Asymptotic Behaviour of the Norm of Input-Output Operators Corresponding to Singularly Perturbed Systems with Multiplicative White Noise

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Abstract

In this paper, we study the problem of asymptotic property of the norm of input-output operators related to a class of singularly perturbed stochastic linear systems. The system is under perturbation of multiplicative white noise. By using reduction order and boundary layer techniques, it is shown that the norm of the operator of the perturbed system is less than a given number γ when the small perturbation ε tends to zero, if both the related norms of the reduced subsystem and the boundary layer subsystem are less than γ.

1. Introduction

Singularly perturbed control systems (SPCS) evolving in discrete time scale arise in many applications as well as in the construction of the difference approximations of SPCS evolving in continuous time. A great amount of effort has been made on singularly perturbed control systems in the past three decades, see, e.g. [1] and the references therein. A popular approach adopted to deal with these systems is based on the so called reduced technique [2]. The composite design based on separate designs for slow and fast subsystems has been systematically reviewed in [3]. Moreover, a number of averaging type methods allowing to treat general SPCS in continuous time were developed recently (see, [4, 5, 6]). These methods are much more adaptable to the discrete time scale. For example, a full analogy between the averaging procedures in problems of optimal control of SPCS evolving in continuous and discrete times was established in [7]. The research on singularly perturbed systems in H_∞ sense is of great practical importance which has attracted a lot of interest in the last fews years, see, e.g. [8]. The state-space solution of the H_∞ control problem [9] was used to approximate the solution of singularly perturbed H_∞ control using slow and fast subproblems [10]. A sequential procedure was described in [8] to decompose the problem into slow and fast subproblems, and a composite compensator was provided. Recently, the H_∞-optimal control of singularly perturbed linear systems, under either perfect state measurements or imperfect state measurements, for both finite and infinite horizons has been investigated by [11, 12] via a differential game theoretic approach. In the meantime, [13] studied the asymptotic expansions for game theoretic Riccati equations and showed how they may be used in singularly perturbed H_∞ control. More recently, [14] considered a construction of high-order approximations to a controller that guarantees a desired performance level on the basis of the exact decomposition of the full-order Riccati equations to the reduced-order slow and fast equations. The problems of H_∞-norms and disturbance attenuation for systems with fast transients have been tackled by [15], and it has been shown that for a singularly perturbed system, the H_∞ of the transfer function tends to the largest of the H_∞ norms for the boundary layer system and for the reduced slow model. A composite linear controller has been designed in [16], based on the slow and fast problems such that both robust stability and a prescribed H_∞ performance for the full-order system are achieved, irrespective of the uncertainties. The problem of H_∞ control for singularly perturbed linear continuous-time systems with Markovian jump parameters has been studied in [17], in which the asymptotic structure of composite mode-dependent controller is characterized.

It is worthwhile to mention that an important issue in the theory of SPCS is a justification of a so called reduction technique approach (RTA). According to this approach the fast variables are replaced by their steady states obtained with “frozen” slow variables and controls, and the slow dynamics is approximated by the corresponding reduced order system. Although the RTA may fail to provide a proper approximation for the SPCS in a general case ([4, 5]), its application was very successful in many important special cases (see [18, 19] and the references therein). In the differential game context the efficiency of the RTA was established for SP linear quadratic games in [20, 21] and for SP H^∞ problem with linear dynamics in [11, 12].

On the other hand, the control of stochastic systems with multiplicate white noise has received much attention in the past half century. For the results concerning the stability for stochastic systems with state dependent noise, we refer readers to, for example, [22, 23, 24, 25] and the references therein. The linear quadratic problem associated to a linear stochastic systems with multiplicative white noise was investigated, for example, [26, 27]. While robust stabilization for the above class of stochastic systems was intensively studied in [28, 29] and the references therein.

In this paper, we investigate the asymptotic behaviour of the input-output operator norm of the singularly perturbed linear continuous-time systems with multiplicative white noise. We consider the norms of both slow/reduced subsystem and fast/boundary layer subsystem. We demonstrate that when the perturbation ε goes to zero, then the input output norm of the original system is less than the maximum of the norms corresponding to the both subsystems.

