Practical stabilization of a class of nonlinear systems. Application to chain systems and mobile robots.

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Abstract

A characterization of the Lie Algebra Rank Condition (LARC) by transverse periodic functions is applied to feedback stabilization of a class of nonlinear systems. The approach is illustrated on controllable homogeneous driftless systems subjected to known additive perturbations. A generalized path tracking problem for mobile robots is then addressed, in connection with some aspects of the path planning problem.

1 Introduction

The objective of this paper is to point out and illustrate, via a selection of examples, how a theorem proved in [4], which relates controllability properties associated with a set of smooth control vector fields (v.f.) to the existence of *transverse* bounded functions, can be used for feedback control design and *practical* stabilization purposes. The feedback control laws proposed herein should thus be regarded as elements of a new and still maturing control approach for nonlinear systems, based on the use of this theorem.

Throughout the paper, $\mathbb{T}^k$ ($k \in \mathbb{N}$) denotes the k-dimensional torus, $B_n(0,\delta)$ denotes the closed ball in $\mathbb{R}^n$ centred at zero and of radius δ, x^T denotes the transpose of x (a vector or a matrix). The main result of [4] is recalled below.

Theorem 1 *[4] Let $g_1, \dots, g_m$ denote smooth v.f. on $\mathbb{R}^n$ such that, in a neighbourhood of 0, the dimension of the distribution $\Delta(x) \triangleq Span\{g(x) : g \in Lie\{g_1, \dots, g_m\}\}$ is constant equal to n_0. Then, the following properties are equivalent:*
1. LARC(0): the system

$$S: \qquad \dot{x} = \sum_{i=1}^{m} g_i(x) u_i$$

satisfies the LARC at the origin (i.e. $n_0 = n$).
2. TC(0): there exist $\bar{n} \in \mathbb{N}$ and a family $(f_\epsilon)_{\epsilon>0}$ of functions $f_\epsilon \in \mathcal{C}^\infty(\mathbb{T}^{\bar{n}-m}; B_n(0,\epsilon))$ such that, for any $\epsilon > 0$, the following Transversality Condition holds:

$$\forall \theta \in \mathbb{T}^{\bar{n}-m}, \quad Rank(H_{f_\epsilon}(\theta)) = n \tag{1}$$

$$\text{with} \quad H_{f_\epsilon}(\theta) = \Big(g_1(f_\epsilon(\theta)) \ \dots \ g_m(f_\epsilon(\theta)) \\ -\frac{\partial f_\epsilon}{\partial \alpha_{m+1}}(\theta) \ \dots \ -\frac{\partial f_\epsilon}{\partial \alpha_{\bar{n}}}(\theta) \Big) \tag{2}$$

and $(\alpha_{m+1}, \dots, \alpha_{\bar{n}})$ coordinates on $\mathbb{T}^{\bar{n}-m}$.

2 Preliminary considerations

Consider a n-dimensional control system

$$(S) \quad : \quad \dot{x} = g_0(x,t) + \sum_{i=1}^{m} g_i(x) u_i \tag{3}$$

such that the control v.f. g_i ($i = 1, \dots, m$) verify the assumptions of Theorem 1 and satisfy LARC(0). The function $g_0 \in C^1(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^n)$ may correspond to either a drift term associated with the natural dynamics of the system, or an additive perturbation acting on the system, or a combination of both terms. We will assume hereafter that g_0 is known. Note that when $m < n$, there exist v.f. g_i $(i = 0, \ldots, m)$ such that the origin $x = 0$ cannot be a stable point of the controlled system, whatever the control. For instance, if for some $j \in \{1, \ldots, n\}$ $g_{0,j}(0,t)$ is a nonzero constant and $g_{i,j}(0) = 0$ for $i \in \{1, \ldots, m\}$, the control can at best render a neighbourhood of the origin attractive and achieve in this way some type of "practical stabilization" of the origin.

