Multiobjective Control Design via Successive Over-bounding of Quadratic Terms [1]

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Abstract

This paper addresses less conservative control design for multiple design specifications. Although problems are described by a set of LMIs, they are solved with non-common LMI solutions to reduce the conservatism arising from seeking a common LMI solution. Noticing that completing the square can split two variables in BMI terms into two different LMI ones, we propose iterative algorithms while replacing non-positive quadratic terms by their upper bounds. A suitable choice of the parameters in these upper bounds guarantees convergence property. An illustrated example is included.

1 Introduction

In the past one decade, much attention has been paid on designing controllers that satisfy multiple design specifications such as mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control [6], $\mathcal{H}_\infty$ control with regional pole placement [2], and synthesis of strictly positive real $\mathcal{H}_2$ controllers [3]. They are called *multiobjective control design* [8], in which the term *multiobjective* means multiple design specifications but does not always mean multiple objective functions.

Recently, LMI-based multiobjective control design has become well-established [7, 8]. However, one drawback remains; a common LMI solution has been sought at the expense of conservatism. In LMI-based multiobjective control design, problems are defined by a combination of LMIs corresponding to multiple design specifications. Those LMIs are individually convex but jointly non-convex. In order to convexify those problems, a common LMI solution has been sought at the expense of conservatism. Many researchers in this field have also been seeking a common LMI solution for multiobjective problems [2, 3, 6]. This choice, however, results in conservatism of design; e.g., robust stability is overemphasized at the expense of performance.

To reduce this conservatism, in this paper, we propose a design procedure to solve multiobjective problems with non-common LMI solutions. We provide a unified framework to design static/dynamic output-feedback or state-feedback controllers based on non-common LMI solutions. Although the theory is comprehensive, we concentrate on two typical problems which are the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control and regional pole placement. Furthermore, we mainly focus on dynamic output-feedback controllers. Noticing that completing the square can split two variables in BMI terms into two different LMI ones, we provide iterative algorithms while replacing non-positive quadratic terms by their upper bounds. A suitable choice of the parameters in these upper bounds guarantees convergence property. In a numerical example, the conservatism will be found to be effectively reduced. In related work, an attempt is made in the same direction but in a different manner, in which the multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control problem is solved with non-common Lyapunov functions based on finite dimensional Q-parametrization with LMIs [4].

[1] This work was supported by the Ministry of Education of Japan under the Grant-in-Aid for Young Scientists No.12750405.

2 Problem Formulation

Let us consider the following linear time-invariant generalized plant.

$$\Sigma_p : \begin{cases} \dot{x} = Ax + Bu + \sum_{j=1}^{N} B_j w_j \\ y = Cx + Du + \sum_{j=1}^{N} F_j w_j, \\ z_j = C_j x + E_j u + D_j w_j, \quad j = 1, 2, \ldots, N \end{cases} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the control input, $y(t) \in \mathbf{R}^p$ is the measurement output, $w_j(t) \in \mathbf{R}^{m_j}$, $j = 1, 2, \ldots, N$ are the disturbance inputs, and $z_j(t) \in \mathbf{R}^{p_j}$, $j = 1, 2, \ldots, N$ are the performance outputs. All matrices in (1) are of compatible dimensions. Without loss of generality, we assume $D = 0$ [5]. Let $H_{z_j w_j}(s)$ be the closed-loop transfer functions from w_j to z_j; $j = 1, 2, \ldots, N$. Every pair $(w_j(t), z_j(t))$ concerns each design specification such as $\mathcal{H}_2$ norm to be minimized or $\mathcal{H}_\infty$ norm to be bounded.

Next, let us consider the n_c-th order dynamic ($n_c > 0$) or static ($n_c = 0$) output-feedback controller:

$$\Sigma_{cd} : \begin{cases} \dot{x}_c = A_c x_c + B_c y \\ u \ = C_c x_c + D_c y \end{cases}, \qquad \Sigma_{cs} : u = Gy, \tag{2}$$

or, if x is available, consider state-feedback controllers ($n_c > 0$ or $n_c = 0$). We simply call one of them "*the controller* Σ_c," which is connected by standard negative feedback to the open-loop plant Σ_p as shown in Figure 1.

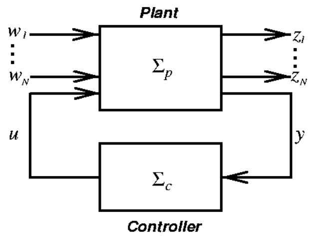

Figure 1: Generalized plant-controller configuration

Note that Σ_{cd} and Σ_{cs} include state-feedback versions as special cases. By setting $y = x$, the state-feedback ones are obtained. For the dynamic controller Σ_{cd}, we denote its transfer function as follows.

$$K_{cd}(s) := C_c(sI - A_c)^{-1}B_c + D_c. \qquad (3)$$

The j-th closed-loop system $\Sigma_{cl}^{(j)}$ can be described by

$$\Sigma_{cl}^{(j)} : \begin{cases} \dot{x}_{cl} = A_{cl}x_{cl} + B_{cl}^{(j)}w_j \\ z_j \;= C_{cl}^{(j)}x_{cl} + D_{cl}^{(j)}w_j \end{cases}, \quad 1 \le j \le N. \qquad (4)$$