2. Problem Formulation

Let us consider the linear controlled system described by its differential equations:

$$\begin{aligned} dx_1(t) &= \Big[A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t)\Big]dt \\ &+ \sum_{j=1}^{N}\Big[A_{11}^j x_1(t) + A_{12}^j x_2(t)\Big]dw_j(t) \\ \varepsilon dx_2(t) &= \Big[A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t)\Big]dt \\ &+ \varepsilon^\nu \sum_{j=1}^{N}\Big[A_{21}^j x_1(t) + A_{22}^j x_2(t)\Big]dw_j(t) \end{aligned} \tag{2.1}$$

and the output

$$y(t) = C_1x_1(t) + C_2x_2(t) + Du(t) \tag{2.2}$$

where $x_i \in \mathbf{R}^{n_i}, i = 1, 2$ are state vectors, $u \in \mathbf{R}^m$ is the input vector, $A_{lk}, A_{lk}^j, B_l, C_k, l, k = 1, 2, j = 1, 2, ..., N, D$are real matrices with corresponding dimensions, $\varepsilon > 0$ small parameter, $\nu > \frac{1}{2}$. $w(t) = (w_1(t), w_2(t), ..., w_N(t)), t \geq 0$ is a standard Wiener process as a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

We consider also, the uncontrolled system associated to (2.1)

$$\begin{aligned} dx_1(t) &= \Big[A_{11}x_1(t) + A_{12}x_2(t)\Big]dt \\ &+ \sum_{j=1}^{N}\Big[A_{11}^j x_1(t) + A_{12}^j x_2(t)\Big]dw_j(t) \\ \varepsilon dx_2(t) &= \Big[A_{21}x_1(t) + A_{22}x_2(t)\Big]dt \\ &+ \varepsilon^\nu \sum_{j=1}^{N}\Big[A_{21}^j x_1(t) + A_{22}^j x_2(t)\Big]dw_j(t) \end{aligned} \tag{2.3}$$

For each $\varepsilon > 0$ fixed, we denote $\Phi(t, t_0, \varepsilon)$ the fundamental random matrix solution associated to (2.3).

Now, we recall the following definition.

Definition 2.1 *We say that the zero solution of the system (2.3) is exponentially stable in mean square (ESMS) if there exist* $\alpha > 0, \beta \geq 1$ *such that* $E|\Phi(t, t_0, \varepsilon)x_o|^2 \leq \beta e^{-\alpha(t-t_0)}|x_0|^2, \forall t \geq t_0 \geq 0, x_o \in \mathbf{R}^{n_1+n_2}$.

For each $t \geq 0$ we denote $\mathcal{F}_t \subset \mathcal{F}$ the smallest σ- algebra containing all sets $S \in \mathcal{F}$ with $\mathcal{P}(S) = 0$ and with respect to which all functions $w_j(s), 0 \leq s \leq t, 1 \leq j \leq N$ are measurable.

Let $L_w^2\{[0, \infty) \times \Omega, \mathbf{R}^d\}$ be the set of all functions $u \in L^2\{[0, \infty) \times \Omega, \mathbf{R}^d\}$ with additional property that $u(t)$ are $\mathcal{F}_t$ - measurable for all $t \geq 0$.

Since $\mathcal{F}_t$ contains all sets $S \in \mathcal{F}$ with $\mathcal{P}(S) = 0$, it can be proved that $L_w^2\{[0, \infty) \times \Omega, \mathbf{R}^d\}$ is closed in $L^2\{[0, \infty) \times \Omega, \mathbf{R}^d\}$ and hence it is itself a real Hilbert space with the inner product

$$< u, v >_{L_w^2} = E\int_0^\infty u^*(t)v(t)dt$$

Let us suppose that the zero solution of the system (2.3) is ESMS.

If $u \in L_w^2\{[0, \infty) \times \Omega, \mathbf{R}^m\}$ we denote by $x(t, \varepsilon, u) = \begin{pmatrix} x_1(t, \varepsilon, u) \\ x_2(t, \varepsilon, u) \end{pmatrix}$ the solution of the system (2.1) with initial zero condition (i.e. $x_1(0, \varepsilon, u) = 0, x_2(0, \varepsilon, u) = 0$).

Applying Proposition 1 in [30] we deduce that $x(t, \varepsilon, u) \in L_w^2\{[0, \infty) \times \Omega, \mathbf{R}^{n_1+n_2}\}$ and $\lim_{t\to\infty} E|x(t, \varepsilon, u)|^2 = 0$.

