Finite Time Stability and Stabilization[1]

W. PERRUQUETTI (*), S. DRAKUNOV (**)
(*) Ecole Centrale de LILLE, LAIL-UPRESA CNRS 8021, BP 48,
59 651 Villeneuve d'Ascq CEDEX - FRANCE.
E-mail : `Wilfrid.Perruquetti@ec-lille.fr`
Phone : (33) +3 20 33 54 50 Fax: (33) +3 20 33 54 18
(**) Dept. of EECS, 204 Stanley Thomas Hall, Tulane University,
New-Orleans, LA 70118, USA.
E-mail: `drakunov@eecs.tulane.edu`
Phone: 504/861-3285 Fax: 504/862-3293

Abstract

In this paper, finite time stability and stabilization are investigated for systems described by ordinary differential equations (ODE) or differential inclusions: some sufficient conditions are given for scalar and $n-$dimensional cases. Then, a stabilization result for i/o linearizable systems is derived from these results.

1 Intoduction

For some applications or control scheme, it is usefull to derive some finite time or practical stability conditions. For example, an interesting finite time stabilizing control design relies upon sliding mode theory (see [13]) and an other one upon optimal control theory (see [11]). In both cases the controller leads to some state discontinuous feedback control laws. The main idea relies on infinite eigenvalue assignation for the closed loop system at the origin. For example, the solutions of $\dot{x} = -|x|^{\frac{-1}{3}} x, x \in \mathbb{R}$ are $x(t) = \mathrm{sign}(x_0)(|x_0|^{\frac{1}{3}} - \frac{t}{3})^3$, if $0 < t < 3|x_0|^{\frac{1}{3}}, x(t) = 0$ if $t \geq 3|x_0|^{\frac{1}{3}}$. Thus they converge in finite time to the origin, note that the eigenvalue of the "linearized" part ($-|x|^{\frac{-1}{3}}$) tends to minus infinity as the state tends to zero. As a consequence of this "infinite eigenvalue assignation", the right hand side of the ODE becomes non Lipschitz while the state reaches zero. These kinds of controllers have gained much attention this last decade (see [3] where an interesting finite time continuous controller is provided for double integrator) for two main reasons: some robustness properties occur (even if sometimes the finite time convergence property is lost), to provide higher order sliding mode control scheme (see [6, 8]).

2 Notations

The following notations are used:

1. $\mathrm{cl}(\mathcal{S}), \overline{\mathrm{conv}}(\mathcal{S})$ denotes respectively the closure and the closure of the convex hull of the set $\mathcal{S}$,
2. for x in a normed vector space $\mathcal{X}$: $\mathcal{B}_\varepsilon(x)$ is the open ball centered at x of radius ε, this is $\mathcal{B}_\varepsilon(x) = \{x \in \mathcal{X} : \|x\| < \varepsilon\}$, for a set $\mathcal{A} \subset \mathcal{X} : \mathcal{B}_\varepsilon(\mathcal{A}) = \{y \in \mathcal{X} : \exists x \in \mathcal{A} \wedge y \in \mathcal{B}_\varepsilon(x)\} = \bigcup_{x \in \mathcal{A}} \mathcal{B}_\varepsilon(x)$,
3. Let $V : \mathbb{R}^{n+1} \to \mathbb{R}_+, (t,x) \mapsto V(t,x)$. When the classical gradient exists, it is denoted by $\nabla V(t,x)$ (gradient of V evaluated at (t,x)). Let Ω be the union of any set of zero measure with the set where V fails to be differentiable, then, if V is locally Lipschitz in (t,x), one can defined the generalized gradient (see [4]) by $\partial_C V(t,x) = \overline{\mathrm{conv}} \left\{ \lim_{(t_i,x_i) \notin \Omega \to (t,x)} \nabla V(t_i,x_i) \right\}$, also called Clarke's gradient for finite dimensional Banach space in short generalized gradient.
4. $\varphi_a(x) = |x|^a \,\mathrm{sign}(x)$.

3 Problem formulation

Consider systems of the form

$$\dot{x} = f(t,x,u), \tag{1}$$

where $t \in \mathbb{R}, x \in \mathcal{X}, u \in \mathcal{U}$, f is defined for almost every $x \in \mathcal{X}$, this is (1) holds for every x in $\mathcal{X} \backslash \mathcal{M}$ where $\mathcal{X} \subset \mathbb{R}^n$ is a manifold and $\mathcal{M}$ is a set of zero Lebesgue measure. Sometimes will also be considered the free system (without control)

$$\dot{x} = f(t,x), t \in \mathbb{R}, x \in \mathcal{X}. \tag{2}$$

We are interested in the study of the finite time stabilization of (1), for this the two following cases will considered: $n = 1$ and the general case : $n \geq 1$.

