On the H^2 and H^∞ Dead-Time Compensation[1]

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Abstract

In the paper the structure of the H^2 and H^∞ suboptimal controllers is studied. In both cases the suboptimal controllers have the form of the dead-time compensator reminiscent of that of the famous Smith predictor. It is shown that unlike the H^2 case, the H^∞ dead-time compensator takes into account the exogenous disturbance signal. Since the disturbance is not measurable, the H^∞ dead-time compensator "predicts" it on the basis of the worst-case scenario.

1 Introduction

The dead-time (DT) compensation is one of the classical approaches to the feedback control of time delay systems. The essence of the approach is to use a predictor block to compensate the delay and draw it out of the feedback loop in the analysis. Since the famous Smith predictor [1], there were numerous works on DT compensators (DTC) and their configurations, see [2] and the references therein. Yet, although the original idea of Smith was motivated by the LQ optimal control, in most further research the structures of the DTC are postulated on the basis of some *ad hoc* reasoning associated with either control goals or implementation requirements. Roughly speaking, DT compensation still lacks rigorous justification.

In this paper the DT controller structures associated with the H^2 and H^∞ suboptimal problems for systems with a single delay in the loop are considered. It follows from the recent works [3, 4, 5] that both H^2 and H^∞ regulators can be presented in the form of DTC with FIR (finite impulse response) predictor blocks (this, in fact, justifies the use of the DTC structure for feedback control of DT systems). There are, however, several differences between H^2 and H^∞ DTC which have to be explained in order to gain further insight in the controller structure. The purpose of this paper is to account for these differences and study properties and interpretations of the H^2 and H^∞ DTC. In particular, it is shown that the H^∞ DTC attempts to compensate DT in the feedback loop by generating the disturbance signal artificially on the basis of the worst-case scenario. This is unlike to the H^2 case, where DT compensation is performed in the disturbance-free setting.

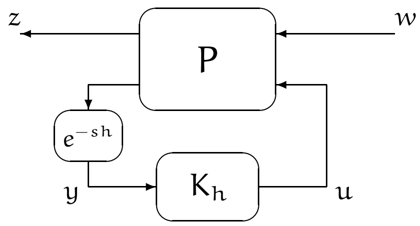

Figure 1: General LFT dead-time setup

Notations* The notations throughout the paper are rather standard. Given an LTI strictly proper system $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$, its conjugate $G^{\sim}(s) = \left[\begin{array}{c|c} -A' & -C' \\ \hline B' & 0 \end{array}\right]$. The *completion* of $e^{-sh}G(s)$ to $[0, h]$ is defined as follows: $\boldsymbol{\pi}_h\{e^{-sh}G(s)\} \doteq \left[\begin{array}{c|c} A & B \\ \hline Ce^{-Ah} & 0 \end{array}\right] - e^{-sh}\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$. It can be seen that $\boldsymbol{\pi}_h\{e^{-sh}G\}$ is entire function whose impulse response has support on $[0, h]$ (FIR system).

2 H^2 and H^∞ control of DT systems

Consider the LFT dead-time setup in Fig. 1, where P is a rational generalized plant and $e^{-sh}K_h$ is a (dead-time) controller. It is supposed that

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right], \qquad (1)$$

where the partitioning is compatible with that in Fig. 1 and the following assumptions hold:

(A1): The pair (A, B_2) is stabilizable and the pair (C_2, A) is detectable.

(A2): $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank $\forall \omega \in \mathbb{R}$.

[1] This research was supported by THE ISRAEL SCIENCE FOUNDATION founded by The Israel Academy of Sciences and Humanities.

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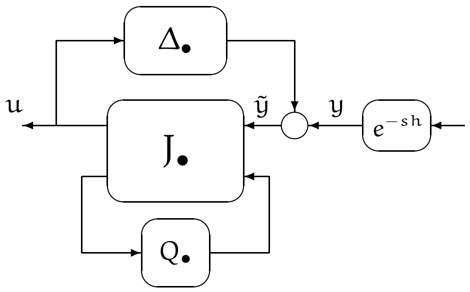

Figure 2: Parametrization of all solutions to $\mathsf{OP}_{\mathrm{H}^\bullet}$ ('$\bullet$' stands for either '2' or '∞').

(A3): $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank $\forall \omega \in \mathbb{R}$.

