An LMI Approach to Optimal Simultaneous System Design of Structure and Controller under $_{\infty}$ Constraint

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Abstract

This paper proposes an linear matrix inequality (LMI) approach to simultaneous optimization design of structure and dynamic output feedback controller under H_∞ constraint. First, a controlled object (plant) and the closed-loop system are described by descriptor form. Secondly, H_∞ constraint represented as a Riccati inequality is decoupled into two low-order Riccati inequalities solved consecutively. Based on the solutions of two inequalities, we can derive a controller and structural parameters for H_∞ performance. The problem can be reduced to convex optimization problem (COP) subject to inequality constraints.

1 Introduction

In conventional system design, the controller is synthesized under a certain specification after modeling of the structure. However, the interaction between structure and controller tends to be closer as system becomes complicated. Repetition synthesis of structure and controller by trial and error is indispensable for the purpose of pursuing optimal system performance. Hence, it is important to synthesize controller and structure of the system simultaneously. This is called inseparability and compatibility[1][2]. And there is possibility that simultaneous optimal design gets better performance than that of the individual design of structure and controller. From this point of view, simultaneous design problem of structural parameters and controller has been paid much attention for the last decade. In fact, the simultaneous design problems based on optimal regulator, pole assignment, H_∞ control have been extensively studied[3][4][5]. Also, matrix inequality approach of great interest in control design is applied to simultaneous design problems[6][7][8]. However, they reduce the problems to bilinear matrix inequalities (BMIs). In general, the existence of solution to BMI problem is not guaranteed and further it is difficult to solve. This fact is well known as the NP-hardness[9]. Even if there exists solution, its solution is not global but local. In order to avoid this difficulty, the authors propose a new approach based on linear matrix inequality (LMI). In LMI problems, the global optimal solution can be obtained because of convex optimization problem, and plural specifications can be dealt with at the same time. When a control problem can be described in terms of LMI, the problem can be considered to be "solved"[10]. In this paper, we present LMI conditions for the simultaneous design problem of structure and controller minimizing the objective function which consists of summention of an upper bound of H_∞-norm of transfer function and a function of structural parameters.

This paper is organized as follows. Section 2 gives the mathematical model of a linear plant whose coefficient matrices contain adjustable structural parameters. The plant is driven by a dynamic controller. And the plant and the closed-loop system are represented by descriptor forms, respectively, because the design parameters appear naturally (linearly) in these forms. In Section 3, the problem is formulated by introducing H_∞ constraint described in the Riccati inequality and a criterion of the structure. The Riccati inequality is decoupled into two low-order Riccati inequalities, and characterizations of the optimal controller and structural parameters are given in terms of the solutions of the two inequalities in Section 4. The problem is reduced to one of optimization problems subject to LMIs. A controller and structural parameters are obtained by solving a certain set of LMIs, which can be solved efficiently by using the software package[11].

The principal symbols used in this paper are listed below:

$\mathbf{R}^n$	:	set of all real n-vectors
$\mathbf{R}^{m\times n}$	:	set of all real $m\times n$-matrices
I_n	:	$n\times n$-identity matrix
A^T	:	transpose of a vector or matrix A
$A>(\geq)0$	:	A is positive (semi-)definite
$A>(\geq)B$	:	$A-B$ is positive (semi-)definite
$\|A\|_\infty$	:	the H_∞ norm of A defined by $\|A\|_\infty := \sup_{\omega\in\mathbf{R}} \sigma_{\max}(A(j\omega))$.
$A^\perp$	:	orthogonal complement of $A\in\mathbf{R}^{n\times m}$ defined by $A^\perp\in\mathbf{R}^{(n-r)\times n}$, $A^\perp A=0$,

