# Exponential Stabilization of Motion and Vibration for a Large Space Structure

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**Abstract**

In this papaer, we propose the exponential stabilizing controller of the motion of the rigid mode and the vibration of flexible modes for two flexible beams connected by a spring as a simple example of the large space structures. As the rigid mode is unstable, the original open-loop flexible system is not exponential stable. We propose a direct sensor output feedback control law for the motion and vibration absorption control. Using the spectral analysis, the exponential stability of the closed-loop system is proven. To demonstrate the validity of the proposed model and effectiveness of the proposed control law experiments have been carried out.

## 1 Introduction

Modeling and control problems of flexible structures have been extensively studied by many researchers[1]-[7]. One of the important problem for the control of the large space structures is to compensate the spillover instability caused by physical parameter uncertainty and residual modes which are neglected at the controller design phase. One solution for the problem is to construct a robust controller, for example a robust $H_\infty$ controller and an optimal controller with low-pass property, based on an approximated finite-dimensional model.

On the other hand, the computational power which can be used in the space is not so high. As the order of a robust $H_\infty$ controller and an optimal controller with low-pass property are not small and they need high computational power, the controllers may not be suitable in the space missions. It is necessary to construct a simple controller for the implementation with robustness based on the distributed parameter system of an infinite-dimensional system. Though a simple PDS control law [8] for flexible structures is proposed, it can ensure only asymptotic stability of the closed-loop system. The exponential stabilizing controller for a one-link flexible arm is dicsussed in [9]. We discussed the exponential stabilization for only the vibration of two flexible beams connected by a spring [10].

In this paper, we consider motion and vibration control for the two flexible beams connected by a spring as a simple example of the large space structures. The flexible beams and the spring can be regarded as an element of the structure with distributed flexibility and a connective part with lumped flexibility, respectively. As we introduce Voigt type damping, the original open-loop vibration system is exponential stable. However the original hybrid system of the motion of the rigid mode and the vibration of the flexible system is unstable. We propose a direct sensor output feedback controller for the motion and vibration absorption control. Using the spectral analysis, exponential stability of the closed-loop system is proven. As we don't need an approximated finite-dimensional model at the controller design phase, the controller based on the original distributed parameter system is robust and simple. To demonstrate the validity of the proposed model and the effectiveness of the proposed control law, experiments have been carried out.

## 2 Distributed Parameter Model

We consider two flexible beams connected by a spring as shown in Fig.1.

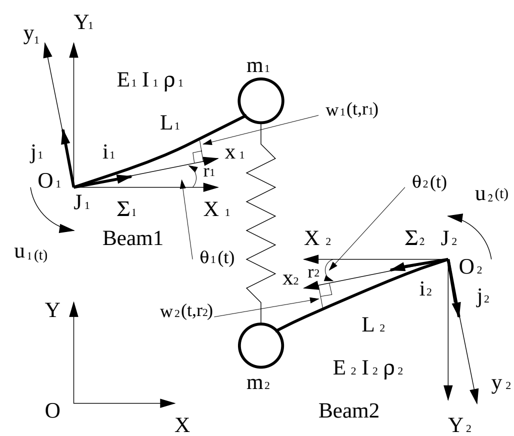



Fig.1 Two flexible beams connected by a spring

The flexible beam $i$ of length $L_i$, having uniform mass density $i$ per unit length, uniform flexural rigidity $E_iI_i$ is clamped on the vertical shaft of the motor $i$ at one end and has a concentrated mass $m_i$ at the other end. The masses $m_1$and $m_2$ are connected by a spring. This spring implies the lumped flexibility of the connected part. Let $\theta_i(t)$ be the angle of rotation of the flexible beam $i$. The flexible beams will deform due to the distributed elasticity as shown in Fig.1. The system of two flexible beams connected by a spring has both lumped and distributed flexibility. As we consider planer motion, the acceleration of gravity can be ignored. Let $w_i(t, r_i)$ denote the transverse displacement at time $t$ and at a spatial point $r$. Let $k$ be a spring constant, the equilibrium point of the spring be the position which satisfies $\theta_1(t) = \theta_2(t) = 0, w_1(t, L_1) = w_2(t, L_2) = 0$. Introducing the assumption $\theta_i(t)$, $(i = 1, 2)$ are small and using the Hamilton's principle yields dynamic equations of vibration $w_i(t, r_i)$. The obtained boundary condition is nonhomogeneous, we introduce new variables

$$v_i(t, r_i) = w_i(t, r_i) - (L_1\theta_1(t) + L_2\theta_2(t))(-3L_i r_i^2 + r_i^3)c_i \quad (1)$$