(A4): $D_{12}' D_{12} = I$ and $D_{21} D_{21}' = I$.

Assumption (A4) is made just to simplify the exposition. In fact, only the nonsingularity of the matrices $D_{12}' D_{12}$ and $D_{21} D_{21}'$ is required. Yet the latter assumptions can easily be reduced to (A4) by an appropriate scaling of the plant and controller parameters, see [6, p. 168].

In the paper the following problems is considered (below, '$\bullet$' stands for either '2' or '∞'):

$\mathsf{OP}_{\mathrm{H}^\bullet}$: Given the DT setup in Fig. 1 with the plant (1) and time delay h, find (if they exist) all causal K_h which internally stabilize the system and guarantee $\|\mathcal{F}_\ell(P, e^{-sh} K_h)\|_{\mathrm{H}^\bullet} < \gamma$ for a given $\gamma > 0$.

Complete solutions to both $\mathsf{OP}_{\mathrm{H}^2}$ and $\mathsf{OP}_{\mathrm{H}^\infty}$ given below are from [5].

The solution to $\mathsf{OP}_{\mathrm{H}^2}$ is based on the stabilizing solutions[1] $X \geq 0$ and $Y \geq 0$ to the following two standard H^2 algebraic Riccati equations:

$$XA + A'X + C_1'C_1 - F_2'F_2 = 0, \tag{2a}$$

where $F_2 \doteq -(B_2'X + D_{12}'C_1)$, and

$$AY + YA' + B_1B_1' - L_2L_2' = 0, \tag{2b}$$

where $L_2 \doteq -(YC_2' + B_1D_{21}')$. Introduce also the following quantity:

$$\gamma_0 \doteq \sqrt{\operatorname{tr}(B_1'XB_1) + \operatorname{tr}(F_2YF_2')},$$

which is the minimal achievable H^2 norm in the delay-free case ($h = 0$). Then,

[1]Whenever assumptions (A1)–(A4) hold, the stabilizing solutions to (2) exist and are unique.

Theorem 1. *The $\mathsf{OP}_{\mathrm{H}^2}$ is solvable iff $\gamma^2 > \gamma_0^2 + \gamma_h^2$, where*

$$\gamma_h \doteq \sqrt{\int_0^h \operatorname{tr}\big(F_2 e^{At} L_2 L_2' e^{A't} F_2'\big)\, dt}.$$

If this condition holds, then the set of all solutions is parametrized as shown in Fig. 2 subject to (below the notation $A_2 \doteq A + B_2F_2 + e^{Ah}L_2C_2e^{-Ah}$ is adopted)

$$J_2(s) = \left[\begin{array}{c|cc} A_2 & -e^{Ah}L_2 & B_2 \\ \hline F_2 & 0 & I \\ -C_2e^{-Ah} & I & 0 \end{array}\right],$$

$\Delta_2(s) = \boldsymbol{\pi}_h\{e^{-sh}P_{22}(s)\}$, and any stable Q_2 such that $\|Q_2\|_{\mathrm{H}^2}^2 < \gamma^2 - \gamma_{0,2}^2 - \gamma_{h,2}^2$.

Similarly, the solution to the $\mathsf{OP}_{\mathrm{H}^\infty}$ requires the following standard H^∞ ARE's:

$$XA + A'X + C_1'C_1 + \gamma^{-2}XB_1B_1'X - F_2'F_2 = 0, \tag{3a}$$

where $F_2 \doteq -(B_2'X + D_{12}'C_1)$, and

$$AY + YA' + B_1B_1' + \gamma^{-2}YC_1'C_1Y - L_2L_2' = 0, \tag{3b}$$

where $L_2 \doteq -(YC_2' + B_1D_{21}')$. Denote also:

$$\Sigma \doteq \exp\left(\begin{bmatrix} A & \gamma^{-2}B_1B_1' \\ -C_1'C_1 & -A' \end{bmatrix} h\right)$$

and then

$$B_Y \doteq \begin{bmatrix} \gamma^{-2}Y & I \end{bmatrix} \Sigma' \begin{bmatrix} C_1'D_{12} \\ B_2 \end{bmatrix},$$

$$C_X \doteq \begin{bmatrix} -\gamma^{-2}D_{21}B_1' & C_2 \end{bmatrix} \Sigma' \begin{bmatrix} -X \\ I \end{bmatrix},$$

$$Z_h \doteq \left(\begin{bmatrix} \gamma^{-2}Y & I \end{bmatrix} \Sigma' \begin{bmatrix} -X \\ I \end{bmatrix}\right)^{-1}.$$