First, we consider output-feedback controllers. For the static controller Σ_{cs}, the state variable is set to $x_{cl} = x$ and the coefficient matrices in (4) are given by

$$\begin{bmatrix} A_{cl} & B_{cl}^{(j)} \\ C_{cl}^{(j)} & D_{cl}^{(j)} \end{bmatrix} = \begin{bmatrix} A & B_j \\ C_j & D_j \end{bmatrix} + \begin{bmatrix} B \\ E_j \end{bmatrix} G[\,C \quad F_j\,]. \qquad (5)$$

For the dynamic controller Σ_{cd}, the state variable is set to $x_{cl} = [\,x' \quad x_c'\,]'$ and the coefficient matrices in (4) are given by

$$\begin{bmatrix} A_{cl} & B_{cl}^{(j)} \\ C_{cl}^{(j)} & D_{cl}^{(j)} \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B}_j \\ \hat{C}_j & \hat{D}_j \end{bmatrix} + \begin{bmatrix} \hat{B} \\ \hat{E}_j \end{bmatrix} \hat{G}[\,\hat{C} \quad \hat{F}_j\,], \qquad (6)$$

where

$$\begin{bmatrix} \hat{A} & \hat{B}_j & \hat{B} \\ \hat{C}_j & \hat{D}_j & \hat{E}_j \\ \hat{C} & \hat{F}_j & \hat{G}' \end{bmatrix} := \left[\begin{array}{cc|c|cc} A & 0 & B_j & B & 0 \\ 0 & 0 & 0 & 0 & I_{n_c} \\ \hline C_j & 0 & D_j & E_j & 0 \\ \hline C & 0 & F_j & D_c' & B_c' \\ 0 & I_{n_c} & 0 & C_c' & A_c' \end{array}\right]. \qquad (7)$$

As pointed out in [5], exactly the same structure is retained for both Σ_{cs} and Σ_{cd}. Once a design method has been established for Σ_{cs}, therefore, it also works for Σ_{cd} with the augmented matrices defined in (7). Before proceeding, we also assume $D_j = 0,\ j = 1, 2, \ldots, N,\ E_jGF_j = 0$ for Σ_{cs}, and $D_c = 0$ for Σ_{cd} for simplicity.

Next, we consider state-feedback controllers. Once a design method has been developed for Σ_{cs} or Σ_{cd}, by setting $C = I$ and $F_j = 0$ in addition to $D = 0$, we immediately have the state-feedback version of the method.

Now we are ready to state the problems to be solved here. To make the presentation as simple as possible, we concentrate on the following two typical problems.

2.1 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control

Let us set $N = 2$, then the first problem is stated as follows.

Problem 1 *Given a scalar $\gamma > 0$, find a controller Σ_c that achieves*

$$\inf\{\ \|H_{z_1w_1}(s)\|_2^2\ :\ \|H_{z_2w_2}(s)\|_\infty < \gamma\ \}. \qquad (8)$$

Hereafter, without loss of generality, we set the $\mathcal{H}_\infty$-norm bound γ to 1 for simplicity.

2.2 Regional Pole Placement

Let us consider the LMI region $\mathcal{S}(\alpha, r, \theta)$ as shown in Figure 2, which is given in [2].

The second problem is stated as follows.

Problem 2 *Given scalars $\alpha > 0,\ r > 0$, and $0 < \theta < \pi/2$, find a controller Σ_c that achieves*

$$\lambda_j(A_{cl}) \in \mathcal{S}(\alpha, r, \theta), \qquad 1 \le j \le n + n_c, \qquad (9)$$

where $\lambda_j(A_{cl})$ denotes the j-th eigenvalue of A_{cl}.

As a variation, we also consider the joint problem of Problems 1 and 2.

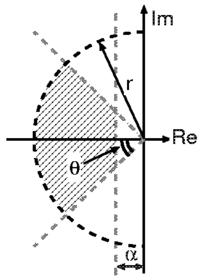

Figure 2: The LMI region $\mathcal{S}(\alpha, r, \theta)$

3 Preliminaries

With the assumption of $D_{cl}^{(j)} = 0,\ j = 1,\ 2$, we have the following lemmas for $\mathcal{H}_2$ optimization and the $\mathcal{H}_\infty$ constraint [1].

Lemma 1 *The following statements are equivalent:*

1. *Σ_c is an $\mathcal{H}_2$ controller for $\Sigma_{cl}^{(1)}$;*
2. *A_{cl} is stable, and $\Sigma_{cl}^{(1)}$ achieves $\inf \|H_{z_1w_1}(s)\|_2^2$;*
3. *Σ_c achieves $\inf\left[\mathrm{Tr}(C_{cl}^{(1)}\Pi_1 C_{cl}^{(1)\prime})\right]$ subject to*

$$\Pi_1 > 0, \quad A_{cl}\Pi_1 + \Pi_1 A_{cl}' + B_{cl}^{(1)}B_{cl}^{(1)\prime} < 0. \qquad (10)$$

Lemma 2 *The following statements are equivalent:*

1. *Σ_c is an $\mathcal{H}_\infty$ controller for $\Sigma_{cl}^{(2)}$;*
2. *A_{cl} is stable, and $\Sigma_{cl}^{(2)}$ satisfies $\|H_{z_2w_2}(s)\|_\infty < 1$;*
3. *There exists $P_2 > 0$ such that*

$$P_2A_{cl} + A_{cl}'P_2 + P_2B_{cl}^{(2)}B_{cl}^{(2)\prime}P_2 + C_{cl}^{(2)\prime}C_{cl}^{(2)} < 0. \qquad (11)$$

Let us define $P_1 := \Pi_1^{-1} > 0$. From (10) and (11), if we perform the Schur complement formula, Problem 1 can be described by a set of inequalities as follows.