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The notion of finite time stabilization to a set is a special case of the notion of practical stability with settling time (see [9] and references herein). From the engineering point of view, it is important to know if starting from an initial time $t_0 \in \mathcal{T}_0$ and from an initial state in $\mathcal{D}_i \subset \mathcal{X}$, solutions:

1. are at least defined over $[t_0, t_0 + T_f]$,
2. stay in $\mathcal{D}_a \subset \mathcal{X}$ during that time,
3. reach $\mathcal{A}$ a compact set and stay in this set for $t \in [t_0 + T_s, t_0 + T_f]$.

Moreover, as we will see later, solutions to (2) can be non unique. So we can summarize these remarks in the following

Definition 1 *The compact set* $\mathcal{A}$ *is* **practically stable** *with respect to* $\{\mathcal{T}_0, T_s, T_f, \mathcal{D}_i, \mathcal{D}_a\}$ *iff* $\forall x_0 \in \mathcal{D}_i, \forall t_0 \in \mathcal{T}_0$ *: any solution of (2) denoted by* $x(t, t_0, x_0)|_{(2)}$ *is defined over* $[t_0, t_0 + T_f]$ *and satisfies*

$$\begin{aligned} x(t, t_0, x_0)|_{(2)} &\in \mathcal{D}_a, \forall t \in [t_0, t_0 + T_f], && (3) \\ x(t, t_0, x_0)|_{(2)} &\in \mathcal{A}, \forall t \in [t_0 + T_s, t_0 + T_f]. && (4) \end{aligned}$$

The compact set $\mathcal{A}$ *is* **finite time practically stable** *with respect to* $\{\mathcal{T}_0, T_s, \mathcal{D}_i, \mathcal{D}_a\}$ *iff it is* **practically stable** *with respect to* $\{\mathcal{T}_0, T_s, \infty, \mathcal{D}_i, \mathcal{D}_a\}$.

Note, that from definition 1: $\mathcal{A} \subset \mathcal{D}_a \subset \mathcal{D}_i$. In a similar way, one can define the notion of practical stabilization if there exists $u \in \mathcal{U}$ such that the solutions $x(t, t_0, x_0, u_{[t_0,t]})$ are satisfying (3) and (4). One may also be interested in the Lyapunov-like version of this last definition (see [3] for autonomous system)

Definition 2 *The compact set* $\mathcal{A}$ *is* **locally finite time stable** *with respect to* $\{\mathcal{T}_0, \mathcal{D}_i\}$ *iff*

1. $\forall t_0 \in \mathcal{T}_0, \forall x_0 \in \mathcal{D}_i$ *: all* $x(t, t_0, x_0)|_{(2)}$ *are defined for all* $t \geq t_0$,
2. $\mathcal{A}$ *is* **locally stable** *with respect to* $\{\mathcal{T}_0, \mathcal{D}_i\}$ *; this is :* $\forall t_0 \in \mathcal{T}_0, \forall \varepsilon > 0, \exists \delta(t_0, \varepsilon) : x_0 \in \mathcal{B}_\delta(\mathcal{A}) \Rightarrow$ *all* $x(t, t_0, x_0)|_{(2)}$ *remain in* $\mathcal{B}_\varepsilon(\mathcal{A})$ *for all* $t \geq t_0$,
3. *all trajectories starting from* $t_0 \in \mathcal{T}_0$ *and any given point in the domain of stability converge in finite time to* $\mathcal{A}$ *; this is :* $\forall t_0 \in \mathcal{T}_0, \forall x_0 \in \mathcal{D}_i, \exists T_s(t_0, x_0) < \infty : x(t, t_0, x_0)|_{(2)} \in \mathcal{A}, \forall t \geq T_s$.

In these definitions, when $\mathcal{D}_i$ is $\mathcal{X}$, the property is said to be **global**. For autonomous systems $\mathcal{T}_0$ does not play any role and is set to $\mathbb{R}$.

4 Sufficient Condition for finite time stability

In this section, we give sufficient conditions for finite time stability: for this we start from some scalar criterions (section 4.1) which are extended to the general case using Lyapunov functions (section 4.2).