$A^{\perp}A^{\perp T} > 0$, $r := \text{rank } A$

A^{+} : the Moore-Penrose pseudo-inverse of A

2 Preliminaries

2.1 Mathematical Model

Consider a linear time-invariant parameter-dependent plant described in the descriptor expression,

$$\left.\begin{aligned} &E(p)\dot{x}_p(t) = A(p)x_p(t) + B_w(p)w(t) + B_u(p)u(t) \\ &z(t) = C_z x_p(t) + D_{zw}w(t) + D_{zu}u(t) \\ &y(t) = C_y x_p(t) + D_{yw}w(t), \end{aligned}\right\} \tag{1}$$

where $x_p(t) \in \mathbf{R}^n$ is the plant state; $y(t) \in \mathbf{R}^r$ the measured output; $z(t) \in \mathbf{R}^m$ the regulated output; $u(t) \in \mathbf{R}^\ell$ $(r, \ell < n)$ the control input (the controller output); $w(t) \in \mathbf{R}^s$ the unknown exogenous disturbance. $E(p)$, $A(p) \in \mathbf{R}^{n\times n}$, $B_w(p) \in \mathbf{R}^{n\times s}$, $B_u(p) \in \mathbf{R}^{n\times \ell}$, C_z, C_y, D_{zw}, D_{zu} and D_{yw} are the system matrices of compatible dimensions. Both matrices $B_u(p)$ and C_y are assumed to be of full column rank and full row rank, respectively, i.e., $\text{rank } B_u(p) = \ell$, $\text{rank } C_y = r$. Also, the plant matrices $E(p)$, $A(p)$, $B_w(p)$ and $B_u(p)$ are rational functions of the vector p of the structural design variables p_i:

$$p := \begin{bmatrix} p_1 & p_2 & \cdots & p_N \end{bmatrix}^T$$
$$0 < p_i < 1 \qquad (i = 1, 2, \cdots, N).$$

The adjustable physical parameters $q_i \in (\underline{q}_i, \overline{q}_i)$ (e.g., mass, spring constant, etc.) appearing at the entries in $E(p)$, $A(p)$, $B_w(p)$, $B_u(p)$ with specified constants $\underline{q}_i$, $\overline{q}_i$ are represented by $q_i = \underline{q}_i + p_i(\overline{q}_i - \underline{q}_i)$. That is, to determine the design variable p is equivalent to determine the physical parameter values. Furthermore, $E(p)$, $A(p)$, $B_w(p)$ and $B_u(p)$ are assumed to have the structures defined as follows:

$$\begin{aligned} E(p) &:= E_0 + \Delta E(p) = E_0 + H_{\dot{x}}\Sigma_{\dot{x}}(p)F_{\dot{x}} \\ A(p) &:= A_0 + \Delta A(p) = A_0 + H_x\Sigma_x(p)F_x \\ B_w(p) &:= B_{w0} + \Delta B_w(p) = B_{w0} + H_w\Sigma_w(p)F_w \\ B_u(p) &:= B_{u0} + \Delta B_u(p) = B_{u0} + H_u\Sigma_u(p)F_u, \end{aligned} \tag{2}$$

where $\{E_0, A_0, B_{w0}, B_{u0}, H_{\dot{x}}, H_x, H_w, H_u, F_{\dot{x}}, F_x, F_w, F_u\}$ are known constant matrices of compatible dimensions that represent the structure of the system, $\text{rank } E_0 = n$ is assumed, and

$$\begin{aligned} \Sigma_{\dot{x}}(p) &= \text{diag}(p_i I_{t_i}) \quad i = 1, \cdots, N_1 \\ \Sigma_x(p) &= \text{diag}(p_i I_{t_i}) \quad i = N_1+1, \cdots, N_2 \\ \Sigma_w(p) &= \text{diag}(p_i I_{t_i}) \quad i = N_2+1, \cdots, N_3 \\ \Sigma_u(p) &= \text{diag}(p_i I_{t_i}) \quad i = N_3+1, \cdots, N. \end{aligned} \tag{3}$$

And the control input $u(t)$ is generated by a linear dynamics

$$\left.\begin{aligned} u(t) &= C_c x_c(t) + D_c y(t) \\ \dot{x}_c(t) &= A_c x_c(t) + B_c y(t), \end{aligned}\right\} \tag{4}$$

where $x_c(t) \in \mathbf{R}^q$ is the controller state; $A_c \in \mathbf{R}^{q\times q}$, $B_c \in \mathbf{R}^{q\times r}$, $C_c \in \mathbf{R}^{\ell\times q}$, and $D_c \in \mathbf{R}^{\ell\times r}$.