Moreover, there exists $\gamma > 0$ such that

$$\begin{aligned} ||x(\cdot; \varepsilon, u)||^2 &= E\int_0^\infty |x(t, \varepsilon, u)|^2 dt \\ &\leq \gamma^2 E\int_0^\infty |u(t)|^2 dt = \gamma^2||u||^2. \end{aligned} \tag{2.4}$$

Thus is well defined the linear operator $\mathbf{T}_\varepsilon : L^2_w([0,\infty)\times \Omega, \mathbf{R}^m) \to L^2_w([0,\infty)\times\Omega,\mathbf{R}^p)$ by

$$(\mathbf{T}_\varepsilon u)(t) = (C_1 \quad C_2)x(t,\varepsilon,u) + Du(t),\ \forall t \geq 0. \quad (2.5)$$

From (2.4) we deduce that T_ε is linear bounded operator and it will be called input-output operator associated to the system (2.1)-(2.2) and the system (2.1)-(2.2) will be termed "state space realization" of the operator $\mathbf{T}_\varepsilon$.

Remark 2.1 *When the system (2.1) is a deterministic one (i.e. $A^j_{lk} = 0, l,k = 1,2, j = 1,2,...,N$), the transfer matrix function G is the frequency domain version of the input output operator. However in the stochastic framework, we are not able to define in a standard way a transfer matrix function associated to system (2.1)-(2.2) and therefore we consider input-output operators instead of transfer matrices even the coefficients of the given system are time invariant.*

Our aim in this paper is to investigate the asymptotic behaviour of the norm of operator $\mathbf{T}_\varepsilon$ when ε approaches zero.

We shall extend the results in [15, 31] to the case of controlled systems described by Ito differential equations of type (2.1)-(2.2).

To this end we associate to system (2.1)-(2.2) two systems with lower dimensions not depending upon small parameter ε, namely the reduced subsystem and boundary layer subsystem.

Setting $\varepsilon = 0$ and assuming that A_{22} is an invertible matrix, we can associated the following reduced subsystem corresponding to system (2.1)-(2.2):

$$\begin{aligned} dx_1(t) &= \Big[A_r x_1(t) + B_r u(t)\Big]dt + \sum_{j=1}^{N} A^j_r x(t) dw_j(t) \\ y_r(t) &= C_r x_1(t) + D_r u(t) \end{aligned} \quad (2.6)$$

where $A_r = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $A^j_r = A^j_{11} - A^j_{12}A_{22}^{-1}A_{21}$, $B_r = B_1 - A_{12}A_{22}^{-1}A_{21}$, $C_r = C_1 - C_2A_{22}^{-1}A_{21}$, $D_r = D - C_2A_{22}^{-1}B_2$. The unforced reduced subsystem is :

$$dx_1(t) = A_r x_1(t)dt + \sum_{j=1}^{N} A^j_r x_1(t)dw_j(t) \quad (2.7)$$

If the zero solution of the system (2.7) is ESMS then we can associate to system (2.6) the corresponding input-output operator :

$$\mathbf{T}_r : L^2_w[(0,\infty)\times\Omega,\mathbf{R}^m] \to L^2_w[(0,\infty)\times\Omega,\mathbf{R}^p]$$

by

$$(\mathbf{T}_r u)(t) = C_r x_1(t,u) + D_r u(t)$$

, where $x_1(\cdot;u)$ is a solution of the system (2.6) with the initial condition $x_1(0,u) = 0$.

To the given system (2.1) we associate the so called boundary layer system, described by

$$\begin{aligned} x'(\tau) &= A_{22}x_2(\tau) + B_2u(\tau) \\ y(\tau) &= C_2x_2(\tau) + Du(\tau) \end{aligned} \quad (2.8)$$

$\tau = \frac{t}{\varepsilon}$, which will be a deterministic system.

The transfer matrix function corresponding to the system (2.8) is $G_f(s) = C_2(sI_{n_2} - A_{22})^{-1}B_2 + D$.

In this paper we shall prove that $||\mathbf{T}_\varepsilon||$ tends to $\max\{||\mathbf{T}_r||, ||G_f||_\infty\}$.