Problem 3 *Find a controller Σ_c that achieves*

$$\inf\,[\mathrm{Tr}(Q)] \text{ subject to} \qquad (12)$$

$$\begin{bmatrix} P_1 & C_{cl}^{(1)\prime} \\ C_{cl}^{(1)} & Q \end{bmatrix} > 0, \qquad (13)$$

$$\begin{bmatrix} P_1A_{cl} + A_{cl}'P_1 & P_1B_{cl}^{(1)} \\ B_{cl}^{(1)\prime}P_1 & -I \end{bmatrix} < 0, \qquad (14)$$

$$P_2 > 0, \qquad (15)$$

$$\begin{bmatrix} P_2A_{cl} + A_{cl}'P_2 & P_2B_{cl}^{(2)} & C_{cl}^{(2)\prime} \\ B_{cl}^{(2)\prime}P_2 & -I & 0 \\ C_{cl}^{(2)} & 0 & -I \end{bmatrix} < 0. \qquad (16)$$

4 Conventional Design for Output-Feedback $\mathcal{H}_2/\mathcal{H}_\infty$ Controllers

In this section, we review the conventional method of a common LMI solution, while focusing on the dynamic output-feedback $\mathcal{H}_2/\mathcal{H}_\infty$ controller Σ_{cd} with $N = 2$. Let us consider the following matrices.

$$P_j := \begin{bmatrix} Y_j & U_j \\ U_j' & \hat{Y}_j \end{bmatrix}, \qquad P_j^{-1} := \begin{bmatrix} X_j & V_j \\ V_j' & \hat{X}_j \end{bmatrix},$$
$$T_j := \begin{bmatrix} X_j & I \\ V_j' & 0 \end{bmatrix}, \qquad S_j \ := \begin{bmatrix} I & Y_j \\ 0 & U_j' \end{bmatrix}. \qquad (17)$$

where X_j and Y_j are $n \times n$ and symmetric, and other matrices are of compatible dimensions. Recalling $A_{cl}, B_{cl}^{(j)}$ and $C_{cl}^{(j)}$ defined in (6) and (7), and noticing the relation $P_j T_j = S_j$, we have the following identities [2, 7, 8].

$$\begin{aligned}
&{T_j}' P_j T_j = {S_j}' T_j = \begin{bmatrix} X_j & I \\ I & Y_j \end{bmatrix}, \\
&{T_j}' P_j A_{cl} T_j = {S_j}' A_{cl} T_j = \begin{bmatrix} AX_j + B\bar{C}_c & A \\ \bar{A}_c & Y_j A + \bar{B}_c C \end{bmatrix}, \\
&{T_j}' P_j B_{cl}^{(j)} = {S_j}' B_{cl}^{(j)} = \begin{bmatrix} B_j \\ Y_j B_j + \bar{B}_c F_j \end{bmatrix}, \\
&C_{cl}^{(j)} T_j = [\, C_j X_j + E_j \bar{C}_c \quad C_j \,], \qquad (18)
\end{aligned}$$

where

$$\begin{aligned}
\bar{A}_c &:= [\, Y_j \quad \bar{B}_c \,] \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} X_j \\ \bar{C}_c \end{bmatrix} + U_j A_c {V_j}', \\
\bar{B}_c &:= U_j B_c, \\
\bar{C}_c &:= C_c {V_j}'. \qquad (19)
\end{aligned}$$

We multiply (13) and (14), from the left, by diag $(T_1, I)'$ and, from the right, by diag (T_1, I). Next, we multiply (15), from the left, by T_2' and, from the right, by T_2. Finally, if we perform a similar congruence transformation with diag (T_2, I) on (16), we then have the following equivalent problem restricted to the class of Σ_{cd}.

Problem 4 *Find a controller Σ_{cd} that achieves*

$$J^{(0)} := \inf [\mathrm{Tr}(Q)] \quad \text{subject to} \qquad (20)$$

$$\begin{bmatrix} {T_1}' P_1 T_1 & {T_1}' {C_{cl}^{(1)}}' \\ C_{cl}^{(1)} T_1 & Q \end{bmatrix} > 0, \qquad (21)$$

$$\begin{bmatrix} {T_1}' P_1 A_{cl} T_1 + {T_1}' A_{cl}' P_1 T_1 & {T_1}' P_1 B_{cl}^{(1)} \\ {B_{cl}^{(1)}}' P_1 T_1 & -I \end{bmatrix} < 0, \qquad (22)$$

$${T_2}' P_2 T_2 > 0, \qquad (23)$$

$$\begin{bmatrix} {T_2}' P_2 A_{cl} T_2 + {T_2}' A_{cl}' P_2 T_2 & {T_2}' P_2 B_{cl}^{(2)} & {T_2}' {C_{cl}^{(2)}}' \\ {B_{cl}^{(2)}}' P_2 T_2 & -I & 0 \\ C_{cl}^{(2)} T_2 & 0 & -I \end{bmatrix} < 0, \qquad (24)$$

in the matrix variables $\bar{A}_c$, $\bar{B}_c$, $\bar{C}_c$, X_j, and Y_j ($j = 1,\ 2$), where ${T_j}' P_j T_j$, ${T_j}' P_j A_{cl} T_j$, ${T_j}' P_j B_{cl}^{(j)}$, and $C_{cl}^{(j)} T_j$ are given as in (18).