where $c_i$ is defined as

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{k}{6E_1I_1 + 2k(L_1^3 + L_2^3\frac{E_1I_1}{E_2I_2})} \\ \frac{k}{6E_2I_2 + 2k(L_1^3\frac{E_2I_2}{E_1I_1} + L_2^3)} \end{bmatrix}$$

and derive a homogeneous boundary condition [10]. By using the new variables (1) the vibration equation can be rewritten as

$$\begin{aligned} \ddot{v}_i(t, r_i) + \delta_i \frac{E_iI_i}{\rho_i}\dot{v}_i''''(t, r_i) + \frac{E_iI_i}{\rho_i}v_i''''(t, r_i) \\ = b_{i1}(r_i)u_1(t) + b_{i2}(r_i)u_2(t) \end{aligned}$$

$$\begin{cases} v_i(t, 0) = 0, v_i'(t, 0) = 0, v_i''(t, L_i) = 0 \\ E_iI_i\{\frac{m_i}{\rho_i}v_i''''(t, L_i) + v_i'''(t, L_i)\} \\ \qquad -k\{v_1(t, L_1) + v_2(t, L_2)\} = 0 \end{cases} \quad (2)$$

where $\delta$ is a small damping coefficient, a dot and a prime denote the time derivative and the derivative with respect to the spatial variable $r_i$, and

$$\begin{aligned} b_{ij}(r_i) &= -L_i(-3L_ir_i^2 + r_i^3)c_j - r_i \qquad (i = j) \\ b_{ij}(r_i) &= -L_i(-3L_ir_i^2 + r_i^3)c_j \qquad (i = j). \end{aligned}$$

We assume that the speed reference type servo amplifiers of the motors are used. The equation of rotation of the motor can be regarded as

$$\ddot{\theta}_i(t) = u_i(t). \quad (3)$$

The angular accerelation $\ddot{\theta}(t)$ of the motor is regarded as the input $u(t)$ of the system.

As we introduce the Voigt type damping of the flexible beams, the original open-loop flexible system is exponential stable. However the hybrid system of the vibration of the flexible beams and the rotation of the motor is unstable. An exponential stabilizing controller for the hybrid system should be constructed.

## 3 Control Law and Closed-loop System

We consider the combining system of the equation (2) of the vibration of the flexible beams and the equation (3) of the rotation of motor. It can be regarded as the hybrid system of the rigid mode and the flexible modes. We construct a direct sensor output feedback controller which can ensure the exponential stability for the motion and the vibration absorption.

We propose the hybrid control law as

$$\begin{aligned} u_i(t) &= -k_d\dot{\theta}_i(t) - k_p\theta_i(t) \\ &+ (-1)^{i+1}k_s\{\frac{E_1I_1}{L_1}\dot{v}_1''(t, 0) - \frac{E_2I_2}{L_2}\dot{v}_2''(t, 0)\} \quad (4) \end{aligned}$$

where $k_p, k_d, k_s > 0$. The closed-loop system for the controller (4) is expressed as

$$\ddot{\boldsymbol{v}} + \mathcal{RA}\dot{\boldsymbol{v}} + \mathcal{A}\boldsymbol{v} = \mathcal{D}_p\boldsymbol{\theta} + \mathcal{D}_d\dot{\boldsymbol{\theta}} \quad (5)$$

$$\ddot{\boldsymbol{\theta}} + k_d\dot{\boldsymbol{\theta}} + k_p\boldsymbol{\theta} = \mathcal{C}\dot{\boldsymbol{v}} \quad (6)$$

where

$$\boldsymbol{v} = [v_1(t, r_1), v_2(t, r_2)]^T, \quad \boldsymbol{\theta} = [\theta_1, \theta_2]^T$$

$$\mathcal{A}\boldsymbol{v} = \begin{bmatrix} \frac{E_1I_1}{\rho_1}v_1''''(r_1) \\ \frac{E_2I_2}{\rho_2}v_2''''(r_2) \end{bmatrix}, \quad \mathcal{RA}\dot{\boldsymbol{v}} = (\delta\mathcal{A} + k_s\mathcal{B})\dot{\boldsymbol{v}}$$

$$\mathcal{B}\dot{\boldsymbol{v}} = \begin{bmatrix} \frac{r_1}{L_1}\{\frac{E_1I_1}{L_1}\dot{v}_1''(0) - \frac{E_2I_2}{L_2}\dot{v}_2''(0)\} \\ \frac{r_2}{L_2}\{-\frac{E_1I_1}{L_1}\dot{v}_1''(0) + \frac{E_2I_2}{L_2}\dot{v}_2''(0)\} \end{bmatrix}$$