Then the following result can be formulated:

Theorem 2. *The $\mathsf{OP}_{\mathrm{H}^\infty}$ is solvable iff its delay-free counterpart is solvable and in addition*

(a) $\|P_{11}\|_{L^2[0,h]} < \gamma$,

(b) $\rho\big(X\Sigma_{12}\Sigma_{22}^{-1}\big) < 1$, *and*

(c) $\rho\big((\Sigma_{22} - X\Sigma_{12})^{-1}(\Sigma_{21} - X\Sigma_{11})Y\big) < \gamma^2$.

If these conditions hold, then Z_h is well defined and the set of all solutions is parametrized as shown in Fig. 2 subject to (below, the notation $A_\infty \doteq A + \gamma^{-2}B_1B_1'X + B_2F_2 + Z_hL_2C_X$ is adopted)

$$J_\infty = \left[\begin{array}{c|cc} A_\infty & -Z_hL_2 & Z_hB_Y \\ \hline F_2 & 0 & I \\ -C_X & I & 0 \end{array}\right],$$

$\Delta_\infty = \boldsymbol{\pi}_h\{e^{-sh}\mathcal{F}_u(P, \gamma^{-2}P_{11}^{\sim})\}$, and any stable Q_∞ such that $\|Q_\infty\|_{\mathrm{H}^\infty} < \gamma$.

Remark 2.1. Similar controller structure has already been reported by Meinsma and Zwart in [3] for the mixed sensitivity H^∞ problem and in [7] for the general four-block problem. Yet the controller formulae there are less transparent and consequently the interpretation of the H^∞ DTC is not evident.

Note, that the controllers in Theorems 1 and 2 both have the structure of the feedback interconnection of a rational LFT part (which is similar to controllers for the corresponding delay-free problems) and an FIR block. The latter in both cases has quite intriguing structure with the internal feedback reminiscent of that of the Smith predictor $(G_1 - e^{-sh}G_2)$. The next section is devoted to its analysis.

3 Controller structure

3.1 The H^2 dead-time compensator

Consider the H^2 γ-suboptimal controller in Fig. 2. The measurement signal entering the controller is $y = e^{-sh}(P_{21}w+P_{22}u)$. This signal is then "preprocessed" by adding $\boldsymbol{\pi}_h\{e^{-sh}P_{22}\}u$ and the resulting signal, $\tilde{y}$, enters the rational part of the controller. The rationale behind such a preprocessing becomes apparent when $\tilde{y}$ is considered. Indeed, using the definition of the completion operator $\boldsymbol{\pi}_h\{\cdot\}$ one can easily see that

$$\tilde{y} = P_{21}w_h + \tilde{P}_{22}u,$$

where $w_h = e^{-sh}w$ is the delayed disturbance and

$$\tilde{P}_{22}(s) \doteq \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 e^{-Ah} & 0 \end{array}\right]$$

is finite dimensional. Hence, the resulting feedback loop ($u \circlearrowleft \tilde{y}$) does *not* contain any delay. In other words, the purpose of the FIR block Δ_2 is to "compensate" the delay in the feedback loop, which is reflected in the term DTC. The rational part of the controller is then just a γ-suboptimal controller for the modified plant $\tilde{P}_{22}$ (see [4]).

This structure is similar to the Smith predictor. The latter uses the block $P_{22} - e^{-sh}P_{22}$ to compensate the delay ("predict" the plant output) and can be applied to open-loop stable systems only. The H^2 DTC uses the exact completion of $e^{-sh}P_{22}$ and can be used for the control of unstable systems. A similar scheme was proposed by Watanabe and Ito [8] (though not in the context of optimal control).

Remark 3.1. It is worth stressing that when the plant P_{22} is stable, the H^2 DTC in Theorem 1 can be recast in the Smith predictor form. Indeed, in the system in Fig. 2 the DTC block $\Delta_2 = \tilde{P}_{22} - e^{-sh}P_{22}$ can be replaced with $P_{22} - e^{-sh}P_{22}$ by transforming the rational part J_2 of the controller to $\left[\begin{smallmatrix} 0 & I \\ I & \tilde{P}_{22}-P_{22}\end{smallmatrix}\right] \star J_2$, where "$\star$" stands for the Redheffer star product [9, §10.4].