4.1 Scalar case

Solutions of $\dot{x} = -\varphi_a(x), a \in]0, 1[$, are $x(t) = \mathrm{sign}(x_0)(|x_0|^{(1-a)} - (1-a)t)^{1/(1-a)}, 0 \leq t \leq \frac{|x_0|^{(1-a)}}{(1-a)}$ and zero after this time ; thus they converge to zero in finite time $t = \frac{|x_0|^{(1-a)}}{(1-a)}$. This fact is used in [3] to derive a finite time continuous stabilizing controller. In fact, the right hand side of the ODE should be a contracting vector field for which locally at the origin a finite time caracterization relies upon the following result:

Lemma 3 *Let us consider*

$$\dot{x} = f(x), x \in \mathbb{R}, \tag{5}$$

where f *is continuous in* x*. Suppose that there exists* $\mathcal{N}(0)$*, an open connected neigbourhood of the origin, such that:*

1. $[x \in \mathcal{N}(0) \wedge f(x) = 0] \Leftrightarrow [x = 0]$,
2. $\forall x \in \mathcal{N}(0) \backslash \{0\}$ *:* $x f(x) < 0$,
3. $|f(x)| \underset{x=0}{\sim} |x|^a$ *with* $a \in]0, 1[$,

then for any $x_0 \in \mathcal{N}(0)$*, there exists at least one solution ; every solution converges to zero in finite time* $T_s(x_0)$ *and if* $\mathcal{N}(0)$ *is bounded there exists a finite positive real* T_s *such that the origin is* **finite time stable** *with respect to* $\{T_s, \mathcal{N}(0), \mathcal{N}(0)\}$. ■

One can obviously extend this lemma for a compact set $\mathcal{A}$.

Proof : ▶ Condition 2 implies that $\mathcal{N}(0)$ is positively invariant thus solutions are forward-complete (defined for $t \geq 0$). If in addition Condition 1 holds, then such solutions converge to zero in a time $T(x_0) \leq \infty$. Moreover, for a given $\varepsilon > 0$ such that the compact set $\mathcal{B}_\varepsilon(0) = \{x : |x| \leq \varepsilon\} \subset \mathcal{N}(0)$, and a point x_0 in $\mathcal{N}(0)$, there exists a finite time $T_\varepsilon(x_0) < \infty$ such that trajectories starting from x_0 reach and stay in $\mathcal{B}_\varepsilon$ (because the origin is attractive). Thus $T(x_0) \leq T_\varepsilon(x_0) + \int_0^\varepsilon \frac{dx}{|f(x)|}$, this integral is definite (finite) if condition 3 holds. Obviously $T_s(x_0)$ is continuous with respect to x_0 and thus the last assertion results form the existence of maximun of a continuous real valued function over compact set (take $\mathrm{cl}(\mathcal{N}(0))$). ◀

Remark 4 *In the above result, condition 3 may be replaced by a weaker one:* $\int_0^{\varepsilon} \frac{dx}{|f(x)|} < \infty$ *for some* $\varepsilon > 0$.

Example 5 *Consider the following system* $\dot{x} = u$. *First let* $u = -\varphi_a(x), a \in]0,1[$, *then hypothesis of lemma (3) are satisfied. Thus, for any real* x_0, *solutions to the associated Cauchy problem converge to zero in finite time* $T_s^1(x_0) = \int_0^{|x_0|} \frac{dx}{|x|^a} = \frac{|x_0|^{1-a}}{1-a}$. *One can note that the control* $u = -\varphi_a(x) - bx, a \in]0,1[, b > 0$ *accelerates the convergence: any solution starting from the initial condition* x_0 *converges in finite time* $T_s(x_0) = \frac{\ln(1+b|x_0|^{1-a})}{b(1-a)} < T_s^1(x_0)$ *(if* $x_0 \neq 0$*), note that* $\lim_{b\to\infty} \frac{\ln(1+b|x_0|^{1-a})}{b(1-a)} = 0$. *This fact can be used for example in sliding mode control (see [13]).*

Lemma 6 *Let us consider*

$$\dot{x} = f(t,x), t \in \mathbb{R}, x \in \mathbb{R}, \tag{6}$$

where f *is measurable in* t *and continuous in* x. *Suppose that there exists* $\mathcal{N}(0)$, *an open connected neigbourhood of the origin, such that:*