$$\mathcal{C}\dot{\boldsymbol{v}} = \begin{bmatrix} \frac{k_s}{L_1}\{\frac{E_1I_1}{L_1}\dot{v}_1''(0) - \frac{E_2I_2}{L_2}\dot{v}_2''(0)\} \\ \frac{k_s}{L_2}\{-\frac{E_1I_1}{L_1}\dot{v}_1''(0) + \frac{E_2I_2}{L_2}\dot{v}_2''(0)\} \end{bmatrix}$$

$$\mathcal{D}_p\boldsymbol{\theta} = \begin{bmatrix} -k_pb_{11}(r_1)\theta_1(t) - k_pb_{12}(r_1)\theta_2(t) \\ -k_pb_{21}(r_2)\theta_1(t) - k_pb_{22}(r_2)\theta_2(t) \end{bmatrix}$$

$$\mathcal{D}_d\dot{\boldsymbol{\theta}} = \begin{bmatrix} -k_db_{11}(r_1)\dot{\theta}_1(t) - k_db_{12}(r_1)\dot{\theta}_2(t) \\ -k_db_{21}(r_2)\dot{\theta}_1(t) - k_db_{22}(r_2)\dot{\theta}_2(t) \end{bmatrix}$$

From the analysis of the operator $\mathcal{A}$ we find that
- $0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots$
- $\mathcal{A}\boldsymbol{\psi}_n = \mu_n\boldsymbol{\psi}_n, \qquad n = 1, 2, \cdots$
- The set $\{\boldsymbol{\psi}_n(\cdot)\}$ of the eigenfunction forms a complete orthonormal system.

## 4 Separation of Closed-loop Operator

We consider the separation of the operator related to the closed-loop system (5), (6) by using the following Lemma 1 [9].

**Lemma 1** *Let $\Phi_1$ be a linear operator which generates the $C_0$-semigroup and ensures the exponential stability, and $\Phi_2$ be a linear compact operator in a Hilbert space $\mathcal{H}$. For the differencial equation*

$$\dot{\boldsymbol{X}} = (\Phi_1 + \Phi_2)\boldsymbol{X} \quad (7)$$

*the following statements are equivalent.*

- *The solution of the equation (7) is exponential stable.*
- *The spectral bound of the operator $\Phi_1 + \Phi_2$ satisfies* $\sup\{\mathrm{Re}\lambda | \lambda \in \sigma(\Phi_1 + \Phi_2)\} < 0$.

Let $\mathcal{H} = H^4(0, L_1) \times H^4(0, L_2) \times H^4(0, L_1) \times H^4(0, L_2) \times R^4$ be a Hilbert space with an inner product

$$\langle \begin{bmatrix} \boldsymbol{f_1} \\ \boldsymbol{f_2} \\ \boldsymbol{z_1} \\ \boldsymbol{z_2} \end{bmatrix}, \begin{bmatrix} \boldsymbol{g_1} \\ \boldsymbol{g_2} \\ \boldsymbol{q_1} \\ \boldsymbol{q_2} \end{bmatrix} \rangle = \langle \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \end{bmatrix} \rangle_E + \boldsymbol{z}_1^T \boldsymbol{q}_1 + \boldsymbol{z}_2^T \boldsymbol{q}_2$$

where

$$\langle \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \end{bmatrix} \rangle_E = \langle \mathcal{A}\boldsymbol{f}_1, \mathcal{A}\boldsymbol{g}_1 \rangle + \langle \mathcal{A}\boldsymbol{f}_2, \boldsymbol{g}_2 \rangle$$

$$\langle \begin{bmatrix} f_{11} \\ f_{12} \end{bmatrix}, \begin{bmatrix} g_{11} \\ g_{12} \end{bmatrix} \rangle = \sum_{i=1}^{2} \rho_i \int_0^{L_i} f_{1i}(r_i) g_{1i}(r_i) dr_i + m_i f_{1i}(L_i) g_{1i}(L_i).$$

