The Application of Monotone Stability to Certain Saturated Feedback Interconnections

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Abstract

Due to the conservatism of the conventional, or affine, Small Gain Theorem, in some situations its use may not give an accurate indication of the true performance achievable in a given system. This is especially true of certain feedback loops with saturation elements present. Here, the recently derived *monotone* Small Gain Theorem is discussed with respect to these types of system, and it is shown how certain constructions of feedback will guarantee *global stability* of the interconnection. This result is then applied to two instances where this type of configuration naturally occurs: anti-windup compensation and systems with “soft” output constraints.

1 Introduction

The conventional Small Gain Theorem (SGT) is a common control tool, used to establish stability of a feedback interconnection of two sub-systems. Essentially it does this by ensuring that the loop gain, defined in terms of some appropriate norm, is less than unity. By loop gain, we mean the product of the two sub-systems' *finite* gains, which is an affine relationship between the norm of the input and the norm of the output.

For many interconnections (and one is particularly reminded of the situation in robust control), this criterion for stability is appropriate, as often little is known about the sub-system in the feedback loop. However, for certain systems, where a fairly accurate mathematical description is available for both sub-systems, this can lead to conservative results, which can hinder the design of compensators to deliver adequate performance.

In a recent paper, [3] proposed a generalisation of the conventional SGT. Their generalisation extended the SGT so that monotone relationships could be used, instead of only affine relationships, to link input and output norms of stable, causal operators. They showed that provided these monotone functions satisfied certain properties, the feedback loop would be stable. This considerably enlarged the class of systems for which stability could be proved.

Certain forms of interconnections containing saturation could take advantage of this new concept of stability, as the saturation nonlinearity is often quite well defined, and simple examples can demonstrate that the conventional SGT can be exceptionally conservative. A recent move in this direction was made by [7] who gave an *asymptotic* $\mathcal{L}_\infty$ Small Gain Theorem. Teel's results were quite general and applied to general *feed-forward* nonlinear systems, of which saturated systems are a special case. The basic idea of his results was to predict asymptotic information about the output of a system, given asymptotic information about the input.

In this paper, we discuss less general types of system than in [7], but we get a rather strong result: in certain types of saturated feedback loops, bounded-input-bounded-output (BIBO) stability can be globally guaranteed for a wide class of feedback elements. Moreover, the class of feedbacks which we construct are simple to implement, compared with some schemes which have been proposed in the literature.

In fact, the simplest of the feedbacks which are constructed are similar to those first introduced in [6] for a chain of integrators, from which we take our inspiration. [6] advocated a feedback of ‘nested saturations’ which could achieve global stabilisation for an integrator chain of arbitrary order; this represented an important advance in the theory of input-saturated control systems. The types of feedback we use are slightly different, and do not apply, in their current form, to systems with closed-right-half plane poles; this could be a direction for future work.

2 Preliminaries

The notation used is standard throughout, with $||(.)||$ representing an arbitrary norm and $\mathbb{R}_+$ representing the positive real scalars. Specifically the $\mathcal{L}_\infty$ norm is defined by

$$||u||_\infty := \sup_{t\geq 0} \max_i |u_i(t)| \tag{1}$$

The $\mathcal{L}_1$ norm is the system norm induced by the $\mathcal{L}_\infty$ norm and is denoted as $||(.)||_1$. The $\mathcal{H}^\infty$ norm of a linear system is defined as

$$||G||_\infty := \sup_{\omega} \bar{\sigma}(G(j\omega)) \tag{2}$$

where the distinction between the $\mathcal{H}^\infty$ norm and the $\mathcal{L}_\infty$ norm will be clear from the context. The rational subspace of systems having a finite $\mathcal{H}^\infty$ norm is denoted $\mathcal{RH}^\infty$.