3. Some Preliminary Results

3.1. A Klimusev-Krasovski type result

In this subsection we extend the results of Klimusev-Krakovski [32] to the singularly perturbed systems of Ito differential equations of type (2.3).

Theorem 3.1 *Assume that the matrix A_{22} has eigenvalues located in the half plane $Res < 0$ and the zero solution of the reduced subsystem (2.7) is ESMS. Then there exists $\varepsilon > 0$ such that for arbitrary $\varepsilon \in (0,\varepsilon_0)]$ the zero solution of the full system (2.3) is ESMS.*

Moreover, if $\begin{pmatrix} \Phi_{11}(t,t_0,\varepsilon) & \Phi_{12}(t,t_0,\varepsilon \\ \Phi_{21}(t,t_0,\varepsilon) & \Phi_{22}(t,t_0,\varepsilon) \end{pmatrix}$ is the partition of the fundamental matrix solution $\Phi(t,t_0,\varepsilon)$ of the system (2.3) the following estimates hold:

$$E|\Phi_{11}(t,t_0,\varepsilon)|^2 \leq \beta_1 e^{-\alpha_1(t-t_0)}$$

$$E|\Phi_{12}(t,t_0,\varepsilon)|^2 \leq \beta_1 \varepsilon e^{-\alpha_1(t-t_0)}$$

$$E|\Phi_{21}(t,t_0,\varepsilon)|^2 \leq \beta_2 e^{-\alpha_1(t-t_0)}$$

$$E|\Phi_{22}(t,t_0,\varepsilon)|^2 \leq \beta_2\Big(e^{\frac{-\alpha_2(t-t_0)}{\varepsilon}} + \varepsilon e^{-\alpha_1(t-t_0)}\Big)$$

$\forall t \geq t_0 \geq 0, \varepsilon \in (0,\varepsilon_0], \alpha_i > 0, \beta_i \geq 1, i = 1,2$ being independent of ε, t, t_0.

The result of the above Theorem can be used to design a stabilizing state feedback for the full system (2.1) based on designing of stabilizing feedback gains for the reduced subsystem (2.6) and for the boundary layer subsystem (2.8) respectively.

Corollary 3.1 *Assume $\tilde{F} \in \mathbf{R}^{m\times n_1}$ is chosen such that the zero solution of the closed- loop system*

$$dx_1(t) = [A_r + B_r\tilde{F}]x_1(t)dt + \sum_{j=1}^{N}[A^j_r + B^j_r\tilde{F}]x_1(t)dw_j(t)$$

is ESMS, and $F_2 \in \mathbf{R}^{m\times n_2}$ is chosen such that $A_{22} + B_2F_2$ is a Hurwitz matrix. Then for arbitrary $\varepsilon > 0$ small enough, the control

$$u(t) = F_1x_1(t) + F_2x_2(t)$$

stabilizes the system (2.1) where $F_1 = (I_m + F_2A_{22}^{-1}B_2)\tilde{F} + F_2A_{22}^{-1}A_{21}$.

3.2. Representation formula for the stabilizing solution of a class of algebraic Riccati equations
Let us consider the controlled system described by the differential Ito equation

$$dx(t) = \Big[Ax(t) + Bu(t)\Big]dt + \sum_{i=1}^{N} A^i x(t)dw_i(t) \quad (3.1)$$

and the output

$$y = Cx(t) + Du(t) \quad (3.2)$$

If the uncontrolled system

$$dx(t) = Ax(t)dt + \sum_{i=1}^{N} A^i x(t)dw_i(t) \quad (3.3)$$

associated to (3.1) generates an exponentially stable evolution, then the system (3.1)-(3.2) defines an input-output operator $T : L^2_w\{(0,\infty)\times\Omega, \mathbf{R}^m\} \to :^2_w \{(0,\infty)\times\Omega, \mathbf{R}^p\}$ by $(\mathbf{T}u)(t) = Cx_u(t) + Du(t), t \geq 0$. $x_u(\cdot) \in L^2_w\{(0,\infty)\times\Omega, \mathbf{R}^n\}$ being the solution of (3.14) with initial condition $x_u(0) = 0$.

The result of the following theorem gives stochastic version of the well known bounded real lemma.