Here, setting $n_c = n$, we consider a full-order dynamic controller Σ_{cd}. Once matrices $\bar{A}_c$, $\bar{B}_c$, $\bar{C}_c$, X_j, and Y_j ($j = 1,\ 2$) have been found, the controller matrices A_c, B_c, and C_c are calculated as follows. Choose a non-singular matrix V_j and calculate U_j such that $U_j {V_j}' = I - Y_j X_j$, then we have

$$\begin{aligned}
A_c^{(j)} &:= U_j^{-1}(\bar{A}_c - [\, Y_j \quad \bar{B}_c \,] \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} X_j \\ \bar{C}_c \end{bmatrix})({V_j}')^{-1}, \\
B_c^{(j)} &:= U_j^{-1} \bar{B}_c, \\
C_c^{(j)} &:= \bar{C}_c (V_j')^{-1}, \quad j = 1,\ 2. \qquad (25)
\end{aligned}$$

The variables $P_j,\ j = 1, 2$ in (17) are not necessarily required to be a common LMI solution ($P_2 = P_1$) but must generate the same controller matrices. Hence the following equality constraints are posed.

$$A_c^{(1)} = A_c^{(2)}, \quad B_c^{(1)} = B_c^{(2)}, \quad C_c^{(1)} = C_c^{(2)}. \qquad (26)$$

Due to these constraints, Problem 4 is not convex. In order to convert this non-convex problem into a convex one, a common LMI solution:

$$P_1 = P_2 = P := \begin{bmatrix} Y & U \\ U' & \hat{Y} \end{bmatrix} > 0, \ P^{-1} := \begin{bmatrix} X & V \\ V' & \hat{X} \end{bmatrix} \qquad (27)$$

has been sought at the expense of conservatism [2, 7, 8].

5 Less Conservative Design for Output-Feedback $\mathcal{H}_2/\mathcal{H}_\infty$ Controllers

In this section, we derive a less conservative design procedure for output-feedback $\mathcal{H}_2/\mathcal{H}_\infty$ controllers. Although we use the description for the coefficient matrices of Σ_{cs}, the following results are directly applicable to Σ_{cd} with the augmented matrices defined in (7).

5.1 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem

Substituting $A_{cl}, B_{cl}^{(j)}$ and $C_{cl}^{(j)}$ defined in (5) into (13)-(16), we can describe the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem with the controller variable G as follows.

Problem 5

$$J_{ac} := \inf [\mathrm{Tr}(Q)] \quad \text{subject to} \qquad (28)$$

$$\begin{bmatrix} P_1 & \text{sym.} \\ C_1 + E_1 G C & Q \end{bmatrix} > 0, \qquad (29)$$

$$\begin{bmatrix} P_1(A + BGC) + (A + BGC)' P_1 & \text{sym.} \\ (B_1 + BGF_1)' P_1 & -I \end{bmatrix} < 0, \qquad (30)$$

$$P_2 > 0, \qquad (31)$$

$$\begin{bmatrix} P_2(A + BGC) + (A + BGC)' P_2 & & \text{sym.} \\ (B_2 + BGF_2)' P_2 & -I & \\ C_2 + E_2 G C & 0 & -I \end{bmatrix} < 0. \qquad (32)$$

This is a BMI problem with respect to the variables G and $P_j,\ j = 1, 2$.

5.2 Enforced Problems

Since the third row and column in (32) are already affine, we focus on the first 2×2 principal diagonal block as a necessary condition.

$$\begin{bmatrix} P_2(A + BGC) + (A + BGC)' P_2 & \text{sym.} \\ (B_2 + BGF_2)' P_2 & -I \end{bmatrix} < 0. \qquad (33)$$

In (30) and (33), the variables G and $P_j,\ j = 1, 2$ in the BMI terms $P_j BGC$ in (1,1)-blocks can be split into two different LMI terms by completing the square $(B'P_j + GC)'(B'P_j + GC)$ and performing the Schur complement formula. The same is true with the BMI terms $P_j BGF_j$ in (1,2)- and (2,1)-blocks, in which after completing the square with respect to $[\, B'P_j \quad GF_j \,]$, we perform the Schur complement formula. These tricks are made at the expense of generating the non-positive quadratic terms $-C'G'GC$, $-P_j BB'P_j$, and $-{F_j}'G'GF_j,\ j = 1, 2$ in diagonal blocks. As a result, we have

$$\begin{bmatrix} \Psi(P_j, G) & & \text{sym.} & \\ {B_j}' P_j & -I - \underline{{F_j}'G'GF_j} & & \\ B'P_j + GC & 0 & -I & \\ B'P_j & GF_j & 0 & -I \end{bmatrix} < 0, \qquad (34)$$

$$\begin{aligned}
\Psi(P_j, G) &:= \Phi_A(P_j) - 2\underline{P_j BB'P_j} - \underline{C'G'GC}, \\
\Phi_A(P_j) &:= P_j A + A' P_j. \qquad (35)
\end{aligned}$$

Due to these non-positive quadratic terms (underlined), the inequalities in (34) are not yet LMIs. Furthermore, since these terms are non-positive definite, we cannot apply the Schur complement formula to them in order to convert (34) into LMIs. To overcome this difficulty, we consider the following lemma (see [9] for early work).