1. $\forall t \in \mathbb{R} : [x \in \mathcal{N}(0) \wedge f(t,x) = 0] \Leftrightarrow [x = 0]$,
2. $\forall t \in \mathbb{R}, \forall x \in \mathcal{N}(0)\backslash\{0\} : xf(t,x) < 0$,
3. $|f(t,x)| \underset{x=0}{\sim} h(t)\,|x|^a$ *with* $a \in]0,1[$ *and* $h(t) > 0$ *for* $t > T_0$,
4. *for any fixed* x *sufficiently small, there exists a function* m *Lebesgue integrable*: $|f(t,x)| \leq m(t)$,

then for any $x_0 \in \mathcal{N}(0)$ *and any* $t_0 > T_0$, *there exists at least one solution passing through* (t_0, x_0) *defined for* $t > t_0$, *every solution converges to zero in finite time, the origin is* **locally finite time stable**. ■

Proof : ► Using Lyapunov's theorem one obtains the forward-completness of solutions and that for a given $\varepsilon > 0$ such that the compact set $\mathcal{B}_\varepsilon(0) = \{x : |x| \leq \varepsilon\} \subset \mathcal{N}(0)$, there exists a finite time $T_\varepsilon(t_0, x_0) < \infty$ such that trajectories starting in $\mathcal{N}(0)$ reach and stay in $\mathcal{B}_\varepsilon(0)$ (because the origin is atractive). Shrinking ε if necessary : $|f(t,x)| = \beta h(t)\,|x|^a\,(1 + r(t,x))$ (condition 3) and as f is Lebesgue integrable with respect to time, one obtains: $0 \leq r_1(t) < r(t,x) < r_2(t)$, with $h(\tau)(1 + r_1(\tau))$ integrable (due to measurablity in t of f and condition 4). Thus,

$$F(T_f - T_\varepsilon) = \int_0^{T - T_\varepsilon} h(\tau)(1 + r_1(\tau))d\tau \leq \int_0^{\varepsilon} \frac{dx}{|x|^a} < \infty,$$

F is continuous strictly increasing which leads to $T_f - T_\varepsilon < \infty$. ◄

Example 7 *Consider* $\dot{x} = f(t,x) = -(2 + g(t))\varphi_a(x)$, *with* $a \in]0,1[$ *and* $g(t) = \sin(t)$ *if* $t \in \mathbb{Q}$, $g(t) = \cos(t)$ *if* $t \in \mathbb{R}\backslash\mathbb{Q}$. *The function* $f(t,x)$ *is continuous in* x *and measurable in* t, *as* $xf(t,x) \leq -|x|^a$, $|f(t,x)| \underset{x=0}{\sim} |2 + g(t)|\,|x|^a$ *and the measurable function* $2 + g(t)$ *is bounded above by 3 and below by 1. Thus conditions 1 to 4 hold. Using the previous result, one obtains that for any* $t \in \mathbb{R}, x \in \mathbb{R}$, *the solutions are defined for any time, bounded and converge in finite time to zero.*

Let us consider

$$\dot{x} \underset{\text{a.e}}{=} f(t,x), \tag{7}$$

defined for almost every $t \in \mathbb{R}, x \in \mathbb{R}$, where f is measurable in t and continuous for $(t,x) \in \mathbb{R}^2\backslash\mathcal{M}$ where $\mathcal{M}$ is a set of zero Lebesgue measure. The solutions of this equation are to be understood in the sense of differential inclusions (see [1, 5, 7]). For this, change (7) into a differential inclusion

$$\dot{x} \in F(t,x), \tag{8}$$

letting $F(t,x) = \bigcap_{\varepsilon>0} \overline{\text{conv}}(f(t, B_\varepsilon(x)\backslash\mathcal{M}))$, then a solution to the associate Cauchy problem ($x(t_0) = x_0$) is an absolutely continuous[1] function ϕ defined on a non empty set $\mathcal{I}(t_0, x_0) \subset \mathbb{R}$ which contains t_0 such that $\phi(t_0) = x_0$ and $\frac{d\phi(t)}{dt} \in F(t, \phi(t)), \forall t \in \mathcal{I}(t_0, x_0)$.

First we give sufficient conditions for finite time stabilization of (8) from which a corollary is derived to obtain sufficient conditions for (7).