2.2 Closed-loop System

By using (2) and (3), (1) is rewritten as

$$\dot{x}_p(t) = \bar{A}x_p(t) + \bar{B}_w w(t) + \bar{B}_u u(t) + \bar{B}_e\Sigma_p e(t), \tag{5}$$

where

$$\begin{aligned} \Sigma_p &:= \text{block diag}\,[\Sigma_{\dot{x}}(p), \Sigma_x(p), \Sigma_w(p), \Sigma_u(p)] \\ &= \begin{bmatrix} p_1 I_{t_1} & & 0 \\ & \ddots & \\ 0 & & p_N I_{t_N} \end{bmatrix} \end{aligned}$$

$$e(t) := Hx_p(t) + Fu(t) + G_1 w(t) + G_2\Sigma_p e(t), \tag{6}$$

where we assume the existence of $(I_{\bar{N}} - G_2\Sigma_p)^{-1}$, $(\bar{N} := t_1 + \cdots + t_N)$, and

$$\begin{aligned} &\bar{A} := E_0^{-1}A_0, \quad \bar{B}_w := E_0^{-1}B_{w0} \\ &\bar{B}_u := E_0^{-1}B_{u0}, \quad \bar{B}_e := E_0^{-1}B_{e0} \\ &B_{e0} := [-H_{\dot{x}}\ H_x\ H_w\ H_u] \end{aligned}$$

$$H := \begin{bmatrix} F_{\dot{x}}\bar{A} \\ F_x \\ 0 \\ 0 \end{bmatrix}, \quad F := \begin{bmatrix} F_{\dot{x}}\bar{B}_u \\ 0 \\ 0 \\ F_u \end{bmatrix}$$

$$G_1 := \begin{bmatrix} F_{\dot{x}}\bar{B}_w \\ 0 \\ F_w \\ 0 \end{bmatrix}, \quad G_2 := \begin{bmatrix} F_{\dot{x}}\bar{B}_e \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Making use of (6), combining (5) with (4), we have the closed-loop system

$$\left.\begin{aligned} E_{c\ell}\dot{x}(t) &= A_{c\ell}x(t) + B_{c\ell}w(t) \\ z(t) &= C_{c\ell}x(t) + D_{c\ell}w(t), \end{aligned}\right\} \tag{7}$$

where

$$x(t) := \begin{bmatrix} x_p^T(t) & x_c^T(t) & e^T(t) \end{bmatrix}^T$$

$$E_{c\ell} := \begin{bmatrix} I_{n+q} & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_{c\ell} := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B_{c\ell} := \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$$
$$C_{c\ell} := \begin{bmatrix} C_{11} & 0 \end{bmatrix}$$
$$D_{c\ell} := D_{zw} + D_{zu} D_c D_{yw} = D_{zw} + B_4 K C_2$$
$$A_{11} := A_1 + B_1 K C_1, \quad A_{12} := B_2 \Sigma_p$$
$$A_{21} := A_2 + B_3 K C_1, \quad A_{22} := -I_{\bar{N}} + G_2 \Sigma_p$$
$$B_{11} := A_3 + B_1 K C_2, \quad B_{21} := G_1 + B_3 K C_2$$
$$C_{11} := A_4 + B_4 K C_1$$
$$A_1 := \begin{bmatrix} \bar{A} & 0 \\ 0 & 0 \end{bmatrix}, \; A_2 := \begin{bmatrix} H & 0 \end{bmatrix}$$
$$A_3 := \begin{bmatrix} \bar{B}_w \\ 0 \end{bmatrix}, \; A_4 := \begin{bmatrix} C_z & 0 \end{bmatrix}$$
$$B_1 := \begin{bmatrix} \bar{B}_u & 0 \\ 0 & I_q \end{bmatrix}, \; B_2 := \begin{bmatrix} \bar{B}_e \\ 0 \end{bmatrix}$$
$$B_3 := \begin{bmatrix} F & 0 \end{bmatrix}, \; B_4 := \begin{bmatrix} D_{zu} & 0 \end{bmatrix}$$
$$C_1 := \begin{bmatrix} C_y & 0 \\ 0 & I_q \end{bmatrix}, \; C_2 := \begin{bmatrix} D_{yw} \\ 0 \end{bmatrix}$$
$$K := \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}.$$

The closed-loop system (7) in the descriptor form by introducing the process $e(t)$ defined by (6) does not include any nonlinear terms in Σ_p.