Lemma 3 *Given L_j, $M \in \mathbf{R}^{(m+n_c)\times(n+n_c)}$, and $N_j \in \mathbf{R}^{(m+n_c)\times m_j}$, the following inequalities hold.*

$$-P_j BB'P_j \le -P_j BL_j - {L_j}'B'P_j + {L_j}'L_j =: \Phi_B(P_j, L_j); \ (36)$$

$$-C'G'GC \le -C'G'M - M'GC + M'M =: \Sigma(G, M); \quad (37)$$

$$-{F_j}'G'GF_j \le -{F_j}'G'N_j - {N_j}'GF_j + {N_j}'N_j =: \Lambda_j(G, N_j). (38)$$

When L_j, M and N_j are set to $L_j = B'P_j$, $M = GC$, and $N_j = GF_j$, respectively, then each inequality reduces to an equality; in other words, the upper bound then recovers the original value.

Proof: First, we have

$$(L_j - B'P_j)'(L_j - B'P_j) \geq 0, \tag{39}$$
$$(M - GC)'(M - GC) \geq 0, \tag{40}$$
$$(N_j - GF_j)'(N_j - GF_j) \geq 0, \tag{41}$$

which directly yield (36), (37), and (38), respectively. When L_j, M and N_j are set to $L_j = B'P_j$, $M = GC$, and $N_j = GF_j$, respectively, each inequality reduces to an equality. This completes the proof. ■

Replacing the non-positive quadratic terms (underlined) in (34) by their upper bounds in (36)-(38), we derive

$$\begin{bmatrix} \Phi(P_j, G, L_j, M) & & \text{sym.} & \\ {B_j}'P_j & \Lambda_j(G, N_j) - I & & \\ H(P_j, G) & 0 & -I & \\ B'P_j & GF_j & 0 & -I \end{bmatrix} < 0, \tag{42}$$

$$\Phi(P_j, G, L_j, M) := \Phi_A(P_j) + 2\Phi_B(P_j, L_j) + \Sigma(G, M), \tag{43}$$
$$H(P_j, G) := B'P_j + GC, \tag{44}$$

where $\Phi_A(P_j)$, $\Phi_B(P_j, L_j)$, $\Sigma(G, M)$, and $\Lambda_j(G, N_j)$ are defined in (35)-(38). If (42) is feasible with respect to G and P_j, then (34) is also feasible for the same variables.

Returning the third row and column of (32) to (42) for $j = 2$, we derive the counterpart corresponding to the original $\mathcal{H}_\infty$ condition in (32) to obtain an enforced version of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem:

Problem 6 *Find a controller Σ_c that achieves*

$$J := \inf\,[\mathrm{Tr}(Q)] \quad \text{subject to} \tag{45}$$

$$\begin{bmatrix} P_1 & {C_{cl}^{(1)}}' \\ C_{cl}^{(1)} & Q \end{bmatrix} > 0, \tag{46}$$

$$\begin{bmatrix} \Phi(P_1, G, L_1, M) & & \text{sym.} & \\ {B_1}'P_1 & \Lambda_1(G, N_1) - I & & \\ H(P_1, G) & 0 & -I & \\ B'P_1 & GF_1 & 0 & -I \end{bmatrix} < 0, \tag{47}$$

$$P_2 > 0, \tag{48}$$

$$\begin{bmatrix} \Phi(P_2, G, L_2, M) & & & & \\ {B_2}'P_2 & \Lambda_2(G, N_2) - I & \text{sym.} & & \\ H(P_2, G) & 0 & -I & & \\ B'P_2 & GF_2 & 0 & -I & \\ C_{cl}^{(2)} & 0 & 0 & 0 & -I \end{bmatrix} < 0. \tag{49}$$

Note that $\Phi_A(P_j)$, $\Phi_B(P_j, L_j)$, $\Sigma(G, M)$, and $\Lambda_j(G, N_j)$ are defined in (35)-(38).

5.3 Iterative Algorithms

Now we apply the results developed in the preceding part of this section to the dynamic controller Σ_{cd}. For this purpose, the description for the coefficient matrices of Σ_{cs} used in these results will be replaced by those of Σ_{cd} with the augmented matrices defined in (7) in the rest of this section. To accommodate to this replacement, we call the corresponding versions of Lemma 3, Problems 5, and 6 for the dynamic controller Σ_{cd}; Lemma 3', Problems 5', and 6', respectively. In Problem 4, when we seek a common LMI solution as in (27) to design a full-order dynamic controller Σ_{cd}, while setting $n_c = n$, if Problem 4 is feasible, then the controller matrices A_c, B_c, and C_c are calculated as follows.

With the optimal solutions X, Y, $\bar{A}_c$, $\bar{B}_c$, and $\bar{C}_c$ to Problem 4, choose a non-singular matrix V and calculate U such that $UV' = I - YX$, then we have

$$\begin{aligned} A_c &:= U^{-1}(\bar{A}_c - [\,Y \quad \bar{B}_c\,] \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} X \\ \bar{C}_c \end{bmatrix})(V')^{-1}, \\ B_c &:= U^{-1}\bar{B}_c, \\ C_c &:= \bar{C}_c(V')^{-1}, \quad j = 1,\ 2. \end{aligned} \tag{50}$$

In what follows, we assume Problem 4 is feasible with a common LMI solution. The superscript (i) denotes the iteration number and takes the values $i = 1, 2, 3, \ldots$ Moreover, let the subscript j takes the values $j = 1,\ 2$ in the rest of this section. Note that $\hat{Y}$ is given by $\hat{Y} = -V^{-1}XU$ from (17).

[Iterative Algorithm I]

Step 1. Set $A_c^{(0)}$, $B_c^{(0)}$, and $C_c^{(0)}$ to A_c, B_c, and C_c calculated as in (50) with the optimal solutions X, Y, $\bar{A}_c$, $\bar{B}_c$, and $\bar{C}_c$ to Problem 4 of a common LMI solution. Then compose the controller variable $\hat{G}^{(0)}$ with $A_c^{(0)}$, $B_c^{(0)}, C_c^{(0)}$, and $D_c = 0$ as in (7).