Theorem 8 *Suppose that* F *is measurable in* t, *upper-semi continuous in* x, *closed, convex and that there exists* $\mathcal{N}(0)$, *an open connected neigbourhood of the origin, such that:*

1. $\forall t \in \mathbb{R} : [x \in \mathcal{N}(0) \wedge 0 \in F(t,x)] \Leftrightarrow [x = 0]$,
2. $\forall t \in \mathbb{R}, \forall x \in \mathcal{N}(0)\backslash\{0\}, \forall y \in F(t,x) : xy < 0$,
3. *for any fixed* x *sufficiently small and for* $t > T_0$ *:* $\text{graph}\,(|F|) \subset \{(t,x,y) : \underline{h}(t)\,|x|^{\underline{a}}\,(1 + \underline{r}(t,x)) \leq y \leq \overline{h}(t)(1 + \overline{r}(t,x))\}$ *with* $\underline{a} \in]0,1[$, $\overline{h}(t) > 0$ *and* $\underline{h}(t) > 0$,
4. *for any fixed* x *sufficiently small, there exists a function* $\overline{m}$ *Lebesgue integrable* : $\sup(|\underline{h}(t)\,|x|^{\underline{a}}\,(1 + \underline{r}(t,x))|\,, |\overline{h}(t)(1 + \overline{r}(t,x))|) \leq \overline{m}(t)$,
5. $\forall t \in \mathbb{R}, \forall x \in \mathcal{N}(0), \forall y \in F(t,x)$, *there exists a function* m *Lebesgue integrable* : $|y| \leq m(t)$,

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[1] $\phi : [\alpha, \beta] \mapsto \mathbb{R}^n$ is **absolutely continuous** if $\forall\varepsilon > 0, \exists\delta(\varepsilon) > 0 : \forall \{]\alpha_i, \beta_i[\}_{i\in\{1..n\}}\,,]\alpha_i, \beta_i[\subset [\alpha, \beta], \sum_{i=1}^n (\beta_i - \alpha_i) \leq \delta(\varepsilon) \Rightarrow \sum_{i=1}^n \|\phi(\beta_i) - \phi(\alpha_i)\| \leq \varepsilon$.

then for any $x_0 \in \mathcal{N}(0)$ and any $t_0 > T_0$, there exists at least one solution passing trough (t_0, x_0) defined for $t > t_0$ and every solution converges to zero in finite time. ■

Proof (sketch) : ▶We just prove the result for any initial positive condition $0 < x_0 < a$ (a is the upper-bound of $\mathcal{N}(0)$). The prove is similar when $x_0 < 0$.

Fact 1 (first let us note that the construction of the multifunction F implies that F is measurable in t, upper-semi continuous (u.s.c.) in x, convex and compact: from a well known theorem for differential inclusions (see [1, 7]), this implies the existence of solutions at least locally). For a general differential inclusion, Hypothesis 5 and the fact that F is measurable in t, upper-semi continuous in x, closed, convex can be used to prove local existence of solutions.

Fact 2 Take any solution to (8) starting from t_0 and $x_0 > 0$: $x(t)$. Since F is u.s.c. an integral representation of solution exists: $x(t) \in x_0 + \int_{t_0}^{t} F(s, x(s))ds$. Thus, using Hypothesis 2, we have $[0, x(s)] \subset [0, x(t)] \subset [0, x_0]$ for any $s > t > t_0$.

Fact 2 implies that the set $[0, x_0]$ is invariant (thus Liapunov stability of the origin is deduced) which combined to Fact 1 imply that solutions are defined over $[t_0, +\infty[$. Now, from Fact 2 and using the property of non-empty decreasing sequence of compact sets, there exists a non-empty compact set $[0, l]$ such that $\lim_{t\to\infty}[0, x(t)] = [0, l]$. If $l \neq 0$ (positive), we have $0 \in \int_{t_0}^{t} F(s, l)ds$ in contradiction with Hypothesis 2 since $\forall t \in \mathbb{R}, \forall y \in F(t, l) : ly < 0$ implies that $\int_{t_0}^{t} F(s, l)ds < 0$. From this, one deduces that for every positive ε such that $0 < \varepsilon < x_0$, there exists a time $T_\varepsilon(t_0, x_0)$ such that $t_0 < T_\varepsilon(t_0, x_0) < \infty$ and such that $0 \le x(t) \le \varepsilon, \forall t \ge T_\varepsilon(t_0, x_0)$. Now for $0 < x < \varepsilon$ sufficiently small, $\forall y \in F(t, x)$, hypothesis 2 implies that $y < 0$ and Hypothesis 4 leads to

$$-\overline{h}(t)(1 + \overline{r}(t, x)) \le y \le -\underline{h}(t)x^{\underline{a}}(1 + \underline{r}(t, x)). \qquad (9)$$

