Identification of a 1-dimensional chaotic system: Expectations and limitations for control

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Abstract

The performance of a linear feedback controller designed to stabilize a periodic-orbit of a one-dimensional chaotic system is examined. The case-study illustrates the interaction of identification and control in the context of control of chaos.

1 Introduction

Due to the limited amount of space available, we bypass any motivational ideas. This and any background material may be found in a publication which may be obtained from the authors [1].

2 Problem setting

Consider the system with control:

$$x_{k+1} = 1 - 2(x_k + u_k)^2 \stackrel{\triangle}{=} f(x_k, u_k), \quad x_k \in [-1, 1] \tag{1}$$

where u_k is the control input and x_k is the state. The system has two unstable fixed points when $u_k = 0$: $x = -1$ and $x = 1/2$. The control objective is the stabilization of the latter.

To build a linear model of (1) about $x = 1/2$ from experimental data, we assume a structure of the form:

$$x^m_{k+1} = a + bx_k + cu_k \stackrel{\triangle}{=} L(x_k, u_k), \quad a, b, c \in \mathbb{R}. \tag{2}$$

The model identification stage involves collecting open-loop data $(x_{k_i+1}, x_{k_i}, u_{k_i})$ when x_{k_i} is close to the operating point $x = 1/2$. This level of closeness, ε is an important experimental parameter representing a trade-off between identification effort and model accuracy. In having estimated a, b and c, a feedback control $u(x_k)$ stabilizing (2) is applied to (1) whenever the state is within ε of the target. The accuracy of the estimates depend on the size of the local neighbourhood ε and the amount of data collected N. This dependence is emphasized by denoting the estimates as $\hat{a}_N(\varepsilon)$, $\hat{b}_N(\varepsilon)$ and $\hat{c}_N(\varepsilon)$.

2.1 Performance criteria for parameter identification

Empirical estimates of $\hat{a}_N(\varepsilon)$, $\hat{b}_N(\varepsilon)$ and $\hat{c}_N(\varepsilon)$ are obtained by collecting a sequence of data points $\{(x_{k_i}, u_{k_i}, x_{k_i+1})\}_{i=1}^N$ where $\left|x_{k_i} - \frac{1}{2}\right| < \varepsilon$ and u_{k_i} is a discrete iid stochastic random variable satisfying

$$\begin{aligned} E\{u^n_{k_i}\} = 0 &\quad \text{for } n \text{ odd} \\ 0 < E\{u^n_{k_i}\} < \infty &\quad \text{for all } n \text{ even} \\ \text{and} \quad E\{x_{k_i} u_{k_i}\} = 0 &\quad \text{for all } i = 1, \ldots, N \end{aligned}$$

The parameters are estimated by minimizing the performance

$$J_N(\varepsilon) = \frac{1}{N} \sum_{i=1}^{N} (x_{k_i+1} - L(x_{k_i}, u_{k_i}))^2 \tag{3}$$

which is an approximation for

$$I(\varepsilon) = E_u \left\{ \int_{1/2-\varepsilon}^{1/2+\varepsilon} [f(x,u) - L(x,u)]^2 \frac{dx}{\pi\sqrt{1-x^2}} \right\} \tag{4}$$

The expressions (4) itself is not the actual performance measure which we should be considering but serves as a good approximation.

Both $I(\varepsilon)$ and $J_N(\varepsilon)$ are used later to obtain expressions for the bias and variance of the estimated parameters respectively. The estimates obtained from (3) are stochastic, depending jointly on the N realizations of x_{k_i} and u_{k_i}, whereas the estimates from (4) are deterministic. They can be thought of as the estimates from (3) as $N \to \infty$.

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3 Bias and variance estimates

The bias estimates are estimated by linearizing the distribution in (4) about $x = 1/2$. Substitution into (4) gives:

$$\begin{aligned} a_\infty(\varepsilon) &\approx 3/2 - 89/90\varepsilon^2 + \mathcal{O}(\varepsilon^3) \\ b_\infty(\varepsilon) &\approx -2 - 16/45\varepsilon^2 + \mathcal{O}(\varepsilon^3) \\ c_\infty(\varepsilon) &= -2 - 8/9\varepsilon^2 \end{aligned}$$

The variances of the estimates are determined from simulation and are assumed to be of the form

$$\begin{aligned} \hat{a}_N(\varepsilon) - a_\infty(\varepsilon) &\sim \mathcal{N}(0, C_a\varepsilon/\sqrt{N}) \\ \hat{b}_N(\varepsilon) - b_\infty(\varepsilon) &\sim \mathcal{N}(0, C_b\varepsilon/\sqrt{N}) \\ \hat{c}_N(\varepsilon) - c_\infty(\varepsilon) &\sim \mathcal{N}(0, C_c\varepsilon/\sqrt{N}) \end{aligned}$$

4 Closed-loop analysis

A control stablizing the fixed point of (2) is implemented on (1). The properties of the resulting feedback system are examined, in particular with the view of target destabilization in mind.