The truncation operator is defined as

$$P(u(t)) := \begin{cases} u(t) & t \le T \\ 0 & t > T \end{cases} \tag{3}$$

All signal vectors in this paper are assumed to belong to an appropriate extended space. If U is a Banach space, then an extended space is defined as

$$\mathcal{U}_e := \{u \in U : \|P(u(t))\| < \infty, \quad \forall T \in [0, \infty)\} \tag{4}$$

A causal operator, G, is said to be *finite-gain* stable if, $\exists \gamma, \beta \in \mathbb{R}_+$ such that

$$\|G(u(t))\| \le \gamma \|u(t)\| + \beta, \quad \forall u(t) \in \mathcal{U}_e \tag{5}$$

This is the type of stability used in the conventional Small Gain Theorem. A weaker form of stability, *monotone stability*, is when a causal operator satisfies the following relationship

$$\|G(u(t))\| \le f(\|u(t)\|) + \beta, \quad \forall u(t) \in \mathcal{U}_e \tag{6}$$

where $\beta \in \mathbb{R}_+$ and $f(.)$ is monotone increasing (i.e. $x > y \Rightarrow f(x) > f(y)$). Properties of monotone functions are given in [3], for example. For later use we shall need the following sets of monotone functions

$$\begin{aligned} \mathcal{M} &:= \{f(.) : \mathbb{R}_+ \to \mathbb{R}_+ : f(.) \text{ monotone} \\ &\quad\ \ \text{increasing homeomorphism}\} \\ \mathcal{M}_0 &:= \mathcal{M} \cup O_F : O_F \text{ is the zero function f} = 0 \\ \mathcal{N} &:= \{f \in \mathcal{M} | \exists c \in \mathbb{R}_+, \exists g \in \mathcal{M} : f(x) \le x - g(x) + c\} \\ \mathcal{N}_0 &:= \mathcal{N} \cup O_F \end{aligned}$$

The composition operator is denoted by $\circ$ and the identity operator by i.

The general type of system we consider is displayed in Figure 1, which can be drawn, equivalently, as Figure 2. Essentially this represents a saturated interconnection with feedback driven by the difference between the saturated output and the nominal linear output. Both G and L are assumed to be stable causal linear operators and $f(.)$ is a memoryless nonlinear operator to be defined later. The following important assumption is made on the closed loop.

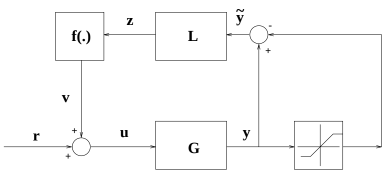

Figure 1: Saturated Interconnection

Assumption 1 (Well Posedness) *The system in Figure 1 (or equivalently in Figure 2) is assumed to be mathematically well-posed. That is for each input r, there exists a unique pair of internal signals, y and v.*

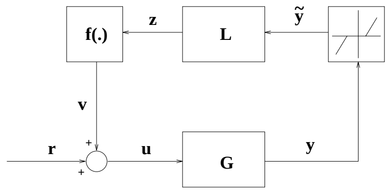

Figure 2: Interconnection with Deadzone

The importance of this assumption lies in the fact that the Monotone SGT we employ later, does not guarantee well-posedness, even though it can guarantee stability. This is in contrast to some finite gain SGT's based on the contraction mapping principle (see [1] or [5] for an incremental gain version), where well-posedness is guaranteed. In the sequel we shall also need the following assumption, which is satisfied in all practical cases.

Assumption 2 *A bound on $\|r\|_\infty$ exists.*

The (standard) saturation operator is defined as

$$sat_\Delta(u) = \begin{bmatrix} sat_{\Delta_1}(u_1) \\ sat_{\Delta_2}(u_2) \\ \vdots \\ sat_{\Delta_m}(u_m) \end{bmatrix} \tag{7}$$

where $sat_{\Delta_i}(u_i) = sign(u_i) \times \min(|u_i|, \Delta_i)$. For the remainder of the paper we assume that the saturation limits are the same in all channels ($\Delta_i = \Delta \forall i \in \{1, 2, \dots m\}$); this can be achieved by transfering gain to other elements in the loop. The deadzone function is defined as

$$Dz_\Delta(u) = \begin{bmatrix} Dz_{\Delta_1}(u_1) \\ Dz_{\Delta_2}(u_2) \\ \vdots \\ Dz_{\Delta_m}(u_m) \end{bmatrix} \tag{8}$$

where $Dz_{\Delta_i}(u_i) = u_i - sat_{\Delta_i}(u_i)$ is the standard scalar dead-zone function. Note that we assume that both the saturation and deadzone elements are de-centralised.