Take any solution to (8) starting from ε. Integral representation of solutions $0 \le x_\varepsilon(t) \in \varepsilon + \int_0^t F(s, x_\varepsilon(s))ds$ combined with (9) imply that solution lies between the solutions of

$$\dot{x} = -\overline{h}(t)(1 + \overline{r}(t, x)), \qquad (10)$$

and

$$\dot{x} = -\underline{h}(t)x^{\underline{a}}(1 + \underline{r}(t, x)), \qquad (11)$$

with initial condition ε. Solutions of (11) reach the origin in finite time (using lemma 6 and Hypothesis 3 ($\underline{h}(t) > 0$) and 4). For sufficiently small ε, solutions of (10) due to Hypothesis 3 ($\overline{h}(t) > 0$) and 4, decrease to zero in time T_f: $F(T_f) = \int_0^T h(\tau)(1 + r_1(\tau))d\tau < \infty$, leading to the conclusion since $F(T_f)$ is strictly increasing. ◀

Corollary 9 *Consider system (7). Suppose that there exists $\mathcal{N}(0)$, an open connected neigbourhood of the origin, such that:*

1. $\forall t \in \mathbb{R} : [x \in \mathcal{N}(0) \wedge f(t, x) = 0] \Leftrightarrow [x = 0]$,
2. $\forall t \in \mathbb{R}, \forall x \in \mathcal{N}(0)\backslash\{0\}$: $xf(t, x) \underset{a.e}{<} 0$,
3. *for any fixed x sufficiently small and for $t > T_0$: $\underline{h}(t)\,|x|^{\underline{a}}(1+\underline{r}(t, x)) \underset{a.e}{\le} f(t, x) \underset{a.e}{\le} \overline{h}(t)(1+\overline{r}(t, x))\}$ with $\underline{a} \in]0, 1[$, $\overline{h}(t) > 0$ and $\underline{h}(t) > 0$,*
4. *for any fixed x sufficiently small, there exists a function $\overline{m}$ Lebesgue integrable : $\sup(|\underline{h}(t)\,|x|^{\underline{a}}(1 + \underline{r}(t, x))|\,, |\overline{h}(t)(1 + \overline{r}(t, x))|) \le \overline{m}(t)$,*

then for any $x_0 \in \mathcal{N}(0)$ and any $t_0 > T_0$, there exists at least one solution passing trough (t_0, x_0) defined for $t > t_0$ and every solution converges to zero in finite time. ■

Proof : ▶Apply the result of theorem 8. ◀

4.2 $n-$dimensional case

There is a simple sufficient criteria for the autonomous case

$$\dot{x} = f(x), \qquad (12)$$

defined for $x \in \mathbb{R}^n$:

Lemma 10 *Suppose that the origin of (12) is locally asymptotically stable. Using $x = rv^T M(\theta)$, where $r = \|x\|\,, v$ is a given constant row-vector and $M(\theta)$ is a rotation matrix, θ is a vector of $(n-1)$ componants, if $\left|\sum_{i=1}^{n} M^T(\theta)vf(rv^T M(\theta))\right| \ge \Psi(r)$ with $\int_0^\varepsilon \frac{dr}{\Psi(r)} < \infty$ (for some $\varepsilon > 0$) then the origin is finite time stable.*■

Example 11 *Consider $a \in]0, 1[$ and system*

$$\begin{aligned} \dot{x}_1 &= -\varphi_a(x_1) - x_1^3 + x_2, \\ \dot{x}_2 &= -\varphi_a(x_2) - x_2^3 - x_1. \end{aligned}$$

Taking $V(x) = \frac{\|x\|^2}{2}$, one obtains $\dot{V} = -\sum_{i=1}^{2}(x_i^4 + |x_i|^{a+1}) \le 0$. And letting $x^T = r(\cos(\theta), \sin(\theta))$ leads to $\left|\sum_{i=1}^{n} M^T(\theta)vf(rv^T M(\theta))\right| \ge r^a$. Thus the origin is finite time stable.

From the first section, the obtained results can be adapted to the $n-$dimensional case. For example:

Let us consider

$$\dot{x} \underset{\text{a.e}}{=} f(t,x), \tag{13}$$

defined for almost every $t \in \mathbb{R}, x \in \mathbb{R}^n$, where f is measurable in t and continuous for $(t,x) \in \mathbb{R}^{n+1} \backslash \mathcal{M}$ where $\mathcal{M}$ is a set of zero Lebesgue measure. As, for the scalar case, this equation is turned into (7) for which now $x \in \mathbb{R}^n$. Take a Lipschitz (continuous) Lyapunov function $V(t,x)$ for which $\frac{dV(t)}{dt} \in G(t,x) = \left\{g : \exists v \in \partial_C V(t,x), \exists f \in F(t,x) : g = \left\langle v, (1,f)^T \right\rangle\right\}$. Thus if this set is included in a set $G_{\sup}(t,V)$, we have $\frac{dV}{dt} \in G_{\sup}(t,V)$ for which using Corollary 9 leads to