Let us define operators $\boldsymbol{\Lambda}$ and $\boldsymbol{\Gamma}$ as

$$\boldsymbol{\Lambda Y} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O} \\ -\mathcal{A} & -\mathcal{R}\mathcal{A} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{O} & \mathcal{C} & \boldsymbol{K}_p & \boldsymbol{K}_d \end{bmatrix} \boldsymbol{Y}, \tag{8}$$

$$\boldsymbol{\Gamma Y} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \mathcal{D}_p & \mathcal{D}_d \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \end{bmatrix} \boldsymbol{Y} \tag{9}$$

where

$$\boldsymbol{Y} = [v_1(t, r_1),\ v_2(t, r_2),\ \dot{v}_1(t, r_1),\ \dot{v}_2(t, r_2),\ \theta_1(t),\ \theta_2(t),\ \dot{\theta}_1(t),\ \dot{\theta}_2(t)\ ]^T.$$

The closed-loop system (5), (6) can be rewritten as

$$\dot{\boldsymbol{Y}} = (\boldsymbol{\Lambda} + \boldsymbol{\Gamma})\boldsymbol{Y}. \tag{10}$$

Now we show that the operators $\boldsymbol{\Lambda}$ and $\boldsymbol{\Gamma}$ satisfy the assumption of Lemma 1. As from the definition of the operators we find that $\boldsymbol{\Gamma}$ is a bounded operator, it is easy to find that $\boldsymbol{\Gamma}$ is a compact operator. Let us consider the differential equation related to (10)

$$\dot{\boldsymbol{Y}} = \boldsymbol{\Lambda Y}. \tag{11}$$

Next we show the exponential stability of (11). In our previous paper [11], the exponential stability of the vibration system

$$\| \begin{bmatrix} \boldsymbol{v} \\ \dot{\boldsymbol{v}} \end{bmatrix} \|_E \le M_1 e^{-\omega_1 t} \| \begin{bmatrix} \boldsymbol{v}_0 \\ \dot{\boldsymbol{v}}_0 \end{bmatrix} \|_E \tag{12}$$

is proven. Using (12) and the property of the operator $\mathcal{A}$

$$\langle \begin{bmatrix} \frac{r_1}{L_1} \\ -\frac{r_2}{L_2} \end{bmatrix}, \mathcal{A} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \rangle = \frac{E_1 I_1}{L_1} v_1''(t, 0) - \frac{E_2 I_2}{L_2} v_2''(t, 0)$$

yields the estimation

$$\begin{aligned} \theta_1(t) &= C_{11} e^{s_{11} t} + C_{12} e^{s_{12} t} \\ &\quad + \frac{k_s}{L_1} \int_0^t \frac{e^{s_{11}(t-\tau)} - e^{s_{12}(t-\tau)}}{s_{11} - s_{12}} \\ &\qquad \times \left(\frac{E_1 I_1}{L_1} \dot{v}_1''(\tau, 0) - \frac{E_2 I_2}{L_2} \dot{v}_2''(\tau, 0)\right) d\tau \\ &\le \overline{C}_{11} e^{s_{11} t} + \overline{C}_{12} e^{s_{12} t} \\ &\quad M_1 \| \begin{bmatrix} \frac{r_1}{L_1} \\ -\frac{r_2}{L_2} \end{bmatrix} \| \ \| \begin{bmatrix} \boldsymbol{v}_0 \\ \dot{\boldsymbol{v}}_0 \end{bmatrix} \|_E \\ &\quad \times \int_0^t \left| \frac{s_{11} e^{s_{11}(t-\tau)} - s_{12} e^{s_{12}(t-\tau)}}{s_{11} - s_{12}} \right| e^{-\omega_1 \tau} d\tau \\ &\le M_2 e^{-\omega_2 t} \end{aligned} \tag{13}$$

where $s_{11}, s_{12} = \frac{-k_d \pm \sqrt{k_d^2 - 4k_p}}{2}$ and $\omega_2 = \min\{|\mathrm{Re}\ s_{11}|, |\mathrm{Re}\ s_{12}|, \omega_1\}$. Following the same procedure we find the exponential stability of the state variable $\theta_2(t)$ and finally we obtain $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$ are exponential stable.

Combining obtained results gives the following Lemma 2.

**Lemma 2** *The closed-loop system (5), (6) is expressed as*

$$\dot{\boldsymbol{Y}} = (\boldsymbol{\Lambda} + \boldsymbol{\Gamma})\boldsymbol{Y} \tag{14}$$

*with the linear operator $\boldsymbol{\Lambda}$ which generates the $C_0$-semigroup and ensure the exponential stability, and the linear compact operator $\boldsymbol{\Gamma}$ in a Hilbert space $\mathcal{H}$.*

## 5 Spectral Analysis

From the property of the operator $\boldsymbol{\Lambda}$ and $\boldsymbol{\Gamma}$, we find that $(\boldsymbol{\Lambda} + \boldsymbol{\Gamma})^{-1}$ exists and is compact, and the spectrum set $\sigma(\boldsymbol{\Lambda} + \boldsymbol{\Gamma})$ of the operator $\boldsymbol{\Lambda} + \boldsymbol{\Gamma}$ consists only of its eigenvalues. We can prove the Lemma 3.