According to Theorem 1, there exists a family of functions $f_\epsilon \in \mathcal{C}^\infty(\mathbb{T}^{\bar{n}-m}; B_n(0,\epsilon))$ which, associated with the v.f. g_i $(i = 1, \ldots, m)$, satisfy the transversality rank condition (1). Now, define the *extended* $\bar{n}$-dimensional control vector $\bar{U} \triangleq [u_1, \ldots, u_m, \dot{\alpha}_{m+1}, \ldots, \dot{\alpha}_{\bar{n}}]^T$ and consider the following feedback control law

$$\bar{U}(x,\theta,t) = -H^\dagger_{f_\epsilon}(\theta)[k(x - f_\epsilon(\theta)) + g_0(x,t)] \quad (4)$$

with $k > 0$ and $H^\dagger_{f_\epsilon} = H^T_{f_\epsilon}(H_{f_\epsilon} H^T_{f_\epsilon})^{-1}$. In view of (1), it is not difficult to prove that this control yields asymptotic convergence of $(x(t) - f_\epsilon(\theta(t)))$ to zero, provided that ϵ and $\|x(0) - f_\epsilon(\theta(0))\|$ are small enough, and k is large enough. Since ϵ can be chosen arbitrarily small, this already illustrates how Theorem 1 can be used to derive a simple feedback control which maintains the state $x(t)$ *near* zero, whatever the perturbation g_0. However, this result is by itself of limited practical value because it does not even imply that a solution originated at $x = 0$ will not grow unbounded (local practical stability of the origin is not ensured). Therefore, the interest of Theorem 1 appears more clearly if one can prove a stronger (more global) result involving a larger set of solutions. This is illustrated in the next section.

3 Practical stabilization of homogeneous driftless systems

We further assume that the v.f. g_i $(i = 1, \ldots, m)$ are homogeneous of degree -1 with respect to some dilation operator (see [2] for definitions). This assumption implies that there exist a system of coordinates x, a dilation $\delta^r_\lambda : x \mapsto (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n)$, with $r_i > 0$, and an integer $p \leq n$ such that the control system (S) writes as

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \\ \vdots \\ \dot{x}^p \end{bmatrix} = \begin{bmatrix} G^1 \\ G^2(x^1) \\ G^3(x^1, x^2) \\ \vdots \\ G^p(x^1, \ldots, x^{p-1}) \end{bmatrix} U + g_0(x,t) \quad (5)$$

with 1. x^i $(i \in \{1, \ldots, p\})$ the state sub-vector whose components (viewed as functions) are homogeneous of same degree $r^i \in \{r_1, \ldots, r_n\}$, with $r^1 < r^2 < \ldots < r^p$, 2. $U \triangleq [u_1, \ldots, u_m]^T$, 3. $dim(x^1) \leq m$, 4. G^1 a full rank matrix, 5. $G^i(x^i, \ldots, x^{i-1})$ a matrix of polynomial functions which are homogeneous of degree $r^i - 1$.

Note that $\theta \triangleq \big(\sin(\alpha_{m+1}), \cos(\alpha_{m+1}), \ldots, \sin(\alpha_{\bar{n}}), \cos(\alpha_{\bar{n}})\big)^T$ can be interpreted as a vector in $\mathbb{R}^{2(\bar{n}-m)}$ whose variation along time is described by

$$(S_\theta) \quad : \quad \dot{\theta} = C(\theta) V \quad (6)$$

with $V \triangleq (\dot{\alpha}_{m+1}, \ldots, \dot{\alpha}_{\bar{n}})^T$, and $C(\theta)$ the matrix-valued function of dimension $2(\bar{n} - m) \times (\bar{n} - m)$ with entries

$$C_{i,j}(\theta) = \begin{cases} \cos(\alpha_{m+j}) & if\ i = 2j - 1 \\ -\sin(\alpha_{m+j}) & if\ i = 2j \\ 0 & otherwise\ . \end{cases}$$

Proposition 1 *Consider the systems (5)-(6). Assume that* $\|g_0(x,t)\| < K_0 < +\infty$, $\forall (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$*. Let* $\bar{U} \triangleq \begin{pmatrix} U \\ V \end{pmatrix}$ *denote the control vector associated with the resulting control system, and define* $\tilde{x} \triangleq x - f_\epsilon(\theta)$*. Apply the following feedback control to this system*

$$\bar{U}(x,\theta,t) \triangleq -H^\dagger_{f_\epsilon}(\theta)\,(k_0 \bar{\tilde{x}} + g_0(x,t)) \quad (7)$$

with

$$\bar{\tilde{x}} = \left(\frac{\tilde{x}^1}{1 + k_1\|\tilde{x}^1\|}, \ldots, \frac{\tilde{x}^p}{1 + k_p\|\tilde{x}^p\|} \right)^T \quad (8)$$

and $k_i > 0$ $(i = 1, \ldots, p)$*. Then, for any initial state value* $(x(0), \theta(0)) \in (\mathbb{R}^n \times \mathbb{T}^{\bar{n}-m})$

