap L_2(-\infty, \infty), x_1 \neq 0} \\ &\quad \inf_{x_0 \in \mathcal{G}_0 \cap L_2(-\infty, \infty), x_0 \neq 0} \frac{\|x_1 - x_0\|_2}{\|x_0\|_2}, \end{aligned} \quad (5.1)$$

and also define the  $\rho$ -gaps

$$\begin{aligned} \vec{\rho}_g(P_0, P_1) &= \limsup_{T \rightarrow \infty} \sup_{x_1 \in \mathcal{G}_1 \cap L_2, ce(-\infty, \infty), x_1 \neq 0} \\ &\quad \inf_{x_0 \in \mathcal{G}_0 \cap L_2, ce(-\infty, \infty)} \frac{\|x_1 - x_0\|_T}{\|x_1\|_T} \\ \vec{\rho}_{L_2}(P_0, P_1) &= \sup_{x_1 \in \mathcal{G}_1 \cap L_2(-\infty, \infty), x_1 \neq 0} \\ &\quad \inf_{x_0 \in \mathcal{G}_0 \cap L_2(-\infty, \infty)} \frac{\|x_1 - x_0\|_2}{\|x_1\|_2}. \end{aligned} \quad (5.2)$$

where  $\mathcal{M}_1 = \mathcal{G}_{P_1}$ .

Similar quantities  $\vec{\delta}_0, \vec{\delta}_{H_2}, \vec{\rho}_0, \vec{\rho}_{H_2}$ , on the semi-infinite axis  $[0, \infty)$  are defined in [7].

For linear systems, each respective pair of  $\delta$  and  $\rho$  gap metrics coincide, [4, proof of Proposition 5]. However, for nonlinear systems they need not be the same, though they are related as follows.

**Lemma 5.1** *Let  $\gamma > 1$ , and set  $\beta = \sqrt{1 - \gamma^{-2}}$ . For each of the two cases  $\vec{\delta} \in \{\vec{\delta}_g, \vec{\delta}_{L_2}\}$ , with respective  $\vec{\rho} \in \{\vec{\rho}_g, \vec{\rho}_{L_2}\}$ , we have  $\vec{\delta}(P_0, P_1) < \gamma^{-1}$  iff  $\vec{\rho}(P_0, \beta^{-2}P_1\beta^2) < \gamma^{-1}$  iff  $\vec{\rho}(\beta^2P_0\beta^{-2}, P_1) < \gamma^{-1}$ . Similarly,  $\vec{\rho}(P_0, P_1) < \gamma^{-1}$  iff  $\vec{\delta}(P_0, \beta^2P_1\beta^{-2}) < \gamma^{-1}$  iff  $\vec{\delta}(\beta^{-2}P_0\beta^2, P_1) < \gamma^{-1}$ . Furthermore, if  $P_0$  and/or  $P_1$  is homogeneous, then for each of the four cases  $\vec{\delta} \in \{\vec{\delta}_0, \vec{\delta}_g, \vec{\delta}_{L_2}, \vec{\delta}_{H_2}\}$ , with respective  $\vec{\rho} \in \{\vec{\rho}_0, \vec{\rho}_g, \vec{\rho}_{L_2}, \vec{\rho}_{H_2}\}$ ,  $\min\{\vec{\delta}(P_0, P_1), \vec{\rho}(P_0, P_1)\} < 1$  implies  $\vec{\delta}(P_0, P_1) = \vec{\rho}(P_0, P_1)$ .*

## 5.2 Robust Stability

The  $\vec{\delta}, \vec{\delta}_0, \vec{\delta}_g, \vec{\delta}_{L_2}$  and  $\nu$ -gap metrics introduced in [4], [13], [12] provide a maximum stability margin expressed as the inverse of the induced norm of the parallel projection operator  $\Pi_{\mathcal{M}_0\|\mathcal{N}}$ ; a result of this type appears in [13, Proposition 2.2] using  $\vec{\delta}_g$ , which is based in that of [4, Theorem 3] using  $\vec{\delta}_0$ .

The  $\rho$  type gap metrics introduced above also lead to a natural stability margin, but this time it is expressed in terms of inverse of the induced norm of the parallel projection operator  $\Pi_{\mathcal{N}\|\mathcal{M}_0}$ , revealing an interesting duality.