Theorem 3.2 *([30, 33]) Under the considered assumptions, the following are equivalent:*
(i) the uncontrolled system (3.3) defines a mean square exponentially stable evolution, and the input-output operator $\mathbf{T}$ *associated to (3.1)-(3.2) verifies*

$$||\mathbf{T}|| < \gamma$$

(ii) $D^*D < \gamma^2 I_m$ *and the algebraic Riccati type equation*

$$A^*X + XA + \sum_{i=1}^{N}(A^i)^*XA^i$$
$$+(XB + C^*D)(\gamma^2 I_m - D^*D)^{-1}(B^*X + D^*C) + C^*C = 0$$

has a unique stabilizing solution $\tilde{X} = \tilde{X}^* \geq 0$.

Recall that $\tilde{X}$ "is a stabilizing solution" of the equation (3.4) if the system:

$$dx(t) = (A + B\tilde{F})x(t)dt + \sum_{i=1}^{N} A^i x(t)dw_i(t)$$

defines a mean square exponentially stable evolution:

$$\tilde{F} = (\gamma^2 I_m - D^*D)^{-1}(B^*\tilde{X} + D^*C)$$

Combining the results in Proposition 3 and Theorem 1 in [30, 33] Morozan we obtain a useful representation formula of the stabilizing solution $\tilde{X}$ of the Riccati type equation (3.4).

Theorem 3.3 *Suppose that (i) in Theorem* **??** *holds. Then the stabilizing solution of the equation (3.4) has the representation formula:*

$$\tilde{X} = \mathcal{P}^0 - \mathcal{P}\mathcal{R}_\gamma^{-1}\mathcal{P}^* \quad (3.4)$$

with $\mathcal{P}^0 : \mathbf{R}^n \to \mathbf{R}^n, \mathcal{P} : L^2_w((0,\infty)\times\Omega, \mathbf{R}^m) \to \mathbf{R}^n, \mathcal{R}_\gamma : L^2_w((0,\infty)\times\Omega, \mathbf{R}^m) \to L^2_w((o,\infty)\times\Omega, \mathbf{R}^m)$ *by*

$$\mathcal{P}^0 = E\int_0^\infty \Phi^*(t,0)C^*C\Phi(t,0)dt$$
$$\mathcal{P}u = E\int_0^\infty \Phi^*(t,0)C^*[C\int_0^t \Phi(t,s)Bu(s)ds + Du(t)]dt$$
$$= E\int_0^\infty \Phi^*(t,0)C^*(Tu)(t)dt,$$
$$\forall u \in L^2_w((0,\infty)\times\Omega, \mathbf{R}^m)$$
$$\mathcal{R}_\gamma = T^*T - \gamma^2 I, \Phi(t,0)$$

being the fundamental random matrix associated to the uncontrolled system (3.3).

4. Main Results

Through this section we assume : **H**1: a) The linear unforced system (2.7) defines an exponentially stable evolution in mean square.

b) A_{22} has the eigenvalues located in half plane $Re(s) < 0$.

Proposition 4.1 *Under the considered assumptions if there exists a sequence* $\{\varepsilon_k\}_{k\in\mathbf{N}}$ *such that* $\lim_{k\to\infty}\varepsilon_k = 0$ *and* $||\mathbf{T}_{\varepsilon_k}|| < \gamma$ *for all* $k \in \mathbf{N}$ *then* $\gamma \geq \max\{||\mathbf{T}_r||, ||\mathbf{G}_f||_\infty\}$.

Proof: Let $\gamma' < \gamma$ be fixed. Then $||\mathbf{T}_{\varepsilon_k}|| < \gamma'$, for all $k \in \mathbf{N}$. Set

$$A(\varepsilon_k) = \begin{pmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{pmatrix}$$
$$A^i(\varepsilon_k) = \begin{pmatrix} A^i_{11} & A^i_{12} \\ \varepsilon_k^{\nu-1}A^i_{21} & \varepsilon_k^{\nu-1}A^i_{22} \end{pmatrix}$$
$$B(\varepsilon_k) = \begin{pmatrix} B_1 \\ \varepsilon_k^{-1}B_2 \end{pmatrix}.$$

Applying Theorem 3.1 we deduce that for all k large enough, the system $dx(t) = A(\varepsilon_k)x(t)dt + \sum_{j=1}^N A^j(\varepsilon_k)x(t)dw_j(t)$ defines an exponentially stable evolution. Using Theorem 3.3 (i)→(ii) we conclude that the algebraic Riccati type equation

$$A^*(\varepsilon_k)X + XA(\varepsilon_k) + \sum_{j=1}^{N} A^{j*}(\varepsilon_k)XA^j(\varepsilon_k)$$
$$+(XB(\varepsilon_k) + C^*D)(\gamma^2 I_m - D^*D)^{-1}$$
$$\cdot(B^*(\varepsilon_k)X + D^*C) + C^*C = 0 \quad (4.1)$$

has a stabilizing solution $X(\varepsilon_k) = X^*(\varepsilon_k) \geq 0$.