The transfer function $T_{zw}(s)$ from $w(t)$ to $z(t)$ in the closed-loop system (7) is equivalent to that of the closed-loop system in the state space representation.

In this paper, we presume the existence of controller gain matrix $K = \{A_c, \; B_c, \; C_c, \; D_c\}$ and structural design variable matrix Σ_p under H_∞ constraint.

3 Problem Statement

First, we need the following definition and lemma that characterize some properties of descriptor systems.

Definition 1[12]: A pair $(E_{c\ell}, \; A_{c\ell})$ is said to be admissible if it is regular, exponentially stable, and impulse-free.

1. A pair $(E_{c\ell}, \; A_{c\ell})$ is regular if
   $\mathbf{det}(sE_{c\ell} - A_{c\ell}) \not\equiv 0$.
2. A pair $(E_{c\ell}, \; A_{c\ell})$ is exponentially stable if
   $\mathbf{det}(sE_{c\ell} - A_{c\ell}) \neq 0, \quad \forall \mathbf{Re}(s) > 0$.
3. A pair $(E_{c\ell}, \; A_{c\ell})$ is impulse-free if
   $\mathbf{rank}\; E_{c\ell} = \mathbf{degdet}(sE_{c\ell} - A_{c\ell})$. ◇

Lemma 1[13]: Given a scalar $\gamma > 0$. The descriptor system (7) is admissible and

$$\| T_{zw}(s) \|_\infty := \| C_{c\ell}(sE_{c\ell} - A_{c\ell})^{-1} B_{c\ell} + D_{c\ell} \|_\infty < \gamma \qquad (8)$$
$$\| D_{c\ell} \|_\infty < \gamma \qquad (9)$$

if and only if there exists $X_{c\ell} \in \mathbf{R}^{(n+q+\bar{N}) \times (n+q+\bar{N})}$ such that

$$E_{c\ell} X_{c\ell} = X_{c\ell}^T E_{c\ell}^T \geq 0 \qquad (10)$$
$$\Phi := A_{c\ell} X_{c\ell} + X_{c\ell}^T A_{c\ell}^T + (X_{c\ell}^T C_{c\ell}^T + B_{c\ell} D_{c\ell}^T) \times D(\gamma)(X_{c\ell}^T C_{c\ell}^T + B_{c\ell} D_{c\ell}^T)^T + B_{cl} B_{cl}^T < 0 \qquad (11)$$
$$D(\gamma) := (\gamma^2 I_m - D_{c\ell} D_{c\ell}^T)^{-1}.$$

◇

In simultaneous design problem, we have to pay attention to any balance between structural design and control performance. By introducing the rational function $f(\Sigma_p)$ (e.g., $f(\Sigma_p) = \mathbf{tr}(\Sigma_p)$) of design variable matrix Σ_p as a criterion in structural synthesis, problems such as a mass minimization problem can be dealt with.

Hence, our problem is one of the γ-suboptimal H_∞ control problems, that is, to find structural design variable matrix Σ_p and controller gain matrix K by minimizing $f(\Sigma_p)$ and an upper bound γ of $\|T_{zw}(s)\|_\infty$ under the inequalities (10) and (11).

Thus, the problem is formulated as follows:

$$\left. \begin{array}{ll} \text{minimize} & \gamma + f(\Sigma_p) \\ \text{subject to} & (10) \text{ and } (11). \end{array} \right\} \qquad (12)$$

4 LMI Approach to Simultaneous Design

In this section, we will convert the H_∞ control problem to a problem of solving LMIs. The matrix inequality (11) is not convex in K, Σ_p and $X_{c\ell}$. According to the following procedure, we reduce (10) and (11) into LMIs, respectively. The Riccati inequality (11) is decoupled into two low-order Riccati inequalities, and then solutions to the inequalities are utilized to synthesize a controller and structural parameters.