1. the solution $(x(t), \theta(t))$ *to the controlled system uniquely exists on* $(0, +\infty)$*,*

2. for any $\sigma > 0$*,*

$$\lim_{t \to \infty} \|\tilde{x}(t)\|\ e^{(k_0 - \sigma)t} \ = \ 0 \quad (9)$$

Proof: Existence and uniqueness of the solutions to the controlled system, for $t \in (0, +\infty)$, result from the differentiability and uniform boundedness of the control. The system (5, 6) is equivalent to the system

$$\left\{ \begin{array}{ll} \dot{\tilde{x}} = & H_{f_\epsilon}(\theta)\bar{U} + \Delta G(\tilde{x}, \theta)\bar{U} + g_0(\tilde{x} + f_\epsilon(\theta), t) \\ \dot{\theta} = & [0 \ C(\theta)] \, \bar{U} \ , \end{array} \right. \tag{10}$$

with $\Delta G(\tilde{x}, \theta) = [G(\tilde{x} + f_\epsilon(\theta)) - G(f_\epsilon(\theta)) \ 0]$ and

$$G(x) = \left[\begin{array}{c} G^1 \\ G^2(x^1) \\ \vdots \\ G^p(x^1, \ldots, x^p) \end{array} \right] \tag{11}$$

Using the control expression (7) in (10), one obtains the following equations for the closed-loop system

$$\left\{ \begin{array}{lll} \dot{\tilde{x}}^i & = & -\frac{k_0 \tilde{x}^i}{1 + k_i \|\tilde{x}^i\|} + \Delta G^i(\tilde{x}, \theta)\bar{U} \ (i = 1, \ldots, p) \\ \dot{\theta} & = & [0 \ C(\theta)] \, \bar{U} \ . \end{array} \right. \tag{12}$$

Now, since $\Delta G^1(\tilde{x}, \theta) = 0$,

$$\dot{\tilde{x}}^1(t) = -\frac{k_0}{1 + k_1 \|\tilde{x}^1(t)\|} \tilde{x}^1(t) \ . \tag{13}$$

This implies that $\lim_{t \to \infty} \|\tilde{x}^1(t)\| = 0$. Therefore, $\Delta G^2(\tilde{x}^1, \theta)\bar{U}$ tends to zero and

$$\lim_{t \to \infty} \left(\dot{\tilde{x}}^2(t) + \frac{k_0}{1 + k_2 \|\tilde{x}^2(t)\|} \tilde{x}^2(t) \right) = 0 \ .$$

This in turn implies that $\lim_{t \to \infty} \|\tilde{x}^2(t)\| = 0$. By iteration, one shows in this way that $\tilde{x}(t)$ asymptotically tends to zero. Note that (13) yields

$$\lim_{t \to \infty} \|\tilde{x}^1(t)\| \ e^{(k_0 - \sigma)t} \ = \ 0 \quad , \forall \sigma > 0 \tag{14}$$

In order to prove (9) from there, it suffices to use the following technical result

$$\left. \begin{array}{l} z_1, z_2 \in C^1(\mathbb{R}^+; \mathbb{R}^+) \\ \frac{d}{dt} z_2 \leq -a z_2 + z_1 \\ \lim_{t \to \infty} z_1(t) e^{bt} = 0 \\ a > 0, \ b > 0 \end{array} \right\} \Rightarrow \begin{array}{c} \lim_{t \to \infty} z_2(t) e^{ct} = 0 \\ c = \inf(a, b) \end{array} \tag{15}$$

in combination with (12) and the following inequalities which result from the boundedness of $\|\tilde{x}(t)\|$ and $\|\bar{U}(t)\|$: $\forall i \in \{2, \ldots, p\}$, $\exists K_i > 0$:

$$\forall t \geq 0 \ \|\Delta G^i(\tilde{x}(t), \theta(t))\bar{U}(t)\| \leq K_i \sum_{j=1}^{i-1} \|\tilde{x}^j(t)\|$$

$\blacksquare$

By specifying the structure of the control system (S) further, one can show uniform exponential stabilization of $(x - f_\epsilon(\theta))$ to zero without assuming boundedness of the perturbation g_0. This is illustrated next in the case of a chain system.