Step 2. In Problem 6', set $L_j = \hat{B}'P_j^{(i-1)}$, $M = \hat{G}^{(i-1)}C$, and $N_j = \hat{G}^{(i-1)}\hat{F}_j$.

Step 3. In Problem 6', set $P_j = P_j^{(i)}$, $\hat{G} = \hat{G}^{(i)}$, and $J = J^{(i)}$.

Step 4. Solve Problem 6' with respect to $P_j^{(i)} > 0$ and $\hat{G}^{(i)}$.

Step 5. If $\|J^{(i-1)} - J^{(i)}\| < \varepsilon$ for some $\varepsilon > 0$, e.g., $\varepsilon = 1 \times 10^{-3}$, then stop.

Step 6. Set $i \rightarrow i + 1$ and return to Step 2.

Then we derive the following theorem.

Theorem 1 *If we carry out* Iterative Algorithm I, *Problem 6' is always feasible, and a sequence of the controller variable $\hat{G}^{(i)}$ is given by Step 4 such that*

(a) $\|H_{z_1w_1}(s)\|_2^2 \leq \cdots \leq J^{(i)} \leq J^{(i-1)} \leq \cdots \leq J^{(1)} \leq J^{(0)}$;
(b) $\|H_{z_2w_2}(s)\|_\infty < 1$.

Proof: Problem 5' reduces to Problem 4 of a common LMI solutions under the constraint of $P_1 = P_2$, hence it is always feasible and has more freedom to generate non-common LMI solutions $P_1 \neq P_2$. From Lemma 3', for the choice of L_j, M, and N_j in Step 2, Problem 6' has a set of feasible solutions $P_j^{(i)} = P_j^{(i-1)}$ and $\hat{G}_j^{(i)} = \hat{G}_j^{(i-1)}$, hence it is always feasible and has more freedom to generate non-common LMI solutions $P_j^{(i)} \neq P_j^{(i-1)}$ and $\hat{G}_j^{(i)} \neq \hat{G}_j^{(i-1)}$ since L_j, M, and N_j have been updated by replacing $(P_j^{(i-2)},\ \hat{G}_j^{(i-2)})$ with $(P_j^{(i-1)},\ \hat{G}_j^{(i-1)})$. This guarantees $J^{(i)} \leq J^{(i-1)}$ and the convergence property. The feasibility of Problems 5' and 6' ensures $\|H_{z_1w_1}(s)\|_\infty < 1$. This completes the proof. ■

Once $\hat{G}$ has been fixed, Problem 5' becomes LMIs and can be solved with non-common LMI solutions $P_1 \neq P_2$. With this property, we can revise Iterative Algorithm I as follows.

[Iterative Algorithm II]

Replace Step 2 of Iterative Algorithm I by the following Step 2a – Step 2c.

Step 2a. In Problem 5', set $P_j = \hat{P}_j^{(i-1)}$, $\hat{G} = \hat{G}^{(i-1)}$, and $J_{ac} = J_{ac}^{(i-1)}$.

Step 2b. Given $\hat{G}^{(i-1)}$, solve Problem 5' with respect to $\hat{P}_j^{(i-1)}$.

Step 2c. In Problem 6', set $L_j = \hat{B}'\hat{P}_j^{(i-1)}$, $M = \hat{G}^{(i-1)}\hat{C}$, and $N_j = \hat{G}^{(i-1)}\hat{F}_j$.

Then we arrive at the following theorem.

Theorem 2 *If we carry out* Iterative Algorithm II*, Problems 5' and 6' are always feasible, and a sequence of the controller variable $\hat{G}^{(i)}$ is given by Step 4 such that*

(a) $\|H_{z_1w_1}(s)\|_2^2 \le \cdots \le J_{ac}^{(i)} \le J^{(i)} \le \cdots \le J_{ac}^{(0)} \le J^{(0)}$*;*
(b) $\|H_{z_2w_2}(s)\|_\infty < 1$.

Proof: The proof is similar to that of Theorem 1, hence it is omitted. ∎

Although Iterative Algorithm II is more complicated than Iterative Algorithm I, it may accelerate the convergence speed of the iterative algorithm.

Let $\mathcal{G}_\infty$ denote the set of all controller variables G that satisfy $\|H_{z_2w_2}(s)\|_\infty < 1$. In the above two algorithms, we used the optimal solutions to Problem 4, say $\hat{G}_{com}$, for their initial values. However, even if Problem 4 is infeasible, whenever we obtain a controller variable $\hat{G} \in \mathcal{G}_\infty$ that well defines $\|H_{z_1w_1}(s)\|_2^2$, then we can carry out both algorithms starting with this $\hat{G}$ instead of $\hat{G}_{com}$.

6 Less Conservative Design for Regional Pole Placement

With regard to Problem 2 of the regional pole placement, we have the following lemma.