The decoupling of (11) hinges upon the partition of its solution $X_{c\ell}$ in accordance with the structure of $E_{c\ell}$. Noting (10), we set the matrix $X_{c\ell}$ from its regularity as

$$X_{c\ell} = \begin{bmatrix} X & 0 \\ Q\Gamma & Q \end{bmatrix}, \quad X = X^T > 0, \qquad (13)$$

for some nonsingular matrices X and Q where $X \in \mathbf{R}^{(n+q)\times(n+q)}$, $Q \in \mathbf{R}^{\bar{N} \times \bar{N}}$, and $\Gamma \in \mathbf{R}^{\bar{N} \times (n+q)}$. A new H_∞ control synthesis formulation for simultaneous design problem is stated in the following theorem.

Theorem 1: Let Γ be given constant matrix of compatible dimension. For the system given by (1) and (4), there exist a pair of suboptimal H_∞ controller gain K and structural design variable matrix Σ_p satisfying $\|T_{zw}(s)\|_\infty < \gamma$ if there exist a symmetric positive-definite matrix $X = X^T > 0$ and a nonsingular matrix Q such that

$$\hat{A} X + X \hat{A}^T + (X C_{11}^T + \hat{B} D_{c\ell}^T) D(\gamma)$$

$$\times(XC_{11}^T + \hat{B}D_{c\ell}^T)^T + \hat{B}\hat{B}^T < 0 \quad (14)$$

$$(\tilde{A} + \tilde{B}\Sigma_p)Q + Q^T(\tilde{A} + \tilde{B}\Sigma_p)^T + Q^T(\Sigma_p\tilde{C} + \Gamma)T(\Sigma_p\tilde{C} + \Gamma)^T Q + \tilde{R} < 0, \quad (15)$$

where

$$\hat{A} := \hat{A}_1 + \hat{B}_1 K C_1, \quad \hat{B} := \hat{A}_2 + \hat{B}_1 K C_2$$

$$\hat{A}_1 := A_1 - \Gamma^T A_2 = \begin{bmatrix} \hat{A}_{11} & 0 \\ -\Gamma_2^T H & 0 \end{bmatrix}$$

$$\hat{A}_2 := A_3 - \Gamma^T G_1 = \begin{bmatrix} \hat{A}_{21} \\ -\Gamma_2^T G_1 \end{bmatrix}$$

$$\hat{B}_1 := B_1 - \Gamma^T B_3 = \begin{bmatrix} \hat{B}_{11} & 0 \\ -\Gamma_2^T F & I_q \end{bmatrix}$$

$$\begin{aligned}
\hat{A}_{11} &:= \bar{A} - \Gamma_1^T H, \quad \hat{A}_{21} := \bar{B}_w - \Gamma_1^T G_1 \\
\hat{B}_{11} &:= \bar{B}_u - \Gamma_1^T F, \quad \tilde{A} := -I_N + S^T T \Gamma^T \\
\tilde{B} &:= G_2 + S^T T B_2 - S^T T \Gamma^T G_2 \\
\tilde{C} &:= B_2^T - G_2^T \Gamma \\
\tilde{R} &:= B_{21} D_{zw}^T D(\gamma) D_{zw} B_{21}^T + B_{21} B_{21}^T \\
&\quad + S^T T S \\
S &:= X A_{21}^T + (X C_{11}^T + \hat{B} D_{c\ell}^T) D(\gamma) D_{zw} B_{21}^T \\
&\quad - \Gamma^T B_{21} B_{21}^T + B_{11} B_{21}^T \\
T &:= -\Phi_{11}^{-1} (= T^T > 0) \\
\Gamma &:= \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix}, \ (\Gamma_1 \in \mathbf{R}^{\bar{N} \times n}, \ \Gamma_2 \in \mathbf{R}^{\bar{N} \times q}).
\end{aligned}$$

◇

Proof : Consider a full rank matrix:

$$P := \begin{bmatrix} I_{n+q} & 0 \\ -\Gamma & I_{\bar{N}} \end{bmatrix}.$$

Multiplying (11) by P^T and P from the left and right hand sides, respectively, and noting (13), we have

$$\begin{aligned} 0 &> P^T \Phi P \\ &= \begin{bmatrix} \Phi_{11}(X;K;\gamma) & \Phi_{12}(X,Q;K,\Sigma_p;\gamma) \\ \Phi_{12}^T(X,Q;K,\Sigma_p;\gamma) & \Phi_{22}(Q;K,\Sigma_p;\gamma) \end{bmatrix}. \end{aligned} \quad (16)$$