4 Practical stabilization of a perturbed chain system

We assume now that $m = 2$, and

$$\left\{ \begin{array}{lll} g_1(x) & = & (1, 0, x_2, \ldots, x_{n-1})^T \\ g_2 & = & (0, 1, 0, \ldots, 0)^T \end{array} \right. \tag{16}$$

so that (S) represents a $(2, n)$ chain system perturbed by g_0. The structure of g_1 and g_2 simplifies the determination of functions $f_\epsilon \in C^\infty(\mathbb{T}^{n-m}, \mathbb{R}^n)$ which satisfy the Theorem's transversality condition, as shown in the following proposition.

Proposition 2 *Let*

$$\left\{ \begin{array}{lll} g_1^l(x) & = & (1, 0, x_2, \ldots, x_{l-1})^T \\ g_2^l & = & (0, 1, 0, \ldots, 0)^T \end{array} \right.$$

denote the control v.f. of a $(2, l)$ chain system. Consider the set of functions $f^l \in C^\infty(\mathbb{T}^{l-2}, \mathbb{R}^l)$, $(l \geq 3)$, obtained via the following iterative construction

$$f^3(\theta_3) = \left[\begin{array}{c} \mu_3 \sin(\alpha_3) \\ \mu_3 \cos(\alpha_3) \\ \mu_3^2 \frac{\sin(2\alpha_3)}{4} \end{array} \right]$$

$$f^l(\theta_l) = \left[\begin{array}{c} f^{l-1}(\theta_{l-1}) + \left[\begin{array}{c} \mu_l \sin(\alpha_l) \\ 0 \\ \vdots \\ 0 \\ \mu_l^{l-2} \cos(\alpha_l) \end{array} \right] \\ \mu_l^{l-1} \frac{\sin(2\alpha_l)}{4} + \mu_l^{l-2} \cos(\alpha_l) f_1^{l-1}(\theta_{l-1}) \end{array} \right]$$

with $\theta_l \triangleq (\sin(\alpha_3), \cos(\alpha_3), \ldots, \sin(\alpha_l), \cos(\alpha_l))^T$ and f_i^l the ith. component of f^l.
Then, there exist coefficients $\mu_3, \ldots, \mu_l$ for which f^l, associated with the v.f. g_1^l and g_2^l, satisfies the transversality rank condition (1) of Theorem 1.
Moreover, given such a function f^l, any function $f_\epsilon^l \triangleq \left[\epsilon f_1^l, \epsilon f_2^l, \epsilon^2 f_3^l, \ldots, \epsilon^{l-1} f_l^l\right]^T$, with $\epsilon > 0$, associated with the v.f. g_1^l and g_2^l, also satisfies the transversality rank condition (1).

Due to space limitations, the proof is omitted.
To simplify notations, we will omit from now on the upper index n of f_ϵ^n, introduced in Proposition 2, and will just denote as $f_\epsilon \in C^\infty(\mathbb{T}^{n-2}, \mathbb{R}^n)$ any function which, associated with the v.f. g_1 and g_2 of a $(2,n)$ chain system, satisfies the transversality rank condition (1).

Proposition 3 *Consider the perturbed $(2,n)$ chain system augmented with the system (S_θ), and let $\bar{U} \triangleq (u_1, u_2, V^T)^T$. Apply the following feedback to this system*

$$\bar{U}(x,\theta,t) = H_{f_\epsilon}^{-1}(\theta)\left[e^{Ax_1}\ (K + g_{0,1}(x,t)A)\right.$$
$$\left. e^{-Ax_1}\tilde{x} - g_0(x,t)\right] \quad (17)$$

with $\tilde{x} \triangleq x - f_\epsilon(\theta)$, K a Hurwitz-stable $(n \times n)$ matrix, and A the $(n \times n)$ matrix whose only non-zero entries are $A_{i+1,i} = 1$ for $i \in \{2, \ldots, n-1\}$. Then, there exist[1] *a class $\mathcal{K}$ function γ and a strictly positive real number k_0 such that the solutions to the closed-loop system satisfy*

$$\forall t \geq 0\ :\ \|\tilde{x}(t)\|\ \leq\ \gamma(\|\tilde{x}(0)\|)\ e^{-k_0 t} \quad (18)$$