Lemma 4 *The eigenvalues of A_{cl} are in the LMI region $\mathcal{S}(\alpha, r, \theta)$ if and only if the following inequalities hold.*

$$P_v > 0, \tag{51}$$

$$P_v A_{cl} + A_{cl}' P_v + 2\alpha P_v < 0, \tag{52}$$

$$P_d > 0, \tag{53}$$

$$\begin{bmatrix} -rP_d & P_d A_{cl} \\ A_{cl}' P_d & -rP_d \end{bmatrix} < 0, \tag{54}$$

$$P_c > 0, \tag{55}$$

$$\begin{bmatrix} \sin\theta(P_c A_{cl} + A_{cl}' P_c) & \cos\theta(P_c A_{cl} - A_{cl}' P_c) \\ \cos\theta(A_{cl}' P_c - P_c A_{cl}) & \sin\theta(P_c A_{cl} + A_{cl}' P_c) \end{bmatrix} < 0. \tag{56}$$

Proof: In Theorem 2.2 of [2], defining $P := X^{-1} > 0$, we pre- and post-multiply (8) in [2] by $I_m \otimes P$ (unlike our definition, this m denotes sizes of LMI functions to describe LMI regions). Adding subscripts v, d, and c to each LMI solution $P > 0$ corresponding to each LMI region of a vertical strip $\mathcal{S}(\alpha)$, a disk $\mathcal{S}(r)$, and a conic sector $\mathcal{S}(\theta)$ to allow for non-common LMI solutions, we have (51)-(56) corresponding to (20)-(22) in [2]. ∎

By using a similar technique of the preceding section, after substituting $A_{cl} = A + BGC$ into (52), (54), and (56), we complete the square in terms of G and perform the Schur complement formula, then we have

$$\begin{bmatrix} \Phi_A(P_v) + 2\alpha P_v - \underline{P_v BB' P_v} - \underline{C'G'GC} & \text{sym.} \\ H(P_v, G) & -I \end{bmatrix} < 0, \tag{57}$$

$$\begin{bmatrix} -rP_d - \underline{P_d BB' P_d} & P_d A & P_d B \\ A' P_d & -rP_d - \underline{C'G'GC} & C'G' \\ B' P_d & GC & -I \end{bmatrix} < 0, \tag{58}$$

$$\begin{bmatrix} \Psi_c(P_c, G) & & & & & \\ \Gamma(P_c)' & \Psi_c(P_c, G) & & \text{sym.} & & \\ \cos\theta B_2' P_c & GC & -I & & & \\ -GC & \cos\theta B_2' P_c & 0 & -I & & \\ H(P_c, G) & 0 & 0 & 0 & -\kappa & \\ 0 & H(P_c, G) & 0 & 0 & 0 & -\kappa \end{bmatrix} < 0, \tag{59}$$

where

$$\Psi_c(P_c, G) := \sin\theta \cdot \Phi_A(P_c) - (\sin\theta + \cos^2\theta)\underline{P_c BB' P_c} - (\sin\theta + 1)\underline{C'G'GC},$$

$$\Gamma(P_c) := \cos\theta(P_c A - A' P_c), \tag{60}$$

$$\kappa := (\sin\theta)^{-1}. \tag{61}$$

In every case, the BMI terms $P_j BGC$, $j = v, d, c$ have been converted into LMI terms at the expense of generating non-positive quadratic terms $-P_j BB' P_j$, $j = v, d, c$ and $-C'G'GC$ in diagonal blocks. Again, we use Lemma 3 to tackle these terms. Replacing the non-positive quadratic terms (underlined) in (57)-(59) by their upper bounds in (36) and (37), we derive the following problem.

Problem 7 *Find a controller Σ_c that achieves*

$$P_v > 0, \tag{62}$$

$$\begin{bmatrix} \Phi_v + 2\alpha P_v & H(P_v, G)' \\ H(P_v, G) & -I \end{bmatrix} < 0, \tag{63}$$

$$P_d > 0, \tag{64}$$

$$\begin{bmatrix} \Phi_B(P_d, L_d) - rP_d & P_d A & P_d B \\ A' P_d & \Sigma(G, M) - rP_d & C'G' \\ B' P_d & GC & -I \end{bmatrix} < 0, \tag{65}$$

$$P_c > 0, \tag{66}$$

$$\begin{bmatrix} \Phi_c & & & & & \\ \Gamma(P_c)' & \Phi_c & & \text{sym.} & & \\ \cos\theta B' P_c & GC & -I & & & \\ -GC & \cos\theta B' P_c & 0 & -I & & \\ H(P_c, G) & 0 & 0 & 0 & -\kappa & 0 \\ 0 & H(P_c, G) & 0 & 0 & 0 & -\kappa \end{bmatrix} < 0. \tag{67}$$

where

$$\Phi_v := \Phi_A(P_v) + \Phi_B(P_v, L_v) + \Sigma(G, M),$$

$$\Phi_c := \sin\theta \cdot \Phi_A(P_c) + (\sin\theta + \cos^2\theta)\Phi_B(P_c, L_c) + (\sin\theta + 1)\Sigma(G, M).$$

Note that $\Phi_A(P_j)$, $\Phi_B(P_j, L_j)$, $\Sigma(G, M)$, and Γ are defined in (35)-(37) and (60), respectively.

Based on Problem 7 and the problem (51)-(56) with $A_{cl} = A + BGC$, we can develop similar iterative algorithms in the same manner as in the preceding section. Besides, Problem 7 can be combined with Problem 6 to solve the joint problem of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control and the regional pole placement.

7 State-Feedback Case

As pointed out in Section 2, if x is available for feedback, all the results developed in Sections 5 and 6 are immediately applicable to state-feedback controllers in both static and dynamic cases by setting $C = I$ and $F_j = 0$.

8 Numerical Example

In order to demonstrate the applicability of the proposed method, we consider a simple example of a *two mass - one spring system* as shown in Figure 3.