Hence, by the Schur complement formula, (16) is equivalent to

$$\Phi_{11} < 0, \quad \Phi_{22} - \Phi_{12}^T \Phi_{11}^{-1} \Phi_{12} < 0. \quad (17)$$

The first matrix inequality in (17) just represents (14). Substituting Φ_{11}, Φ_{12} and Φ_{22} defined in (16) into the second inequality in (17), after some algebraic manipulation, we have the inequality (15). □

4.1 Determination of Controller Gain

By the Schur complement formula, the inequality (14) is equivalent to the following matrix inequality:

$$\begin{bmatrix} (\hat{A}_1 + \hat{B}_1 K C_1) X + X(\hat{A}_1 + \hat{B}_1 K C_1)^T & X(A_4 + B_4 K C_1)^T & \hat{A}_2 + \hat{B}_1 K C_2 \\ (A_4 + B_4 K C_1) X & -\gamma^2 I_m & D_{zw} + B_4 K C_2 \\ (\hat{A}_2 + \hat{B}_1 K C_2)^T & (D_{zw} + B_4 K C_2)^T & -I_s \end{bmatrix} < 0. \quad (18)$$

This matrix inequality (18) is not an LMI in the unknown matrices X and K because of the existence of the product term KC_1X. Inequality (18) can be reduced to LMI as follows.

First, we introduce the standard assumptions;

$$D_{zu}^T D_{zu} > 0, \quad D_{yw} D_{yw}^T > 0. \quad (19)$$

The matrix inequality (18) is rewritten as

$$\Psi(X;\gamma) + \hat{U}^T K \hat{V}(X) + \hat{V}^T(X) K^T \hat{U} < 0, \quad (20)$$

where

$$\Psi(X;\gamma) := \begin{bmatrix} \hat{A}_1 X + X \hat{A}_1^T & X A_4^T & \hat{A}_2 \\ A_4 X & -\gamma^2 I_m & D_{zw} \\ \hat{A}_2^T & D_{zw}^T & -I_s \end{bmatrix} \quad (21)$$

$$\hat{U} := \begin{bmatrix} \hat{B}_1^T & B_4^T & 0 \end{bmatrix} \quad (22)$$

$$\hat{V}(X) := \begin{bmatrix} C_1 X & 0 & C_2 \end{bmatrix}. \quad (23)$$

Here, we need the following well-known lemma.
Lemma 2[14]: Given $U \in \mathbf{R}^{n \times m}$, $V \in \mathbf{R}^{k \times n}$, $\Sigma = \Sigma^T \in \mathbf{R}^{n \times n}$, suppose that

$$\text{rank } U < n, \quad \text{rank } V < n. \quad (24)$$

Then there exists a matrix $Z \in \mathbf{R}^{m \times k}$ satisfying

$$UZV + (UZV)^T + \Sigma < 0 \quad (25)$$

if and only if U, V and Σ satisfy the conditions

$$U^\perp \Sigma U^{\perp T} < 0, \quad V^{T\perp} \Sigma V^{T\perp T} < 0. \quad (26)$$

◇

By virtue of this lemma, the solvability condition (20) for the matrix K is replaced by the following two matrix inequalities in terms of X in which K does not appear explicitly[1] :

$$\hat{U}^{T\perp} \Psi(X;\gamma) \hat{U}^{T\perp T} < 0 \quad (27)$$

$$\hat{V}^{T\perp}(X) \Psi(X;\gamma) \hat{V}^{T\perp T}(X) < 0. \quad (28)$$

We can choose

$$\hat{U}^{T\perp} = \begin{bmatrix} I_n & 0 & -\hat{B}_{11} D_{zu}^+ & 0 \\ 0 & 0 & D_{zu}^\perp & 0 \\ 0 & 0 & 0 & I_s \end{bmatrix} \quad (29)$$

$$\hat{V}^{T\perp}(X) = \left[\begin{array}{cc|c|c} I_n & 0 & 0 & -C_y^T D_{yw}^{+T} \\ \hline 0 & 0 & 0 & D_{yw}^{T\perp} \\ \hline 0 & 0 & I_m & 0 \end{array}\right] \begin{bmatrix} X^{-1} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_s \end{bmatrix} \quad (30)$$