Proof : It just consists in verifying that the application of the proposed control yields $\dot{z} = Kz$ with $z \triangleq e^{-Ax_1}\tilde{x}$. Since K is Hurwitz-stable, there exist $\gamma_0 > 0$ and $k_0 > 0$ such that any solution to the above differential equation satisfies

$$\forall t \geq 0\ ,\ \|z(t)\| \leq \gamma_0 \|z(0)\| e^{-k_0 t}$$

Then, inequality (18) readily follows. ∎

5 Application to mobile robot path tracking

We first would like to acknowledge the reference [1] which addresses the problem of path tracking for a unicycle-type mobile robot with the same "philosophy" of practical stabilization as in the present paper. Although the general existence of bounded functions transverse to the v.f. of a controllable driftless system is not identified or suggested in this reference, one can recognize in $f^3(\theta_3)$ of Proposition 2 the function which is involved in the control law proposed in [1] in order to achieve "global practical stabilization" of the path tracking error vector. This reference has inspired our idea for Theorem 1.
In what follows, we show how Proposition 3 applies to the path tracking problem for unicycle-type and car-like vehicles. Not surprisingly, one of the two control solutions proposed here presents strong similarities with the one in [1]. But it is also more general because the assumption according to which the reference vehicle is subjected to the same nonholonomic constraints as the controlled vehicle is not needed. Imposing no motion constraint on the reference vehicle makes, in our opinion, an important difference and confers a particular interest to the solution. The possibility of extending the proposed solution to a car pulling trailers will not be missed by the reader. It is only a matter of prolonging the changes of coordinates here considered, which transform the original systems into perturbed chain systems.

5.1 Unicycle-type case

Consider a reference vehicle (B_r), moving on a plane and whose posture coordinates with respect to a fixed frame $(O; \vec{\imath}_0, \vec{\jmath}_0)$ are (x_r, y_r), the position coordinates of a point P_r on the body, and ν_r, the orientation angle of the body (see Fig. 1).
Consider also a unicycle-type vehicle (B) moving on the same plane, with P the point at mid-distance of the rear wheels' axle, v the intensity of the velocity of point P (i.e. $\vec{V}_P = v\vec{\imath}$), and ν the orientation angle of the vehicle. The control inputs are v and $\dot{\nu}$.

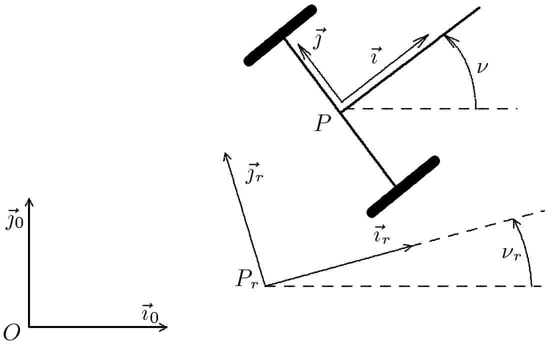

Figure 1: unicycle-type vehicle viewed from above

The control objective is to have (B) track (B_r) "closely", whatever the initial relative posture of

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[1] Recall that a continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $k(0) = 0$.

(B_r) with respect to (B), and whatever the velocity of (B_r), characterized by the three (*possibly independent*) variables $\dot{x}_r$, $\dot{y}_r$, and $\dot{\nu}_r$.

Let x and y denote the coordinates of the vector $\overrightarrow{P_rP}$ expressed in the basis of a mobile frame $(P_r; \vec{\imath}_r, \vec{\jmath}_r)$ attached to (B_r): $\overrightarrow{P_rP} = x\vec{\imath}_r + y\vec{\jmath}_r$. Let a_r and b_r denote the coordinates of $\vec{V}_{P_r}$, expressed in the basis of $(P_r; \vec{\imath}_r, \vec{\jmath}_r)$ so that $\vec{V}_{P_r} = a_r\vec{\imath}_r + b_r\vec{\jmath}_r$ with

$$\begin{pmatrix} a_r \\ b_r \end{pmatrix} = \begin{pmatrix} \cos\nu_r & \sin\nu_r \\ -\sin\nu_r & \cos\nu_r \end{pmatrix} \begin{pmatrix} \dot{x}_r \\ \dot{y}_r \end{pmatrix} \tag{19}$$