**Lemma 3** *The follwoing two conditions are equivalent*

**Condition 1** $\lambda \in \sigma(\Lambda + \Gamma), \quad \lambda^2 + k_d \lambda + k_p \neq 0$

**Condition 2** *There exists a nonzero $\boldsymbol{\psi} = [\psi_1, \psi_2]^T \in \mathcal{D}(\mathcal{A})$ such that*

$$\begin{cases} (\lambda^2 + \lambda \delta \mathcal{A} + \mathcal{A}) \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \lambda^2 \begin{bmatrix} -\frac{r_1}{L_1} \\ \frac{r_2}{L_2} \end{bmatrix} \\ \lambda^2 + \lambda k_d + k_p = \lambda k_s \left(\frac{E_1 I_1}{L_1} \psi_1''(0) - \frac{E_2 I_2}{L_2} \psi_2''(0)\right) \end{cases} \tag{15}$$

*where $\mathcal{A}$ is the operator defined in (5).*

**Proof:** Appendix A ■

For the exponential stability of (14), the property that any $\lambda \in \sigma(\mathbf{\Lambda}+\mathbf{\Gamma})$ $(\lambda^2+k_d\lambda+k_p \neq 0)$ satisfy the condition

$$\sup\{\text{Re}\lambda\} < 0$$

should be proven. If $\lambda^2 + k_d\lambda + k_p = 0$, it is easy to find that $\sup\{\text{Re}\lambda\} < 0$. We should prove that any $\boldsymbol{\psi} \in \mathcal{D}(\mathcal{A})$ satisfy the following conditions.

- $\text{Re}\lambda < 0$
- $\sup\{\text{Re}\lambda\} \neq 0$

### 5.1 Proof of Re $\lambda < 0$

As $\lambda$ satisfies the first equation in (15) and $\lambda \neq 0$, we obtain

$$\lambda^2\langle\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}, \mathcal{A}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}\rangle + \lambda\langle\delta\mathcal{A}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}, \mathcal{A}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}\rangle$$
$$+\langle\mathcal{A}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}, \mathcal{A}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}\rangle = \lambda^2\langle\begin{bmatrix}-\frac{r_1}{L_1}\\ \frac{r_2}{L_2}\end{bmatrix}, \mathcal{A}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}\rangle$$

Using the relation

$$\langle\begin{bmatrix}-\frac{r_1}{L_1}\\ \frac{r_2}{L_2}\end{bmatrix}, \mathcal{A}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}\rangle = -\frac{E_1I_1}{L_1}\psi_1''(0) + \frac{E_2I_2}{L_2}\psi_2''(0)$$

gives

$$\lambda^2 + \lambda k_d + k_p + k_s\lambda\langle\boldsymbol{\psi}_n, \mathcal{A}\boldsymbol{\psi}_n\rangle$$
$$+k_s\langle\delta\mathcal{A}\boldsymbol{\psi}_n, \mathcal{A}\boldsymbol{\psi}_n\rangle + \frac{k_s}{\lambda}\langle\mathcal{A}\boldsymbol{\psi}_n, \mathcal{A}\boldsymbol{\psi}_n\rangle = 0 \quad (16)$$

and using (18) and the property (3) of the operator $\mathcal{A}$ gives

$$\lambda^3 + (k_d + k_s\mu_n)\lambda^2 + (k_p + k_s\delta\mu_n^2)\lambda + k_s\mu_n^2 = 0. \quad (17)$$

A sufficient condition for the polynomial of the left side of (17) to be Hurwitz is

$$k_d > \delta. \quad (18)$$

If $\mu_n \to \infty$ the equation (17) can be rewritten as

$$k_d\delta\lambda + k_s = 0 \quad (19)$$

then we obtain $\lambda = -\frac{1}{\delta}$. We find that if $k_d > \delta$, then $\text{Re}\lambda < 0$ for any $\lambda \in \sigma(\mathbf{\Lambda}+\mathbf{\Gamma})$.