**Theorem 5.2** *Assume  $H_{P_0, K}$  is stable. If*

$$\vec{\rho}_g(P_0, P_1) < \|\Pi_{\mathcal{N}\|\mathcal{M}_0}\|_{\infty}^{-1}, \quad (5.3)$$

where  $\mathcal{M}_0 = \mathcal{G}_{P_0}$  and  $\mathcal{N} = \mathcal{G}_K$ , then  $H_{P_1, K}$  is stable and

$$\|\Pi_{\mathcal{N}\|\mathcal{M}_1}\|_{\infty} \leq \|\Pi_{\mathcal{N}\|\mathcal{M}_0}\|_{\infty} \frac{1 + \vec{\rho}_g(P_0, P_1)}{1 - \|\Pi_{\mathcal{N}\|\mathcal{M}_0}\|_{\infty} \vec{\rho}_g(P_0, P_1)}, \quad (5.4)$$

The  $L_2$  gaps are not by themselves sufficient to prove stability. However, if stability is known for a plant  $P_1$  near a nominal  $P_0$ , then the norms of the parallel projections can be estimated. This was done in [13, Theorem 2.1] for the  $\vec{\delta}_{L_2}$  gap and the  $\Pi_{\mathcal{M}\|\mathcal{N}}$  projection. The corresponding result for the  $\vec{\rho}_{L_2}$  gap and the  $\Pi_{\mathcal{N}\|\mathcal{M}}$  projection is as follows.

**Theorem 5.3** *Assume  $H_{P_0, K}$  is stable. If  $H_{P_1, K}$  is stable and*

$$\vec{\rho}_{L_2}(P_0, P_1) < \|\Pi_{\mathcal{N}\|\mathcal{M}_0}\|_{\infty}^{-1}, \quad (5.5)$$

where  $\mathcal{M}_0 = \mathcal{G}_{P_0}$  and  $\mathcal{N} = \mathcal{G}_K$ , then

$$\|\Pi_{\mathcal{N}\|\mathcal{M}_1}\|_{\infty} \leq \|\Pi_{\mathcal{N}\|\mathcal{M}_0}\|_{\infty} \frac{1 + \vec{\rho}_{L_2}(P_0, P_1)}{1 - \|\Pi_{\mathcal{N}\|\mathcal{M}_0}\|_{\infty} \vec{\rho}_{L_2}(P_0, P_1)}, \quad (5.6)$$

where  $\mathcal{M}_1 = \mathcal{G}_{P_1}$ .

## 5.3 Evaluation

The following theorem shows how the  $\rho$ -gaps can be evaluated in terms of the right and left symbols from the plant representations (only results for  $\vec{\rho}_{L_2}$  are given here).

**Theorem 5.4** *Assume  $L_{e_0}$  is a coinner left representation for  $P_0$  and  $R_{e_1}$  is an inner right representation for  $P_1$ . Then*

$$\vec{\rho}_{L_2}(P_0, P_1) = \|L_{e_0}R_{e_1}\|_{\infty}. \quad (5.7)$$

## 6 Robust Stability and Representation Uncertainty

In [3], [8], [11] relationships between coprime factor uncertainty, robust stabilization, and gap uncertainty were established for linear systems. Some partial results for nonlinear systems have been obtained [1], [14]. In this section we give two dual results relating representation uncertainty to certain gap balls.

**Lemma 6.1** *Let  $L_{e0}$  be a coinner left representation of  $P_0$ , and let  $L_{e1}$  be a left representation of  $P_1$  (not necessarily coinner). Then*

$$\vec{\rho}_{L_2}(P_0, P_1) \leq \|L_{e0} - L_{e1}\|_\infty. \quad (6.1)$$

**Theorem 6.2** *Assume  $H_{P_0, K}$  is stable. Let  $L_{e0}$  be a normalized left representation of  $P_0$ , and let  $L_{e1}$  be a left representation of  $P_1$  (not necessarily normalized). If*

$$\|L_{e0} - L_{e1}\|_\infty < \|\Pi_{\mathcal{N}|\mathcal{M}_0}\|_\infty^{-1}, \quad (6.2)$$

where  $\mathcal{M}_0 = \mathcal{G}_{P_0}$  and  $\mathcal{N} = \mathcal{G}_K$ , then  $H_{P_1, K}$  is stable,

$$\vec{\rho}_{L_2}(P_0, P_1) < \|\Pi_{\mathcal{N}|\mathcal{M}_0}\|_\infty^{-1}, \quad (6.3)$$

and by Theorem 5.3 the norm bound (5.6) holds, where  $\mathcal{M}_1 = \mathcal{G}_{P_1}$ .

Analogous results for right representation uncertainty and  $\vec{\delta}_{L_2}$  also hold [7].

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