Theorem 12 *Suppose that there exists $\mathcal{N}(0)$, an open connected neigbourhood of the origin, such that:*

1. $\forall t \in \mathbb{R}, \forall x \in \mathcal{N}(0) : F(t,x)$ *is upper-semi continuous and convex,*
2. $\forall t \in \mathbb{R} : [0 \in G(t,x)] \Leftrightarrow [x = 0]$,
3. $\forall t \in \mathbb{R}, \forall g \in G_{\sup} : g < 0$,
4. *for any fixed V sufficiently small and for $t > T_0$:* $\operatorname{graph}(|G_{\sup}|) \subset \{(t,V) : \underline{h}(t)\,|V|^{\underline{a}}(1+\underline{r}(t,V)) \underset{a.e}{\leq} y \underset{a.e}{\leq} \overline{h}(t)(1+\overline{r}(t,V))\}$ *with* $\underline{a} \in]0,1[$, $\overline{h}(t) > 0$ *and* $\underline{h}(t) > 0$,
5. *for any fixed x sufficiently small, there exists a function m Lebesgue integrable*: $\forall y \in F(t,x), |y| \leq m(t)$,

then for any $x_0 \in \mathcal{N}(0)$ and any $t_0 > T_0$, there exists at least one solution passing trough (t_0, x_0) defined for $t > t_0$ and every solution converges to zero in finite time. ■

Example 13 *Consider*

$$\begin{aligned} \dot{x}_1 &= -\operatorname{sgn}(x_1) - x_1^3 + x_2, \\ \dot{x}_2 &= -\operatorname{sgn}(x_2) - x_2^3 - x_1, \end{aligned}$$

with sgn *the multifunction defined by* $\operatorname{sgn}(x) = 1$ *if* $x > 0$, $\operatorname{sgn}(x) = -1$ *if* $x < 0$ *and* $\operatorname{sgn}(x) = [-1,1]$ *if* $x = 0$. *Taking* $V(x) = \frac{\|x\|^2}{2}$, *one obtains* $\dot{V} = -\sum_{i=1}^{2}(x_i^4 + |x_i|) \leq -(V^2 + \sqrt{V})$, *thus the origin is finite time stable.*

Remark 14 *For autonomous ODE, one can derived other sufficient conditions using Lyapunov function see [10].*

5 Stabilization in finite time

In [10], it is shown that any non-linear system diffeomorfic to

$$\begin{aligned} \dot{x}_1 &= \varphi_{a_2}(x_2), \\ \dot{x}_i &= \varphi_{a_{i+1}}(x_{i+1}), i = 2, \ldots, n-1, \\ \dot{x}_n &= u, \end{aligned} \tag{14}$$

with $a_i \in]0,1[$, can be stabilized in finite time with control $u = -\sum b_i \varphi_{a_i}(x_i)$, with $a_1 \in]0,1[$ for appropriate b_i.

Now let us consider the following non-linear system:

$$\dot{x} = f(x) + g(x)u. \tag{15}$$

Suppose that, for this system, there exists $h(x)$ such that (15) is an i/o linearizable system: $\mathcal{L}_g \mathcal{L}_f^i h(x) = 0, i = 0, 1, \ldots, n-2$, $\mathcal{L}_g \mathcal{L}_f^{n-1} h(x) \neq 0$. Now defining for a given $a > 1$

$$\begin{aligned} z_1 &= h(x), \\ z_2 &= z_1 + \left(\frac{a}{na-1}\right) \varphi_{n-1/a}(\mathcal{L}_f z_1), \\ &\ldots \\ z_n &= z_{n-1} + \left(\frac{a}{2a-1}\right) \varphi_{2-1/a}(\mathcal{L}_f z_{n-1}), \end{aligned}$$

thus

$$\begin{aligned} \mathcal{L}_f z_1 &= -\varphi_{\frac{a}{na-1}}\left(\left(\frac{na-1}{a}\right)(z_1 - z_2)\right), \\ &\ldots \\ \mathcal{L}_f z_{n-1} &= -\varphi_{\frac{a}{2a-1}}\left(\left(\frac{2a-1}{a}\right)(z_{n-1} - z_n)\right). \end{aligned}$$