3 Motivating Example

To corroborate our assertion of the conservatism of the conventional SGT and to motivate the remainder of our work, this section introduces a simple example. Consider the feedback loop in Figure 2, where $L = k$ and G is given by

$$G(s) = \frac{1}{s+2} \tag{9}$$

where s is the Laplace transform variable. First assume that the deadzone and $f(.)$ elements are actually unity gain elements. As $G(s)$ is a first order transfer function, it is easy to verify that $\|G\|_1 = \|G\|_\infty = \frac{1}{2}$. For the conventional Small Gain Theorem to predict stability we must have - even in the absence of any nonlinear elements - that $k_{max} < 2$. From the root-locus of $G(s)$ it is obvious that the system is stable for $k \in [0, \infty]$.

Even when the nonlinearities are re-introduced, it can be verified that stability holds regardless of the gain k. In fact, only when we change the sign of the feedback does the SGT have any meaning for the purely linear case. This type of conservatism is acceptable and useful in the analysis of uncertain systems, but we can see that for simple nonlinear systems of this type it can be virtually useless.

4 Monotone Small Gain Theorem

Here the objective is to review the Monotone SGT from [3] and one of their important corollaries; this forms the basis of our work.

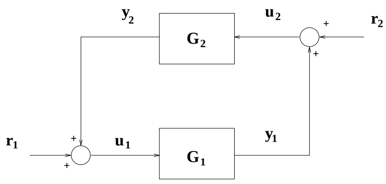

Figure 3: Small Gain Feedback System

Theorem 1 ([3]) *Consider the feedback system in Figure 3, where G_1 and G_2 are stable causal operators with monotone gains:*

$$\|y_1\| \leq g_1(\|u_1\|) + \beta_1 \quad (10)$$
$$\|y_2\| \leq g_2(\|u_2\|) + \beta_2 \quad (11)$$

Suppose that Assumption 1 holds, then the feedback system is monotone stable provided

$$g_1, g_2 \in \mathcal{M}_0, \quad \exists g \in \mathcal{M} : g_2 \circ (i+g) \circ g_1 \in \mathcal{N}_0 \quad (12)$$

The details of this theorem are quite technical and a full exposition can be found in [3]. However, we shall not use this general version; we use two less general results, which are also proved in [3]. For the first result, we make use of the following corollary.

Corollary 1 ([3]) *Consider the feedback system in Figure 3, where the sub-systems are monotone stable and satisfy*

$$\|y_1\| \leq \sigma_1 \|u_1\|^{p1} + \beta_1 \quad (13)$$
$$\|y_2\| \leq \sigma_2 \|u_2\|^{p2} + \beta_2 \quad (14)$$

Suppose Assumption 1 holds, then the system is monotone stable providing any one of the following conditions is true:

1. $\sigma_1 \sigma_2 = 0$;
2. $p_1 p_2 < 1$;
3. $p_1 p_2 = 1, \sigma_1 \sigma_2^{p_1} < 1, \sigma_2 \sigma_1^{p_2} < 1$

In particular we shall be using condition (2) of the above corollary. For the second approach used in the paper we shall use the following theorem (a special case of Theorem 1). In our case G_1 will be our linear system (and hence have an unbiased $\mathcal{L}_p$ induced norm), and G_2 will be our feedback element.

Theorem 2 ([3]) *Consider the feedback system in Figure 3, where G_1 is unbiased ($\beta_1 = 0$) and $r_2 = 0$. Suppose Assumption 1 holds, then the interconnection is monotone stable provided*

$$g_1, g_2 \in \mathcal{M}_0, \quad g_2 \circ g_1 \in \mathcal{N} \quad (15)$$

5 Saturated Interconnections

5.1 Simple Approach

We now apply the results of Corollary 1 to the interconnection in Figure 2. Recall that this is equivalent to Figure 1 and is basically a linear system with saturation, with feedback generated from $y - sat_\Delta(y) =: Dz_\Delta(y)$ fed through another linear system with a nonlinearity at its output. In this section, we let $f(.) = sat_\Pi(.)$. The saturation element after L is reminiscent of [6] (and also [8]) and the reason for its presence will become clear shortly.

Essentially our idea is as follows: if we can restrict the gain from $\tilde{y}$ to v by limiting the signal v by saturating it, we should be able to prove some sort of stability. Note that, if the conventional affine SGT is used, conservatism is again introduced. We now state our main result.