Let $\tilde{\nu} \triangleq \nu - \nu_r$ denote the difference of orientation between (B) and (B_r). It is simple to show that the following kinematic equations characterize the motion of (B) relative to (B_r)

$$\left\{ \begin{array}{rcl} \dot{x} & = & v\cos\tilde{\nu} + y\dot{\nu}_r - a_r \\ \dot{y} & = & v\sin\tilde{\nu} - x\dot{\nu}_r - b_r \ . \end{array} \right. \tag{20}$$

In view of these equations, defining $u_1 \triangleq v\cos\tilde{\nu}$ and $u_2 \triangleq \frac{\dot{\tilde{\nu}}}{(\cos\tilde{\nu})^2}$ as new control inputs, and introducing the new coordinates $X \triangleq (x, \tan\tilde{\nu}, y)^T \triangleq (x_1, x_2, x_3)^T$, we obtain a perturbed $(2,3)$ chain system with perturbation

$$g_0(X,t) = (x_3\dot{\nu}_r - a_r, 0, -x_1\dot{\nu}_r - b_r)^T \ .$$

It only remains to apply Proposition 3 with $n = 3$ and $f_\epsilon(\theta) = f_\epsilon^3(\theta)$ (as given in Proposition 2) to obtain a control which ensures exponential stabilization of $(X - f_\epsilon(\theta))$ to zero. Note that the convergence is global in the coordinates X, but limited to $\tilde{\nu} \in (-\frac{\pi}{2}, +\frac{\pi}{2})$ in the original coordinates. This means in practice that the initial orientation error $\tilde{\nu}(0)$ has to belong to $(-\frac{\pi}{2}, +\frac{\pi}{2})$. The control maintains $\tilde{\nu}(t)$ in this interval thereafter.

In the unicycle case, there exists a simple way of overcoming the abovementioned limitation. It consists in obtaining the perturbed $(2,3)$ chain system via a *global* change of coordinates. Such a change of coordinates is, for instance

$$X \triangleq \begin{pmatrix} 0 & 0 & -1 \\ -\sin\tilde{\nu} & \cos\tilde{\nu} & 0 \\ \cos\tilde{\nu} & \sin\tilde{\nu} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \tilde{\nu} \end{pmatrix}$$

This is equivalent to replacing the coordinates (x,y) of $\overrightarrow{P_rP}$ by other coordinates $(\bar{x}, \bar{y})$ expressed in the basis of the frame attached to (B), and setting $X \triangleq (-\tilde{\nu}, \bar{y}, \bar{x})^T$. Then, defining $u_1 \triangleq v + \bar{y}\dot{\nu}$ and $u_2 \triangleq -\dot{\tilde{\nu}}$, one obtains another perturbed $(2,3)$ chain system, with perturbation

$$g_0(X,t) = (0, -x_3\dot{\nu}_r - \dot{y}_r, -\dot{x}_r)^T \ .$$

5.2 Car-like case

The tracking problem is the same as previously, except that the controlled vehicle, modelled as a tricycle, is now equipped with a front steering wheel whose angle with respect to the axis perpendicular to the rear wheels' axle is denoted by α ($\alpha = 0$ means that the vehicle moves along a straight line).

In this case, the kinematic equations of the system are given by (20), with an additional equation which relates $\dot{\nu}$, the velocity of rotation of the vehicle's body, to the steering wheel angle α and the advancement velocity v: $\dot{\nu} = (v\tan\alpha)/d$, where d is the distance between the rear wheels' axle and the center of the front steering wheel.

Defining $u_1 \triangleq v\cos\tilde{\nu}$ and
$u_2 \triangleq \frac{\dot{\alpha}}{d\cos^3\tilde{\nu}\cos^2\alpha} + \frac{3\tan^2\alpha\sin\tilde{\nu}}{d^2\cos^4\tilde{\nu}}v - \frac{3\tan\alpha\sin\tilde{\nu}}{d\cos^4\tilde{\nu}}\dot{\tilde{\nu}}_r$, and introducing the new coordinates $X \triangleq (x, \frac{\tan\alpha}{d(\cos\tilde{\nu})^3}, \tan\tilde{\nu}, y)^T \triangleq (x_1, x_2, x_3, x_4)^T$, we obtain a perturbed $(2,4)$ chain system, with perturbation

$$g_0(X,t) = \big(x_4\dot{\nu}_r - a_r, 0, -(1 + x_3^2)\dot{\nu}_r, -x_1\dot{\nu}_r - b_r\big)$$

By applying Proposition 3 with $n = 4$ and $f_\epsilon(\theta) = f_\epsilon^4(\theta)$ (as given in Proposition 2), one obtains a control which ensures exponential stabilization of $X - f_\epsilon(\theta)$ to zero, provided that $\tilde{\nu}(0)$ and $\alpha(0)$ belong to $(-\frac{\pi}{2}, +\frac{\pi}{2})$.