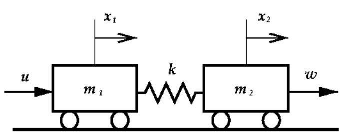

Figure 3: Uncertain two mass - one spring system

In this system, the masses $m_1 = m_2 = 1$ are fixed, the spring constant k can vary from 1.0 to 1.5, and the input matrix B includes ±2% uncertainty. The coefficient matrices are given as follows:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1.25 & 1.25 & 0 & 0 \\ 1.25 & -1.25 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C = [0 \;\; 1 \;\; 0 \;\; 0], \quad F_1 = 0.1, \quad F_2 = [0 \;\; 0],$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.25 & 0.1 \\ 0.25 & 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix},$$

based on the following uncertainty structure:

$$[\Delta A \;\; \Delta B] = B_2 \Delta(t) [C_2 \;\; E_2], \;\; \Delta(t)'\Delta(t) \le I, \quad (68)$$

where $\Delta(t) = \mathrm{diag}(\delta_2(t), \delta_1(t))$, $|\delta_i(t)| \le 1$. From the *small gain theorem*, the robust stability condition is equivalent to $\|H_{z_2 w_2}(s)\|_\infty < 1$. Under this $\mathcal{H}_\infty$ constraint, we minimize $\|H_{z_1 w_1}(s)\|_2^2$ by the proposed iterative algorithms that allow for non-common LMI solutions.

When we seek a common LMI solution as in (27) in Problem 4, we have a dynamic output-feedback controller Σ_{cd} as follows.

$$K_{cd}^{(0)}(s) = \frac{-7.7853(s+0.1471)(s-1.7190)(s-3.1178)}{(s^2+0.7253s+11.7766)(s^2+2.9310s+4.7692)},$$

which yields an upper bound on $\|H_{zw}(s)\|_2^2 \le 26.5497$. With this $K_{cd}^{(0)}(s)$, the actual $\mathcal{H}_2$ cost is found to be $\|H_{zw}(s)\|_2^2 = 12.0765$.

With Iterative Algorithm II and Theorem 2, the controller variable $\hat{G}^{(i)}$ are updated iteratively from the initial value $\hat{G}^{(0)}$ with the $\mathcal{H}_2$ upper bound $J^{(i)}$ and the actual $\mathcal{H}_2$ cost $J_{ac}^{(i)}$ as shown in both Table 1 and Figure 4.

Table 1: The result calculated by Iterative Algorithm II

i	upper bound $J^{(i)}$	actual $\mathcal{H}_2$ cost $J_{ac}^{(i)}$	pure $\mathcal{H}_2$ cost (LQG)
0	26.5497	12.0765	4.0617
1	11.2872	10.8001	|
2	10.2750	9.9452	|
3	9.5842	9.3669	|
⋮	⋮	⋮	|
71	5.5034	5.5027	|
72	5.5024	5.5015	4.0617

For $i = 72$, the algorithm stops under the convergence tolerance $\varepsilon = 1 \times 10^{-3}$, and we have

$$K_{cd}^{(72)}(s) = \frac{-7.0027(s+0.2833)(s^2-1.1067s+3.9208)}{(s^2+0.2123s+12.2955)(s^2+1.6698s+2.2456)},$$

which yields an upper bound on $\|H_{zw}(s)\|_2^2 \le 5.5024$. With this $K_{cd}^{(72)}(s)$, the actual $\mathcal{H}_2$ cost is found to be $\|H_{zw}(s)\|_2^2 = 5.5015$, which is a considerable improvement. For the reference, the $\mathcal{H}_2$ cost of the pure $\mathcal{H}_2$ controller, namely, that of the LQG controller is calculated as $\|H_{zw}(s)\|_2^2 = 4.0617$. However, it is no longer an $\mathcal{H}_\infty$ controller.

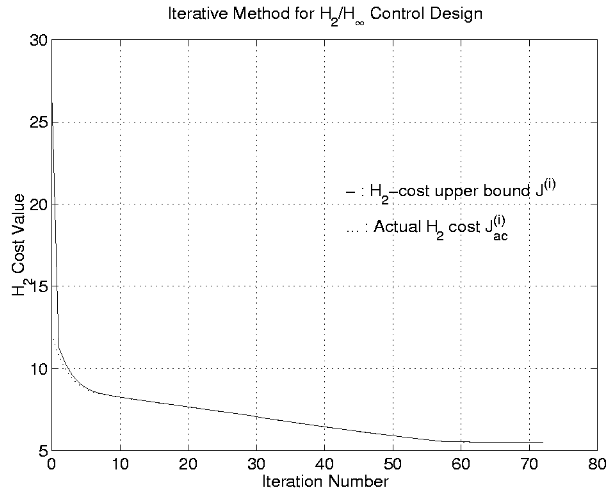

Figure 4: The result calculated by Iterative Algorithm II

9 Conclusion

In this paper, we have proposed a less conservative design procedure for multiobjective control design. Although problems are described by a set of LMIs, they are solved with non-common LMI solutions to reduce the conservatism arising from seeking a common LMI solution. Noticing that completing the square can split two variables in BMI terms into two different LMI ones, we propose iterative algorithms while replacing non-positive quadratic terms by their upper bounds. In addition, it was shown that a suitable choice of the parameters in these upper bounds guarantees their convergence property. Although we have assumed $D_{cl}^{(j)} = 0,\ j = 1, 2, \ldots, N$ for simplicity, it is straightforward to remove this assumption to obtain more general results.

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