From Theorem 3.4 we deduce that the stabilizing solution $X(\varepsilon_k\gamma)$ of the equation (4.1) has the representation formula

$$X(\varepsilon_k,\gamma')=\mathcal{P}^0(\varepsilon_k)-\mathcal{P}(\varepsilon_k)\mathcal{R}^{-1}_{\gamma'}(\varepsilon_k)\mathcal{P}^*(\varepsilon_k).$$

Let $\begin{pmatrix} X_{11}(\varepsilon_k,\gamma') & X_{12}(\varepsilon_k,\gamma') \\ X^*_{12}(\varepsilon_k,\gamma') & X_{22}(\varepsilon_k,\gamma')\end{pmatrix}$ the partition of the solution $X(\varepsilon_k,\gamma')$ compatible with the partition of the coefficient matrix of the system (2.1).

Taking into account the estimations in Theorem 3.1 we shall deduce estimations for $X_{ij}(\varepsilon_k,\gamma')$.

Firstly we remark that

$$-\mathcal{R}_{\gamma'}(\varepsilon_k)=(\gamma'^2-\gamma^2)I-\mathcal{R}_{\gamma}(\varepsilon_k)\geq(\gamma'^2-\gamma^2)I.$$

Since $\mathcal{R}_{\gamma'}(\varepsilon_k)=\mathcal{R}^*_{\gamma'}(\varepsilon_k)$ we deduce that $\mathcal{R}_{\gamma'}(\varepsilon_k)$ is invertible on $L^2_w((0,\infty)\times\Omega,\mathbf{R}^m)$ with bounded inverse. Moreover we have

$$||\mathcal{R}^{-1}_{\gamma'}(\varepsilon_k)||\leq(\gamma'^2-\gamma^2)^{-1/2}$$

for all $k\in\mathbf{N}$ large enough. On the other hand we may write $\mathcal{P}(\varepsilon_k)=\begin{pmatrix}\mathcal{P}_1(\varepsilon_k)\\ \mathcal{P}_2(\varepsilon_k)\end{pmatrix}, \mathcal{P}_j(\varepsilon_k : L^2_w((0,\infty)\times\Omega;\mathbf{R}^m)\to\mathbf{R}^{n_j}, j=1,2,\ \mathcal{P}_j(\varepsilon_k)u = E\int_0^\infty[\Phi^*_{1j}(t,0,\varepsilon_k)C^*_1+\Phi^*_{2j}(t,0,\varepsilon_k)C^*_2]y(t,\varepsilon_k)dt$ where $y(t,\varepsilon_k)=(C_1\quad C_2)\int_0^t\Phi(t,0,\varepsilon_k)B(\varepsilon_k)u(s)ds+Du(t)$.

Using the estimates in Theorem 3.1 we obtain that there exist the constants $c_1>0, c_2>0$ not depending on k but possibly depending on γ', such that:

$$\|\mathcal{P}_1(\varepsilon_k)\|\leq c_1,\quad \|\mathcal{P}_2(\varepsilon_k)\|\leq c_2\varepsilon_k,\quad (\forall)\ k\geq 1.$$

With these inequalities we may conclude that the stabilizing solution $X(\varepsilon_k)$ of the equation (4.1) has the following asymptotic structure: $X(\varepsilon_k)=\begin{pmatrix}X_{11}(\varepsilon_k) & \varepsilon_k X_{12}(\varepsilon_k)\\ \varepsilon_k X^*_{12}(\varepsilon_k) & \varepsilon_k X_{22}(\varepsilon_k)\end{pmatrix}$ with $|X_{ij}(\varepsilon_k)|\leq c_3(\gamma')$ where $c_3(\gamma')>0$ does not depend of k.