Next we show the property $\sup\{\text{Re}\lambda\} \neq 0$. Let $\lambda = a + jb$ be the solution of (16). Using (16) and the property (3) of the operator $\mathcal{A}$ yields

$$a^2 - b^2 + ak_d + k_p + k_sa\mu_n$$
$$+k_s\delta\mu_n^2 + \frac{k_sa}{|\lambda|^2}\mu_n^2 = 0 \quad (20)$$

$$2ab + k_db + k_sb\mu_n - \frac{k_sb}{|\lambda|^2}\mu_n^2 = 0 \quad (21)$$

(i). In the case $b \neq 0$

From (20), (21) we find

$$|\lambda|^2 \geq k_p + k_s\delta\mu_n^2 - \frac{1}{4}(k_d + \mu_n)^2 \quad (22)$$

and from (21)(22)

$$2a \leq -k_d - k_s\mu_n + \frac{4k_s\mu_n^2}{4k_p + 4k_s\delta\mu_n^2 - (k_d+\mu_n)^2} \quad (23)$$

If we take $\mu_n \to \infty$, we find that the third term of the right side of (23) is bounded and

$$\lim_{\mu_n\to\infty} a \to -\infty. \quad (24)$$

We can conclude

$$\sup\{\text{Re}\lambda\} \neq 0. \quad (25)$$

(ii). In the case $b = 0$

From (20) we obatin

$$a^3 + (k_d + k_s\mu_n)a^2 + (k_p + k_s\delta\mu_n^2)a + k_s\mu_n^2 = 0 \quad (26)$$

Let $a_1, a_2, a_3$ be the solutions of (26). The relation of the solution and coefficients of the polynomial of (26) is obtained as

$$\begin{cases} a_1 + a_2 + a_3 = -(k_d + k_s\mu_n) \\ a_1a_2 + a_2a_3 + a_1a_3 = k_p + k_s\delta\mu_n^2 \\ a_1a_2a_3 = -k_s\mu_n^2 \end{cases} \quad (27)$$

From the property of $\mathcal{A}$ the order estimation of the eigenvalue $\mu_n$ is given as $\mu_n = \mathcal{O}(n^4)$ and the order estimation of (27) as

$$a_1 + a_2 + a_3 = \mathcal{O}(n^4), \quad (28)$$
$$a_1a_2 + a_2a_3 + a_1a_3 = \mathcal{O}(n^8), \quad a_1a_2a_3 = \mathcal{O}(n^8)$$

From the relation of the order of (29), we can prove the statement $\sup a_i \neq 0$ $(\mu_n \to \infty)$.

Combining obtained results gives the following Theorem 1.

**Theorem 1** *If the feedback gain $k_d$ is taken as $k_d > \frac{1}{\delta}$, then the output feedback control law (4) can ensure the exponential stability of the closed-loop system for the vibration equation (2) of the flexible beams and the equation of rotation of the motor (3).*

## 6 Implementation

Let $V_{refi}(t)$ be the speed reference voltage for the amplifier of the motor $i$. As we use the speed-control type amplifier of the motor, we obtain

$$V_{refi}(t) = \dot{\theta}_i(t). \quad (29)$$

As we can set $\dot{\theta}_i(0) = 0$, the controller (4)

$$V_{refi}(t) = -k_d\theta_i(t) - k_p\int_0^t \theta_i(\tau)d\tau$$
$$+(-1)^{i+1}k_s\left(\frac{E_1I_1}{L_1}w_1''(t,0) - \frac{E_2I_2}{L_2}w_2''(t,0)\right) \quad (30)$$

As the value of $w_i''(t,0)$ can be measured by a strain gage sensor, the controller (30) is simple and easy to implement. As the control law consists of the PI feedback of the joint angle and the additional strain feedback, it is called as PIS controller. The controller does not need the value of the spring constant of the connected part with uncertainty and is robust against the uncertainty of the connected part.

## 7 Experimental Validation

### 7.1 Experimental Setup

Figure 2 shows two interconnected flexible beams designed for experiments. The physical parameters of the system are as follows: $L_1 = L_2 = 600[\text{mm}], E_1 = E_2 = 2.06 \times 10^{11}[\text{Nm}^2], I_1 = I_2 = 4.17 \times 10^{-12}[\text{m}^4], \rho_1 = \rho_2 = 0.38[\text{kg/m}], M = 0.1[\text{kg}]$.

The strain gage foils are cemented at the root of the beams and the bending moment $w_i''(t,0)$ is measured. Up/down pulses from the encoder give the data corresponding to the angle of rotation $\theta_i(t)$ and the angular velocity $\dot{\theta}_i(t)$. On the basis of the proposed control law, the output signals $\theta_i(t), \int_0^t \theta_i(\tau)d\tau, w_i''(t,0)$ from the sensors are fed back to the driving motor $i$. The sampling period was specified as 1[ms] for real time tasks.