Theorem 3 *Consider the system in Figure 2, where G and L are both finite-gain stable linear operators; $\tilde{y} = Dz_\Delta(y)$; and $v = sat_\Pi(z)$, where Δ and Π are not necessarily the same. Assume that Assumptions 1 and 2 hold and that G and Δ are fixed. Then the feedback loop is bounded-input-bounded-output stable for all stable L and $\Pi \in [0, \infty)$.*

Proof: First note that by finite-gain stability we can write

$$v = sat_\Pi(y_l) \Rightarrow \|v\|_\infty \leq \|y_l\|_\infty \quad (16)$$
$$y_l = L\tilde{y} \Rightarrow \|y_l\|_\infty \leq \|L\|_1 \|\tilde{y}\|_\infty \quad (17)$$

Next consider the deadzone $\tilde{y} = Dz_\Delta(y)$. In [2] it was shown that

$$|Dz_\Delta(y_i)| \leq \frac{y_i^2}{4\Delta} \quad (18)$$

From this and the definition of the $\mathcal{L}_\infty$ norm, we obtain

$$\|Dz(y)\|_\infty \leq \frac{\|y\|_\infty^2}{4\Delta} \quad (19)$$

Thus it follows that

$$\|v\|_\infty \leq \frac{\|L\|_1}{4\Delta} \|y\|_\infty^2 \quad (20)$$

Next, also note that by finite-gain stability we can write that

$$\|y\|_\infty \leq \|G\|_1 \|u\|_\infty \quad (21)$$
$$= \|G\|_1 \|u\|_\infty^{1-p_g} \|u\|_\infty^{p_g}, \quad \forall p_g \in [0,1] \quad (22)$$

The idea now is to bound this gain by a function of the form $\|y\|_\infty \leq \gamma_g \|u\|_\infty^{p_g}$. Now, as $v = sat_\Pi(y_l)$, it follows that $\|v\|_\infty = \Pi$. Also, by Assumption 2, $\exists \bar{r} : \|r\|_\infty \leq \bar{r}$. Hence,

$$\|u\|_\infty \leq \|v\|_\infty + \|r\|_\infty \leq \Pi + \bar{r} \quad (23)$$

Using this relationship, it thus follows that we can write

$$||y||_\infty \leq ||G||_1(\Pi + \bar{r})^{1-p_g}||u||_\infty^{p_g} \tag{24}$$

Note that equations (20) and (24) are in the form of equations (13) and (14). Thus direct application of Corollary 1, ensures that stability holds if $p_g < \frac{1}{2}$. Note that p_g can always be chosen so that this relationship holds. ♢♢

Remark 1: The essence of the argument of the above theorem is to saturate the feedback, using $sat_\Pi(.)$ so that we can bound $||v||_\infty$. As we assume that $||r||_\infty$ is also bounded (Assumption 2) this allows us to bound $||u||_\infty$. In turn, as we have bounded the size of $||u||_\infty$ we can therefore bound $||y||_\infty$ as a function of $||u||_\infty^{p_g}$, where p_g could be less than 1. Without actually bounding the size of $||u||_\infty$ (which requires the feedback to be saturated), we could not show this, as $||y|| \leq \gamma||u||$ does not imply $||y|| \leq \delta||u||^p, \quad p < 1$ (although the converse is true).

Remark 2: At first the result does not seem very strong as it requires both $||v||_\infty$ and $||r||_\infty$ to be bounded. However, the strength of the result comes from the fact that, as stability only requires $p_g < \frac{1}{2}$ to be satisfied, the bounds on $||v||_\infty$ and $||r||_\infty$ can be *arbitrarily large*; from which we deduce that stability holds $\forall L \in \mathcal{RH}^\infty, \Pi \in [0, \infty)$, and bounded inputs r. [1]

5.2 Scheduled Approach

Here we use Theorem 2 to provide stability results for a special class of feedbacks, which are intuitive in their construction. The idea is to use a certain form of gain-scheduling to form a control law which guarantees global stability. Note that the idea of scheduling to improve performance in the face of input constraints was considered in [4].

Theorem 4 *Consider the feedback interconnection in Figure 2, where G and L are both finite-gain stable linear operators and $\tilde{y} = Dz_\Delta(y)$. Suppose that Assumption 1 is satisfied and that $||G||_1||L||_1 = \beta$. Then the following choice for the nonlinear block provides monotone $\mathcal{L}_\infty$ stability for the feedback system:*

$$v = f(z) = \frac{1}{1+\mu||z||}z + \left\{1 - \frac{1}{1+\mu||z||}\right\}\beta^{-1}\epsilon z \tag{25}$$

where $||z|| = \max_i |z_i|$, $\mu \in (0, \infty)$ *and* $0 < \epsilon < 1$.