[1] From (27) and (28), we obtain an arbitrary order controller gain subject to BMI by using several computational approaches (e.g., the XY-centring optimization algorithm[15]).

which are consistent with the definition of orthogonal complement matrix, and from (19) we can set D_{zu}^{+} and D_{yw}^{+} as

$$D_{zu}^{+} = (D_{zu}^T D_{zu})^{-1} D_{zu}^T$$
$$D_{yw}^{+} = D_{yw}^T (D_{yw} D_{yw}^T)^{-1}$$

because D_{zu} and D_{yw} have full column and row ranks, respectively.

Partition X and its inverse as

$$\left.\begin{array}{l} X := \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}, \quad X^{-1} := \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \\ \qquad\qquad X_{11} \in \mathbf{R}^{n \times n}, \quad Y_{11} \in \mathbf{R}^{n \times n} \end{array}\right\} \tag{31}$$

$$\begin{aligned} XX^{-1} &= \begin{bmatrix} X_{11}Y_{11} + X_{12}Y_{12}^T & X_{11}Y_{12} + X_{12}Y_{22} \\ X_{12}^T Y_{11} + X_{22}Y_{12}^T & X_{12}Y_{12} + X_{22}Y_{22} \end{bmatrix} \\ &= \begin{bmatrix} I_n & 0 \\ 0 & I_q \end{bmatrix}. \end{aligned} \tag{32}$$

By using (29), (30), (31) and the definitions of $\hat{A}_1$, $\hat{A}_2$ and A_4, it follows from (27) and (28) that

$$\hat{\mathcal{U}}\Theta_1(X_{11};\gamma)\hat{\mathcal{U}}^T < 0, \; \hat{\mathcal{V}}\Theta_2(Y_{11}, Y_{12};\gamma)\hat{\mathcal{V}}^T < 0 \tag{33}$$

$$X_{11} \geq Y_{11}^{-1} > 0 \iff \begin{bmatrix} X_{11} & I_n \\ I_n & Y_{11} \end{bmatrix} \geq 0, \tag{34}$$

where

$$\hat{\mathcal{U}} := \begin{bmatrix} I_n & -\hat{B}_{11}D_{zu}^{+} & 0 \\ 0 & D_{zu}^{\perp} & 0 \\ 0 & 0 & I_s \end{bmatrix}$$

$$\hat{\mathcal{V}} := \begin{bmatrix} I_n & 0 & -C_y^T D_{yw}^{+T} \\ 0 & 0 & D_{yw}^{T\perp} \\ 0 & I_m & 0 \end{bmatrix}$$

$$\Theta_1(X_{11};\gamma) := \begin{bmatrix} \hat{A}_{11}X_{11} + X_{11}\hat{A}_{11}^T & X_{11}C_z^T & \hat{A}_{21} \\ C_z X_{11} & -\gamma^2 I_m & D_{zw} \\ \hat{A}_{21}^T & D_{zw}^T & -I_s \end{bmatrix}$$

$$\Theta_2(Y_{11}, Y_{12};\gamma) := \begin{bmatrix} Y_{11}\hat{A}_{11} + \hat{A}_{11}^T Y_{11} - Y_{12}\Gamma_2^T H - H^T\Gamma_2 Y_{12}^T & C_z^T & Y_{11}\hat{A}_{21} - Y_{12}\Gamma_2^T G_1 \\ C_z & -\gamma^2 I_m & D_{zw} \\ \hat{A}_{21}^T Y_{11} - G_1^T \Gamma_2 Y_{12}^T & D_{zw}^T & -I_s \end{bmatrix},$$

where (34) is equivalent to $X > 0$.