6 Application to mobile robot path planning

Path planning is here understood as the problem of calculating an *admissible* trajectory (i.e. solution to the system) which connects two state values and does not cross regions corresponding to physical obstacles. This complex problem has motivated in the past intense research. Let us just cite [3] as an example of seminal contribution in the domain. An issue, commonly encountered in publications on the subject, concerns the calculation of admissible trajectories which approximate another trajectory, determined for a vehicle subjected to less (or even no) nonholonomic constraints. Typically, the distance between the initial non-admissible trajectory and its admissible approximation should be kept smaller than a given threshold corresponding to a security margin with respect to obstacles. The underlying idea is to first solve the path planning problem for a mobile robot with more degrees of mobility (a simpler problem) and, in a second step, determine an acceptable approximation of the obtained solution in the form of an admissible trajectory. The approximation step is traditionally performed by applying open-loop control techniques repeatedly and "patching" pieces of trajectory together until a complete feasible trajectory which connects the two end points is obtained. Most of the time, the number of operations involved in this process is not known in advance. It seems to us that using a feedback control scheme, as described in the previous section, could constitute a viable alternative to open-loop control techniques. For instance, the reference vehicle (B_r) would represent the "unconstrained" system associated with the initial trajectory, and an approximating trajectory for the nonholonomic system (B) could be obtained via a single numerical integration of the closed-loop system equations. In this case, the "quality" of the approximation would be directly related to the upperbound of $\|f_\epsilon\|$, a value which can be chosen as small as desired.

7 Concluding comments

For non-holonomic vehicles, classical tracking feedback controllers, such as those derived from a linear approximation of the tracking error system, are known to break down when the reference vehicle is motionless. In this case, other control strategies have to be considered, such as continuous time-varying feedbacks or discontinuous feedbacks. But these latter controllers are themselves not well suited when the reference vehicle keeps moving. Modifying them in order to achieve unconditional and uniform boundedness of the tracking errors is a possibility which has seldom been explored so far. Therefore, until now, a policy of switching from one controller to another was needed in order to cover all possible motions of the reference vehicle. Finding a "good" switching strategy is not a simple problem. A foreseen practical advantage of the control approach outlined in [1] and the present paper is that it alleviates the necessity of switching between several controllers, at least when *perfect* tracking is not the most critical issue.

Another meaningful feature is the possibility, here demonstrated for chain systems, of tracking any differentiable trajectory in the state space with arbitrary precision. This could also be obtained by modifying existing time-varying controllers, but it would be at the expense of simplicity. In a way, this is what the control approach suggested by Theorem 1 achieves in an elegant manner. The feedback controls of Propositions 1 and 3 are indeed much related to time-varying feedbacks, which are themselves related to *oscillatory* open-loop control techniques. This is clear when considering that the function f_ϵ, which is part of the control expression, is periodic. If the *control frequencies* $\dot{\alpha}_i$, with $i \in \{m+1, \ldots, \bar{n}\}$, were constant then the obtained control would be a time-periodic feedback. The possibility of tracking a given trajectory with predetermined precision is closely related to the selection of adequate frequencies. Typically, higher frequencies allow for better trajectory approximation. As pointed out in the previous section, while the problem of approximating a trajectory has usually been addressed via open-loop control techniques involving *constant* frequencies (determined in a conservative manner), the control laws derived in the present paper can be viewed as smooth feedbacks with a frequency *self-adapting* capacity.

References

[1] W.E. Dixon, D.M. Dawson, E. Zergeroglu, and F. Zhang. Robust tracking and regulation control for mobile robots. *Int. J. of Robust and Nonlinear Cont.*, 10(4):199–216, 2000.

[2] H. Hermes. Nilpotent and high-order approximations of vector field systems. *SIAM Review*, 33:238–264, 1991.

[3] J.-C. Latombe. *Robot motion planning.* Kluwer Academic Publishers, 1991.

[4] P. Morin and C. Samson. A characterization of the lie algebra rank condition by transverse periodic functions. Technical Report 3873, INRIA, 2000. Submitted for publication.