Proof: To prove stability, it is sufficient to ensure that the conditions in Theorem 2 are satisfied. First note that as G is linear it is finite gain, and hence monotone, stable and thus is unbiased. Next we must prove that the map from y to v is also monotone stable. Note that

$$\begin{aligned} v &= f(z) && (26)\\ &= \frac{z}{1+\mu||z||} + \left\{\frac{1+||z||-1}{1+\mu||z||}\right\}\beta^{-1}\epsilon z && (27)\\ &= \frac{z + ||z||z\mu\beta^{-1}\epsilon}{1+\mu||z||} && (28)\end{aligned}$$

[1]As pointed out in later sections, it may not be prudent to make certain choices for L and Π.

It now follows that

$$\begin{aligned} ||v||_\infty &\leq \sup_{t\geq 0}\max_i \left|\left(\frac{z + ||z||z\mu\beta^{-1}\epsilon}{1+\mu||z||}\right)_i\right| && (29)\\ &\leq \sup_{t\geq 0}\left\{\frac{||z|| + ||z||^2\mu\beta^{-1}\epsilon}{1+\mu||z||}\right\} && (30)\\ &= \frac{||z||_\infty + ||z||_\infty^2\mu\beta^{-1}\epsilon}{1+\mu||z||_\infty} && (31)\\ &=: f_v(z) && (32)\end{aligned}$$

where the last but one equality follows from the fact that $\frac{||z||+||z||^2\mu\beta^{-1}\epsilon}{1+\mu||z||}$ is a monotonically increasing function, so its supremum is obtained at the supremum of $||z||$. Next recall that $||z||_\infty \leq ||L||_1||y||_\infty$, which is also monotonically increasing. As the composition of two monotone functions is also monotone, $f_v(||L||_1||y||_\infty)$ is a monotonically increasing function of $||y||_\infty$. In the notation of Theorem 2, $g_2(.) = f_v(||L||_1(.))$, and $g_1(.) = ||G||_1(.)$. Finally we need to show that $f_v(||L||_1||G||_1(.)) \in \mathcal{N}$ ($g_2 \circ g_1 \in \mathcal{N}$):

$$\begin{aligned} f_v(||u||_\infty) &= \frac{\beta||u||_\infty(1+\mu\epsilon||u||_\infty)}{1+\mu\beta||u||_\infty} && (33)\\ &\leq \frac{\beta||u||_\infty(1+\mu\epsilon||u||_\infty)}{\mu\beta||u||_\infty} && (34)\\ &= \mu^{-1} + \epsilon||u||_\infty && (35)\end{aligned}$$

Now as $\epsilon \in (0, 1)$

$$\exists\delta \in (0,1) : \epsilon = 1 - \delta \tag{36}$$

so we can write

$$f_v(||u||_\infty) \leq ||u||_\infty - \delta||u||_\infty + \mu^{-1} \tag{37}$$

which is exactly the form of functions in $\mathcal{N}$, and the proof is complete. ♢♢

Remark 3: The basic idea of this gain-scheduled scheme is to enable large gains to be used when the z is small and gains which respect the conventional affine small gain theorem when z is large. To see this note that when $z = 0$, $f_z = z$; the feedback interconnection functions linearly except for the deadzone. When $z >> 1$, $f_z \approx \beta^{-1}\epsilon z$, meaning that the total, affine, loop gain is less than one.

Remark 4: Along with the simple method introduced earlier, this result essentially allows one to design the linear part of the feedback loop, L, such that linear objectives are met. Stability is ensured by the nonlinear scheduler, $f(.)$, which, when the output gets sufficiently large takes corrective measures to reduce the gain of the loop. This theorem has the advantage that Assumption 2 does not need to be satisfied, and allows more flexibility in design. Furthermore, similar gain-schedules to this could be used with similar effect.

5.3 Choice of Parameters

Although stability of the system in Figure 2 (equivalently Figure 1) has been established for any finite saturation limit, Π, or an appropriate schedule function, $f(.)$, and for any stable $L(s)$, no insight into particular choices of these parameters has been given. This will be the subject of continuing research, but a few guidelines will be given below.