Whenever there are no direct couplings between the control input and regulated output and also between exogenous input and measured output, i.e., $D_{zu} = 0$, $D_{yw} = 0$, the matrices $\hat{U}^{T\perp}$ and $\hat{V}^{T\perp}$ reduce, respectively, to

$$\hat{U}^{T\perp} = \begin{bmatrix} \hat{B}_{11}^{\perp} & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_s \end{bmatrix}$$

$$\hat{V}^{T\perp}(X) = \left[\begin{array}{cc|c|c} C_y^{T\perp} & 0 & 0 & 0 \\ \hline 0 & 0 & I_m & 0 \\ \hline 0 & 0 & 0 & I_s \end{array}\right] \begin{bmatrix} X^{-1} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_s \end{bmatrix}.$$

Thus, the condition (14) yields three LMIs, (33), (34), each of which is the LMI in X_{11} or in both Y_{11}, Y_{12}, or in both X_{11}, Y_{11}, respectively. Furthermore, the following rank condition for an arbitrary order controller synthesis has been imposed as[16]

$$\operatorname{rank} \begin{bmatrix} X_{11} & I_n \\ I_n & Y_{11} \end{bmatrix} \leq n + q. \tag{35}$$

This rank condition is not convex. But, if the controller dimension q is larger than or equal to the plant dimension n, the constraint (35) is automatically satisfied and we can remove the equality constraint in (34). The problem is to find the matrix K subject to the constraints (33) and (34).

The procedure is performed by the convex optimization programming approach in the following three steps:

Step 1: Evaluate the matrix X_{11}, Y_{11} and Y_{12} so as to

$$\left.\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & (33) \text{ and } (34). \end{array}\right\} \tag{36}$$

Step 2: By evaluating the matrices X_{12} and X_{22} from (32) using X_{11}, Y_{11}, Y_{12} evaluated in *Step 1*, we can obtain the matrix X.

Step 3: With the matrix X obtained in *Step 2*, solve the LMI (18) for K.

4.2 Determination of Structural Parameters

In this subsection, we determine structural design variable matrix Σ_p. By the Schur complement formula, we have from (15) that

$$\Omega(Q) + \tilde{U}^T \Sigma_p \tilde{V}(Q) + \tilde{V}^T(Q)\Sigma_p \tilde{U} < 0, \tag{37}$$

where

$$\Omega(Q) := \begin{bmatrix} \tilde{A}Q + Q^T\tilde{A}^T + \tilde{R} & Q^T\Gamma \\ \Gamma^T Q & -T^{-1} \end{bmatrix}$$

$$\tilde{U} := \begin{bmatrix} \tilde{B}^T & \tilde{C} \end{bmatrix}, \quad \tilde{V}(Q) := \begin{bmatrix} Q & 0 \end{bmatrix}.$$

By applying Lemma 2 to (37), the matrix inequality (37) is equivalent to

$$\tilde{U}^{T\perp}\Omega(Q)\tilde{U}^{T\perp T} < 0, \quad \tilde{V}^{T\perp}\Omega(Q)\tilde{V}^{T\perp T} < 0 \tag{38}$$

which are two matrix inequalities with respect to Q, in which Σ_p does not appear explicitly. From the regularity of Q and the definition of orthogonal complement matrix, since we can choose $\tilde{V}^{T\perp}$ as $\tilde{V}^{T\perp} := \begin{bmatrix} 0 & I_{n+q} \end{bmatrix}$, the second inequality in (38) is rewritten as $-T^{-1} < 0$, and becomes self-evident.

Hence, we can obtain the structural design variable matrix Σ_p in the following procedure:

Step 4: Compute the matrix Q from the LMI

$$\tilde{U}^{T\perp}\Omega(Q)\tilde{U}^{T\perp T} < 0.$$

Step 5: With the matrix Q obtained in *Step 4*, evaluate the matrix Σ_p so as to

$$\left.\begin{array}{ll} \text{minimize} & f(\Sigma_p) \\ \text{subject to} & (37). \end{array}\right\} \qquad (39)$$

5 Conclusion

In this paper, we have proposed a simultaneous design method of the system which has the adjustable parameters and the controller for H_∞ performance. A prominent feature of the proposed approach is that the LMI conditions for this problem have been derived. Since the conditions for the existence of the H_∞ controller and structural design parameters are given in terms of LMIs which form a convex set, we can obtain such a controller and structural design parameters by means of an existing convex optimization algorithm.

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