Note that the delay compensation above is based on the *open-loop* plant model and does not depend on the rest of the controller. Intuitively, this fits well into the H^2 methodology in which disturbances are supposed to be known and modeling is perfect.

3.2 The H^∞ dead-time compensator

Although the H^∞ controller has the structure similar to that of the H^2 controller, the FIR block of the former is more complicated. Instead of completing the open-loop plant $e^{-sh}P_{22}$, it compensates the delay for the rather artificial system $P_a \doteq e^{-sh}\mathcal{F}_u(P, \gamma^{-2}P_{11}^{\sim})$. The natural question then is what is the rationale behind this.

To answer this question, note that

$$P_a = P_{22} + \tfrac{1}{\gamma^2}P_{21}\big(I - \tfrac{1}{\gamma^2}P_{11}^{\sim}P_{11}\big)^{-1}P_{11}^{\sim}P_{12}$$

and thus the relationship $y = P_a u$ can equivalently be written as

$$y = P_{22}u + P_{21}w_*,$$

where the "disturbance" w_* satisfies:

$$w_* = \tfrac{1}{\gamma^2}P_{11}^{\sim}\big(P_{11}w_* + P_{12}u\big).$$

Thus, if the disturbance w in Fig. 1 were equal to the w_* above and were known, then the block $\Delta_\infty = \boldsymbol{\pi}_h\{e^{-sh}P_a\}$ would just compensate the dead time in both feedback and feedforward loops ("predict" y subject to a given w). This agrees well with the result of Palmor and Powers [10], who proposed to transmit measured disturbances into the Smith predictor to improve its disturbance attenuation properties (feedforward Smith predictor). Although the H^∞ DTC does not measure the disturbance, it *generates* if artificially.

Taking into account the worst-case nature of the H^∞ methodology, one would expect that the disturbance w_* is generated on the basis of a worst-case scenario. It turns out that this indeed happens. To see this, consider the state-space realization of P_a:

$$P_a(s) = \left[\begin{array}{cc|c} A & \frac{1}{\gamma^2}B_1B_1' & B_2 \\ -C_1'C_1 & -A' & -C_1'D_{12} \\ \hline C_2 & \frac{1}{\gamma^2}D_{21}B_1' & 0 \end{array}\right]. \tag{4}$$

It is seen now that the relationship $y = P_a u$ can be written as follows:

$$\begin{cases} \dot{x} = Ax + B_1w_* + B_2u \\ y = C_2x + D_{21}w_* \end{cases}$$

and w_* is generated by the system

$$\begin{cases} -\dot{\lambda} = A'\lambda + C_1'z \\ w_* = \frac{1}{\gamma^2}B_1'\lambda \end{cases},$$

where $z = C_1x + D_{12}u$. Using the standard arguments from the calculus of variations [6], the w_* above can roughly be thought of as the maximizing disturbance for the index

$$J = \int_0^\infty (z'z - \gamma^2 w'w)\,dt$$

subject to any fixed u. Thus, the H^∞ DTC *attempts to compensate the dead time* h *assuming that the disturbance* w *is the worst-case one for the* open-loop *problem*.

It is worth stressing that despite all the differences between the H^2 and H^∞ DTC's, both of them are independent of the rest of the controller or, more precisely, of the way the loop is closed by $J_\bullet$. This means that dead-time compensation is inherently an *open-loop* operation.

Remark 3.2. Unlike the H^2 case, the H^∞ DTC cannot in general be recast in the Smith predictor form, i.e., with the irrational block $P_a - e^{-sh}P_a$. The reason is that the latter must be stable in order to guarantee the internal stability of the system [4]. Yet unless the case[2] $P_{11} = 0$, realization (4) has a Hamiltonian "A" matrix which means that P_a always has unstable poles.

4 Concluding remarks

In this paper the structures of the (sub) optimal H^2 and H^∞ controllers for systems with a single delay in the feedback loop have been studied. It has been shown that both controllers include a dead-time compensation block that attempts to draw the delay out of the feedback loop. While the H^2 DTC attempts to do this without taking into account the plant disturbances, the H^∞ DTC compensates the delay assuming that the disturbance is the worst-case one for the open-loop problem.

[2]This is possible in some robust stability problems.

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