5.3.1 Nominal Linear System: One convenient way to view the design of L is from a purely linear systems standpoint. This view may be restrictive, but it gives the designer a plethora of mathematical tools to choose from. In this case, consider Figure 1, and ignore the $f(.)$ element (which can be considered linear for small signals); also instead of considering $\tilde{y} = sat(y) - y$, take this to be $\tilde{y} = y + d$, where d is some sort of bounded disturbance. Then possible design objectives are:

1. Ensure that the nominal linear system is stable and well-posed i.e. ensure

$$P := \begin{bmatrix} I & -G \\ -L & I \end{bmatrix}^{-1} \in \mathcal{RH}^\infty \quad (38)$$

2. Enforce a performance objective on $y + d$, and/or v i.e. minimise

$$||y + d||_p, \quad ||v||_p \quad (39)$$

By considering the simple motivating example in Section 3, it can easily be deduced that choosing $P \in \mathcal{RH}^\infty$ for that system is equivalent to choosing k with negative feedback. Simulation shows that for large values of k, choosing negative feedback instead of positive gives much more attractive results, even though both cases are stable (by Theorem 3).

Choosing $P \notin \mathcal{RH}^\infty$ can be thought of as artificially inducing stability, in an otherwise unstable system, through use of the saturation function. If $P \notin \mathcal{RH}^\infty$, the output, y, sometimes "winds-up" to an excessive value, which is closely linked to the saturation limit, Π.

5.3.2 Saturated Feedback: The choice of the level of saturation, Π, to give to the feedback, is a more complex problem, and at this time we have no concrete answers to this question. We have found that, provided $P \in \mathcal{RH}^\infty$, a reasonably large Π will suffice and allow good performance to be achieved - in the sense that $\tilde{y}$ will be small if $||L||$ is large. An overly small Π in this case prevents effective use of the feedback.

Conversely, in the case that $P \notin \mathcal{RH}^\infty$, to prevent $\tilde{y}$ winding up to an excessively large value, it seems prudent to keep Π fairly small to limit the effect that the feedback can have on the system.

5.3.3 Scheduled Feedback: The choice of parameters in the scheduled approach does not appear to be as intuitive or straight-forward as in the saturated case. Limited simulation on examples has suggested that, for the case when $P \in \mathcal{RH}^\infty$ the following choices yield acceptable results. If ϵ is fixed as a number slightly less than unity (e.g. 0.95), it will mean that f_v is "only just" a member of $\mathcal{N}$. This implies that a certain level of performance will still be delivered when the signal, z is large; choosing ϵ less than this would decrease the performance attainable for large z. Realistically this would have to be compared against robustness requirements, but this is beyond the scope of this paper. A suitably small choice for μ seems to be compatible with the requirements that the feedback gain should be as large as possible without causing instability.

When $P \notin \mathcal{RH}^\infty$, which simulation so far suggests is fraught with problems, different choices for ϵ and μ are required. Our suggestions in this instance are not complete, but indications so far suggest that ϵ should be chosen small and μ large, although the reasons are not wholly clear.

6 Applications

6.1 Weston-Postlethwaite Anti-Windup

In [10] a new interpretation of the anti-windup problem was given in terms of stable transfer function $M(s)$, assuming that the plant in question is also stable. The basic result is that by conditioning with M, the anti-windup problem illustrated in Figure 4 may viewed as several sub-systems, as shown in Figure 5. Assuming that the nominal linear loop is stable, and as $G, M \in \mathcal{RH}^\infty$, the stability of the system depends purely on the stability of the nonlinear loop.