### 7.2 Experimental Results

We select the following initial position errors of the joint angles $\theta_1(0) = \theta_2(0) = 5[\text{deg}]$. Figures 3-6 show the transient responses of the input torque $\tau_i(t)[\text{Nm}]$, the joint angle $\theta_i(t)[\text{deg}]$, and the strain $w_i''(t,0)[1/\text{m}]$ of the beam $i$ with the initial position errors. In each figure, (a) and (b) represent transient responses for the beam 1 and 2, respectively. In our experiments, we set the controller gains as follows: PI Control : $k_p = 3, k_d = 1, k_s = 0$, PIS Control : $k_p = 3, k_d = 1, k_s = 0.1$.

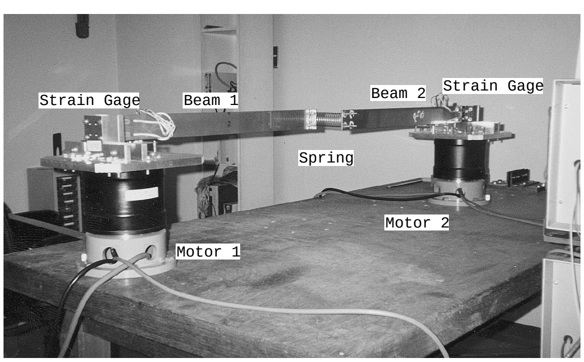
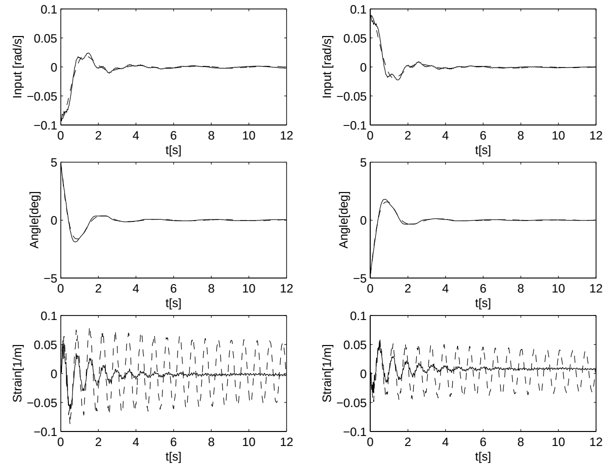
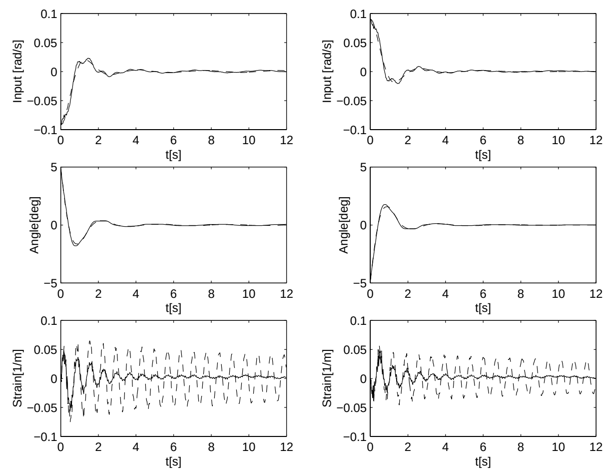



Fig. 2 Experimental Setup



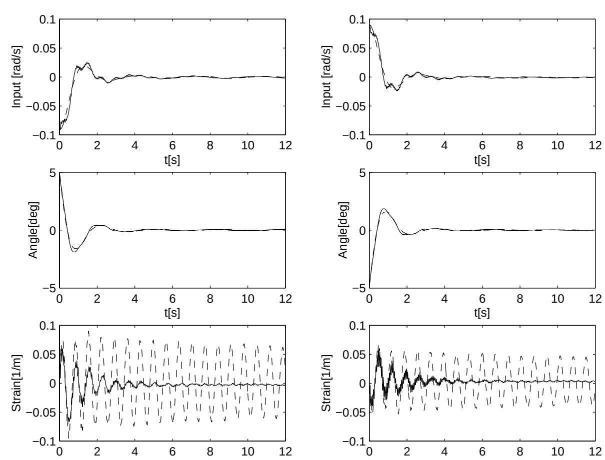

(a) Beam 1 (b) Beam 2

Fig. 3 Transient responses for PI and PIS Feedback (PI - - -, PIS —)