4.1 Closed-loop system destabilization

To obtain the closed-loop control of (2), the linear model fixed-point target x_f^m is determined. In order that the linear feedback controller for (2) stabilizes x_f^m exponentially with ratio $|r| < 1$, we require

$$u(x) = -(b-r)(x - a/(1-b)/c.$$

Using this on (1), the closed-loop system is

$$x_{k+1} = 1 - 2(\alpha x_k + \beta)^2, \tag{5}$$

where $\alpha = 1 - (b-r)/c$ and $\beta = a(b-r)/(c(1-b))$. The fixed-point x^*_{CL} of (5) can be calculated, and its stability determined by examining the derivative $-4\alpha(\alpha x^*_{\mathrm{CL}} + \beta)$. To see the effect of the neighbourhood size ε and data size N on this measure of stability, we substitute:

$$\begin{aligned} a &:= 3/2 - 89/90\varepsilon^2 + C_a\varepsilon s/\sqrt{N} \\ b &:= -2 - 16/45\varepsilon^2 + C_b\varepsilon s/\sqrt{N} \\ c &:= -2 - 8/9\varepsilon^2 + C_c\varepsilon s/\sqrt{N} \end{aligned}$$

into the derivative, where $s \sim \mathcal{N}(0,1)$, and make a Taylor series expansions about $\varepsilon = 0$:

$$\begin{aligned} r + \frac{s\varepsilon}{\sqrt{N}}\frac{p_2(r)}{1-r} + \frac{\varepsilon^2}{90}\frac{5r^2 - 62r - 48}{1-r} + \\ \frac{s^2\varepsilon^2}{N}\frac{p_4(r)}{(1-r)^3} + \mathcal{O}\left(\frac{s^3\varepsilon^3}{N^{3/2}}, \varepsilon^3\right) \qquad (6) \\ \xrightarrow{N\to\infty} \quad r + \frac{\varepsilon^2}{90}\frac{5r^2 - 62r - 48}{1-r}, \end{aligned}$$

where $p_i(r)$ is a degree-i polynomial in r.

4.1.1 Discussion

By writing (6) as

$$r + \frac{\varepsilon^2}{90}\frac{5r^2 - 62r - 48}{1-r} + \mathcal{O}\left(\frac{s\varepsilon}{\sqrt{N}}, \frac{\varepsilon^4}{(1-r)^3}\right) \tag{7}$$

we see that only the first two terms remain when $N \to \infty$. By making $\sqrt{N} \gg \mathcal{O}(1/\varepsilon)$, the higher order terms will have little effect on the stability of the system, and thus collecting much more than $\mathcal{O}(1/\varepsilon^2)$ data points will not result in significant improvement on the stability margin.

Based on the model, the control design may infer a satisfactory probability of successful stabilization provided the effect of the error in identification is less than the robustness margin of the control law. This can be captured by

$$\text{Bias} + \text{Variance} < \text{Robustness margin},$$

or

$$\mathcal{O}(\varepsilon^2) + \mathcal{O}\left(\frac{\varepsilon}{\sqrt{N}}\right) < \mathcal{O}(1 - |r|).$$

This is a re-interpretation of (7) where

$$\left| r + \frac{\varepsilon^2}{90}\frac{5r^2 - 62r - 48}{1-r} + \mathcal{O}\left(\frac{s\varepsilon}{\sqrt{N}}\right)\right| < 1$$

for stability.

5 Conclusion

A 1-dimensional chaotic map possessing an absolutely continuous invariant ergodic measure has been studied. The system was subjected to analysis into the limitations of identifying local linear map parameters. It was found that the errors were located in two terms: a bias term, which scaled quadratically with the local neighbourhood size ε, and a variance term which scaled according to $\varepsilon/\sqrt{N}$.

A closed-loop analysis of the control system utilizing the controller based on the identified parameters found that the closed-loop stability also contained bias and variance terms with similar scaling laws.From a control perspective this places bounds on the data requirements for system identification.

A more in-depth discussion of the study, covering additional scaling laws and generalization of the method is covered in a publication which can be obtained from the authors [1].

References

[1] E. Lim and I. Mareels. Identification of a one-dimensional chaotic system. `http://www.ee.mu.oz.au/pgrad/elim/1dim.ps`. Department of Electrical and Electronic Engineering, the University of Melbourne, 2000.