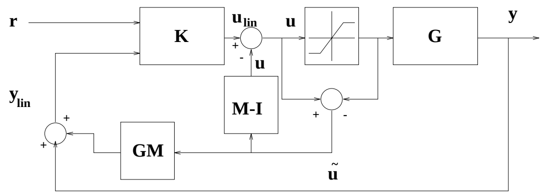

Figure 4: Conditioning with $M(s)$

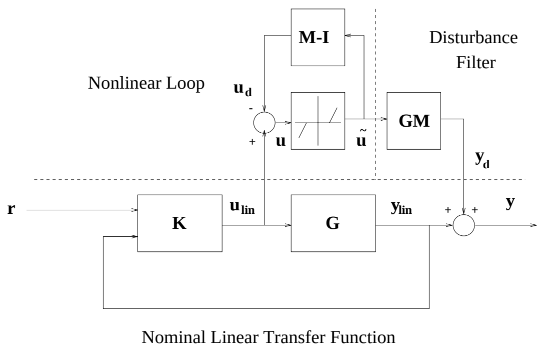

Figure 5: Equivalent Representation of Conditioning with $M(s)$

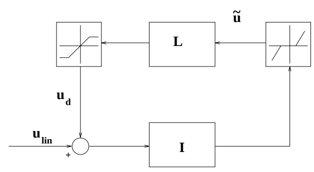

Figure 6: Equivalent Representation of Nonlinear Loop

With M being purely linear, several conditions have to be satisfied, based on Absolute Stability criteria or the affine SGT. However, it actually turns out that we can replace

M by a stable nonlinear operator $f_M(.)$ (see Appendix A). Then if we choose

$$f_M(\tilde{u}) := \tilde{u} + sat_{\Pi}(L\tilde{u}) \tag{40}$$

we can re-draw the nonlinear loop in Figure 5 as the feedback loop in Figure 6, which is in the same form as Figure 2. Hence Theorem 3 can be applied and thus stability follows for all $L \in \mathcal{RH}_\infty$ and all finite Π. The guidelines above can then be followed to choose L and Π, if desired.

6.2 Output "anti-windup"

The output "anti-windup" problem for systems with soft constraints has not been given much attention in the literature, but can be important in some instances. This problem occurs when the output of a system exceeds a certain value for a given period of time, but with the assumptions that:

1. It is allowed to exceed given values for short amounts of time.
2. It is desirable for the output to be regulated to below these values as far as possible.

This situation is described in more detail in [9], but occurs for example when an engine over-torques and, while not causing immediate damage can have an effect long-term; hence it is desirable to keep these over-torques as small and infrequent as possible. We stress this type of approach may not be the best to use if hard constraints are active on the output.

This type of configuration is exactly that described by Figure 1, where the aim is to keep $\tilde{y}$ as small as possible, which in turn means that $sat(y) \approx y$. Note that in this situation it is an unrestrictive assumption to assume that G is stable, as here G represents the nominal **closed loop** system. Theorem 3 can thus be applied and the guidelines above used in choosing appropriate L and Π. Note that for minimum phase SISO closed loops, choosing L as a very large constant may give very satisfactory results and avoids using extra dynamics. This result could not be arrived at using the affine SGT.

7 Conclusion

In this paper we have applied the Monotone SGT to the stability analysis of a type of saturated system which occurs commonly in control problems. Our main results have shown that a certain class of feedbacks solves the stabilisation problem, and we have also given guidelines for choosing parameters to solve this problem in a practically appealing way. One of the strengths of these results is that through the use of the Monotone SGT we have been able to construct feedbacks which would have been prohibited by the affine SGT.

There is a need to extend these results in several areas. For use in an anti-windup situation it seems appropriate to consider the situation when the plant in question is open-loop unstable. Also noting the results of [6], it would be interesting to see if a construction similar to the one proposed there could be arrived at using the Monotone SGT

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A Modified WP Scheme

Here we show how we can modify the original WP scheme, which was a purely linear conditioning scheme, so that instead of conditioning with M, we condition with a nonlinear operator, $f_M(.)$. As with the original scheme, we assume that the plant G is stable. First, substituting M for $f_M(.)$ in Figure 4, note that

$$\begin{aligned} y_{lin} &= y + y_d \\ &= Gf_M(\tilde{u}) + Gsat_\Delta(u) \\ &= Gf_M(\tilde{u}) + G(u - Dz_\Delta(u)) \\ &= Gf_M(\tilde{u}) + G(u_{lin} - [f_M(.) - I]\tilde{u}) - G\tilde{u} \\ &= Gu_{lin} \end{aligned} \tag{41}$$

Also note that

$$y = y_{lin} - y_d \tag{42}$$

$$y_d = Gf_M(\tilde{u}) \tag{43}$$

$$\tilde{u} = Dz_\Delta(u) \tag{44}$$

$$u = u_{lin} - [f_M(.) - I]\tilde{u} \tag{45}$$

Combining equations (41) - (45) allows us to draw the block diagram as in Figure 5 (with $f_M(.)$ replacing M, of course). Choosing $f_M(.) = (.) + sat_\Pi(.)$ allows us to arrive at the expressions in the main body of the paper.