(a) Beam 1 (b) Beam 2

Fig. 4 Transient responses when the spring constant is changed from 10[gf/mm] to 25[gf/mm] (PI - - -, PIS —)




(a) Beam 1 (b) Beam 2

Fig. 5 Transient responses when the spring constant is changed from 10[gf/mm] to 1[gf/mm] (PI - - -, PIS —)

**Table 1:** Performance of Controller

| No. | Controller | Rise Time[s] | ISE [s/m$^2$] |
|---|---|---|---|
| 3 | PIS ($K_s = 0.1$) | 0.34 | 3.2 $\times 10^{-3}$ |
| | PI ($K_s = 0$) | 0.32 | 33.0 $\times 10^{-3}$ |
| 4 | PIS ($K_s = 0.1$) | 0.34 | 3.3 $\times 10^{-3}$ |
| | PI ($K_s = 0$) | 0.32 | 43.3 $\times 10^{-3}$ |
| 5 | PIS ($K_s = 0.1$) | 0.34 | 2.1 $\times 10^{-3}$ |
| | PI ($K_s = 0$) | 0.32 | 22.2 $\times 10^{-3}$ |

Figures 3 shows the transient responses for the PI controller and for the PIS controller. In order to demonstrate the robustness of the PIS controller, we change the spring constant from 10[gf/mm] to 25[gf/mm] (Fig. 4) and 1[gf/mm] (Fig.5), and pose the same experiments for the same controller as that of the case presented in Fig. 3. Figure 6 shows the responses for the absolute value of the strain when the strain feedback gain is changed. Table 1 shows the values of the ISE of the responses of the strains for 12 seconds to compare the performances of the controllers.

The experimental results demonstrate that the proposed PIS controller is robust for the parameter uncertainty of the connective part.

## 8 Conclusions

In this papaer, we have discussed the exponential stabilizing controller of the motion of the rigid mode and the vibration of flexible modes for two flexible beams connected by a spring as a simple example of the large space structures. We proposed the direct sensor output feedback control law (PIS controller) for the motion and vibration absorption control. Using the spectral analysis, the exponential stability of the closed-loop system is proven. As we don't need an approximated finite-dimensional model at the controller design phase, the controller based on the original distributed parameter system is robust and simple. The experimental results demonstrate the validity of the proposed model and the effectiveness of the proposed PIS feedback control law. The next problem to be tacked is an extension of this research to flexible structures with several connective parts.

## 9 Appendix

*Proof of Lemma 3:*

Condition 2 ($\Rightarrow$) Condition 1

Let us define

$$[v_1,\ v_2,\ v_3,\ v_4,\ z_1,\ z_2,\ z_3,\ z_4]^T$$
$$= [\psi_1,\ \psi_2,\ \lambda\psi_1,\ \lambda\psi_2,\ 1/L_1,\ -1/L_2,\ \lambda/L_1,\ -\lambda/L_2]^T$$

where $[\psi_1, \quad \psi_2]^T \in \mathcal{D}(\mathcal{A})$. Using the definition of $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ gives

$$(\Lambda + \Gamma)\boldsymbol{Y} = \lambda \boldsymbol{Y}$$

Condition 1 ($\Rightarrow$) Condition 2

From Condition 1 we obtain

$$(\lambda^2 + \lambda k_d + k_p)(L_1 z_1 + L_2 z_2) = 0 \tag{31}$$

$$(\lambda^2 + \lambda k_d + k_p)(L_1 z_1 - L_2 z_2) = 2\lambda K_s\left(\frac{E_1 I_1}{L_1} v_1''(0) - \frac{E_2 I_2}{L_2} v_2''(0)\right) \tag{32}$$

$$(\lambda^2 + \lambda\delta\mathcal{A} + \mathcal{A}) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda^2 \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{33}$$

Let us introduce $[\psi_1, \quad \psi_2]^T$ such that

$$v_1 = L_1 z_1 \psi_1, \qquad v_2 = -L_2 z_2 \psi_2, \tag{34}$$

then we obtain

$$\lambda^2 + \lambda k_d + k_p = \lambda k_s\left(\frac{E_1 I_1}{L_1}\psi_1''(0) - \frac{E_2 I_2}{L_2}\psi_2''(0)\right) \tag{35}$$

$$(\lambda^2 + \lambda\delta\mathcal{A} + \mathcal{A}) \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \lambda^2 \begin{bmatrix} -\frac{r_1}{L_1} \\ \frac{r_2}{L_2} \end{bmatrix} \tag{36}$$

Using (35) and (36) yields (15).

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