We define

$$F(\varepsilon_k)=(F_1(\varepsilon_k)\qquad F_2(\varepsilon_k))=(\gamma'^2 I_m-D^*D)^{-1}$$
$$\cdot[(B^*_1\qquad \frac{1}{\varepsilon_k}B^*_2)X(\varepsilon_k)+D^*(C_1\qquad C_2)].$$

With this notation we obtain that the equation (4.1) is equivalent to a system with unknowns $X_{11}(\varepsilon_k), X_{12}(\varepsilon_k), X_{22}(\varepsilon_k), F_1(\varepsilon_k), F_2(\varepsilon_k)$ and the rest of the proof can be carried out along the same line as that in deterministic framework, [13]. ∇∇∇

Based on the result in Proposition 4.1, we can easily show the following theorem.

Theorem 4.2 *Under the assumptions* $\mathbf{H_1}-\mathbf{H_2}$ *the norm of the input-output operator defined by the system (2.1)-(2.2) verifies*

$$\liminf_{\varepsilon\searrow 0}\|\mathbf{T}_\varepsilon\|\geq max\Big\{\|\mathbf{T}_r\|,\|\mathbf{G}_f\|_\infty\Big\}.$$

The remainder of this section we will consider a controlled system described by

$$dx_1(t)=\Big[A_{11}x_1(t)+A_{12}x_2(t)+B_1u(t)\Big]dt + \sum_{i=1}^N\Big[A^i_{11}x_1(t)+\varepsilon^\mu A^i_{12}x_2(t)\Big]dw_i(t)$$
$$\varepsilon dx_2(t)=\Big[A_{21}x_1(t)+A_{22}x_2(t)+B_2u(t)\Big]dt + \varepsilon^\nu\sum_{i=1}^N\Big[A^i_{21}x_1(t)+A^i_{22}x_2(t)\Big]dw_i(t) \tag{4.2}$$

and the output

$$y(t)=C_1x_1(t)+C_2x_2(t)+Du(t) \tag{4.3}$$

where $x_i, A_{ij}, B_i, C_j, D, \nu$ and ε are as in the system (2.1)-(2.2) and $\mu>0$ is independent of ε.

When $\mu=0$ the system (4.2)-(4.3) is just the system (2.1)-(2.2).

In this case (when $\mu>0$) the reduced subsystem obtained setting $\varepsilon=0$ in (4.2)-(4.3) is (2.6) where $A^j_r=A^j_{11}, j=1,2,...,N$. The corresponding boundary layer subsystem is (2.8).

It is easy to see that the result of Theorem 4.2 holds also for the input-output operator corresponding to system (4.2)-(4.3). In the case of the system (4.2)-(4.3) we may derive a result concerning the superior limit of the norm of the operator $\mathbf{T}_\varepsilon$ defined by this system.

First we present the following result.

Proposition 4.3 *Assume that* $\mathbf{H_1}-\mathbf{H_2}$ *hold for the system (4.2)-(4.3). Then for all*

$$\gamma > max\{\|\mathbf{T}_r\|,\|\mathbf{G}_f\|_\infty\} \tag{4.4}$$

there exist $\varepsilon_0(\gamma)>0$ *such that for arbitrary* $\varepsilon\in(0,\varepsilon_0(\gamma))$ *we have* $\|\mathbf{T}_\varepsilon\|<\gamma$.

From Proposition 4.3, it follows the following theorem.

Theorem 4.4 *Under* $\mathbf{H_1}-\mathbf{H_2}$ *the norm of the input-output operator defined by the system (4.2)-(4.3) verifies*

$$\limsup_{\varepsilon\searrow 0}\|\mathbf{T}_\varepsilon\|\leq max\Big\{\|\mathbf{T}_r\|,\|\mathbf{G}_f\|_\infty\Big\}.$$

Combining the results of Theorem 4.2 and Theorem 4.4 we have

Theorem 4.5 *Under* $\mathbf{H_1}-\mathbf{H_2}$ *the norm of the input-output operator defined by the system (4.2)-(4.3) verifies*

$$\lim_{\varepsilon\searrow 0}\|\mathbf{T}_\varepsilon\|= max\Big\{\|\mathbf{T}_r\|,\|\mathbf{G}_f\|_\infty\Big\}.$$

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