One obtains (note that $\frac{a}{na-1} \in]0,1[$, for $n \geq 2$):

$$\begin{aligned} \dot{z}_1 &= -\varphi_{\frac{a}{na-1}}\left(\left(\frac{na-1}{a}\right)(z_1 - z_2)\right), \\ &\ldots \\ \dot{z}_{n-1} &= -\varphi_{\frac{a}{2a-1}}\left(\left(\frac{2a-1}{a}\right)(z_{n-1} - z_n)\right), \\ \dot{z}_n &= \mathcal{L}_{f+gu} z_n = a(z) + b(z)u. \end{aligned}$$

Using the control

$$u = -\frac{a(z) + \varphi_{\frac{1}{a}}(z_n)}{b(z)}$$

leads to a finite time stabilization of z_n as soon as $z(t)$ does not enter the manifold $\mathcal{M} = \{z : b(z) = 0\}$. Then, $\dot{z}_{n-1} = -\varphi_{\frac{a}{2a-1}}\left(\frac{2a-1}{a} z_{n-1}\right)$ thus z_{n-1} converges to zero in finite time, leading recursively to a finite time stabilization of z to the origin. Concerning the set $\mathcal{M}$, one can see that if $z(t)$ lies in $\mathcal{M}$, there exists an index $i \leq (n-1)$ such that $\dot{z}_i = 0$. Thus $z_i = z_{i+1}$ which implies $\dot{z}_{i+1} = \dot{z}_i = 0$. Recursively one obtains $\dot{z}_j = 0, z_j = z_i, i \leq j \leq n$ which finally leads to $a(z) + b(z)u = 0$. And from which we can deduce finite time convergence of the remaining states. However the control may blow up during that time. This is the reason why, if $z_1 \neq 0$ and $z(t)$ lies in $\mathcal{M}$, one must apply a control to leave the set $\mathcal{M}$ in finite time (classic).

Example 15 *Consider the following second order system:*

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2, \\ \dot{x}_2 &= u, \end{aligned}$$

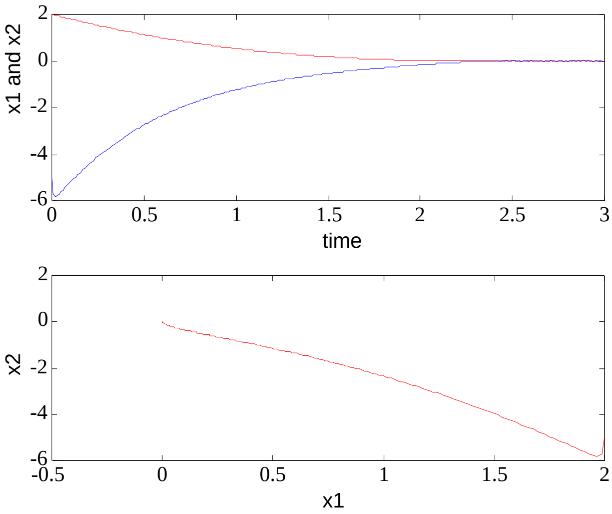
setting ($a = 3$) $z_1 = x_1$, $z_2 = x_1 + \frac{3}{5}\varphi_{\frac{5}{3}}(x_1^2 + x_2)$, *leads to*

$$\begin{aligned} \dot{z}_1 &= -\varphi_{\frac{3}{5}}(5(z_1 - z_2)/3), \\ \dot{z}_2 &= x_2 + \left|x_1^2 + x_2\right|^{\frac{2}{3}} (2x_1(x_1^2 + x_2) + u). \end{aligned}$$

Using the control

$$u = -\left(2x_1(x_1^2 + x_2) + \frac{x_2 + 100\varphi_{\frac{1}{3}}\left(x_1 + \frac{3}{5}\varphi_{\frac{5}{3}}(x_1^2 + x_2)\right)}{\left|x_1^2 + x_2\right|^{\frac{2}{3}}}\right) \tag{16}$$

leads to the following simulations



In that case, the biggest invariant set in $\mathcal{M}$ is the origin. Thus there is no need to change the control (16).

6 Conclusion

Here, some preliminary results concerning finite time stability have been provided: first some sufficient conditions have been derived for scalar systems (autonomous, non autonomous and differential inclusions). Then, these results are translated to $n-$dimensional case via Lyapunov functions. The last section concerns some of the results about finite time stabilization provided in [10]. In a future work, such results may be extended to obtain higher order sliding mode controllers ([8, 6]).

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