# LINEAR PREDICTIVE POLE-PLACEMENT CONTROL: PRACTICAL ISSUES

**Peter J Gawthrop** *

* *Centre for Systems and Control and Department of Mechanical Engineering, University of Glasgow, GLASGOW. G12 8QQ Scotland.* ***P.Gawthrop@eng.gla.ac.uk***

Abstract: Some of the theoretical properties of predictive-pole-placement control (a form of model-based predictive control) are given a practical interpretation and corresponding design rules suggested.

Keywords: Model-based predictive control; pole-placement.

## 1. INTRODUCTION

Most work on stabilising Model-based Predictive Control (MPC) can be seen as the approximation of infinite horizon linear quadratic control using a finite horizon optimisation with constraints. Stability results (for example (Clarke and Scattolini, 1991; Muske and Rawlings, 1993; Rawlings and Muske, 1993; Chen and Allgöwer, 1998)) are based on the idea of showing that, with suitable terminal constraints, this approximation is equivalent to the solution of a related infinite horizon cost function.

The paper of Gawthrop and Ronco (1999 submitted) (upon which this paper is based) takes a different approach. Although optimisation via model-based prediction is used, it is *not* as an approximation to a linear-quadratic cost function but rather as a means of solving linear constrained problems by approximating the behavior of the classical linear state feedback control of a linear system with chosen closed-loop pole locations. These pole locations can be determined by any linear design method (including linear-quadratic). For this reason, the algorithm is named *Predictive Pole Placement* (PPP). In common with many other MPC papers (Muske and Rawlings, 1993; Rawlings and Muske, 1993; Gawthrop *et al.*, 1998; Chen and Allgöwer, 1998), a state (as opposed to output) feedback approach is used; thus the method can be categorised as manipulating input-output behavior using state feedback. In the linear context of this paper, output feedback may be readily accomplished using standard observer techniques.

Within a continuous-time setting, the basic algorithm for a linear unconstrained system has the following features:

(1) The open-loop control signal (within the moving horizon) is constrained to be the linear sum of prespecified basis functions.
(2) Optimisation is used to make the open-loop output response become nearly constant at the end of the optimisation horizon.
(3) A particular choice of basis functions, corresponding to the transient response of a stable linear dynamic system, leads to approximately equal open-loop (within the moving horizon) and closed-loop responses both corresponding to the regulator and tracking response of a stable closed-loop system with prespecified stable poles.

Feature 1 is not new. The usual discrete-time choice of the control (or control move) at each sample time can be viewed as one such choice (Rawlings and Muske, 1993). A polynomial (in time) set of basis functions has been used in the continuous context ((Demircioglu and Gawthrop, 1991; Gawthrop *et al.*, 1998) and Laguerre functions have also been used.

Feature 2 can be viewed as an output-orientated version of the terminal state constraint shown in a number of papers to be important for stability (Clarke and Scattolini, 1991; Muske and Rawlings, 1993; Rawlings and Muske, 1993; Chen and Allgöwer, 1998); we believe that this output-orientated approach is appropriate given the input-output focus of the paper. Not surprisingly, it is likewise important in creating a stable moving-horizon controller.

Scokaert and Rawlings (Scokaert and Rawlings, 1998) show that (discrete-time) "Constrained Linear Quadratic Regulation" has the property that nominal closed-loop performance is identical to the open-loop predictions and thus shares feature 3 with the (continuous-time) PPP algorithm (Gawthrop and Ronco, 1999 submitted).

Of course, standard techniques are available to design controllers for the linear unconstrained case and, in this case, the PPP algorithm would be yet another way of achieving the same result. However, the strength of the method is in its extension to constrained systems; the linear unconstrained case representing an ideal situation with the corresponding nice properties listed above. The paper also provides this extension to the constrained case. Although outside the scope of this paper, we note that the method extends in principle to the nonlinear case and some preliminary results are available elsewhere (Ronco *et al.*, 1999).

This paper takes a closer look at the asymptotic results of Gawthrop and Ronco (1999 submitted) to give practical design guidelines for predictive pole placement control and provide further illustration of its properties.

The paper is organised as follows. Section 2 gives a brief overview of the unconstrained optimisation problem (based on (Gawthrop and Ronco, 1999 submitted)) and Section 3 reviews the conditions under which open and closed loop control are the same and gives the main asymptotic results. Section 4 contains an investigation of the asymptotic properties of PPP. Section 5 concludes the paper.

## 2. PREDICTIVE POLE-PLACEMENT CONTROL

Following Gawthrop and Ronco (1999 submitted), this section introduces the class of systems considered in this paper, the corresponding unconstrained optimisation problem and gives an explicit formula for its solution.

The linear systems considered in this paper are described by:

$$\begin{cases} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases} \tag{1}$$

where $x \in R^{n_x}$, $y \in R^{n_y}$ and $u \in R^{n_u}$. Given the state $x(t)$ at time $t$, we are interested in the evolution of the *moving horizon* state $x^\star(t,\tau)$ and output $y^\star(t,\tau)$ where

$$\begin{cases} \frac{d}{d\tau}x^\star(t,\tau) &= Ax^\star(t,\tau) + Bu^\star(t,\tau) \\ y^\star(t,\tau) &= Cx^\star(t,\tau) \end{cases} \tag{2}$$

The differential equations 1 and 2 are related by having the *same* state space matrices and by imposing the *cross-coupling* conditions:

$$\begin{cases} x^\star(t,0) &= x(t) \\ u(t) &= u^\star(t,0) \end{cases} \tag{3}$$

In this paper, the *moving horizon* control signal $u^\star(t,\tau)$ is *linearly parameterised* by the $n_U$ components of the column vector $U(t)$ so that:

$$u^\star(t,\tau) = U^\star(\tau)U(t) \tag{4}$$

where $U^\star(\tau)$ is a $n_u \times n_U$ matrix of functions of $\tau$. The components of $U^\star(\tau)$ can be regarded as a set of *basis functions* for the control signal $u^\star(t,\tau)$ and the components of $U(t)$ the corresponding weights. The *moving horizon* setpoint $w^\star(t,\tau)$ is linearly parameterised similarly.

The vector $U(t)$ is to be chosen to minimise (at a given time $t$) the (unconstrained) quadratic cost function:

$$\begin{aligned} &J(U(t), x(t), W(t)) \\ &= \frac{1}{2}\int_{\tau_1}^{\tau_2} (y^\star(t,\tau) - w^\star(t,\tau))^T \\ &Q(\tau)(y^\star(t,\tau) - w^\star(t,\tau))d\tau \end{aligned} \tag{5}$$

where $Q(\tau)$ is an $n_y \times n_y$ positive-definite output weighting matrix which is possibly a function of $\tau$.

The derivatives of this cost function are denoted by $J_U = \frac{\partial}{\partial U}J(U(t),x(t),W(t))$, $J_{UU} = \frac{\partial^2}{\partial U^2}J(U(t),x(t),W(t))$, $J_{UX} = \frac{\partial^2}{\partial U \partial X}J(U(t),x(t),W(t))$ and $J_{UW} = \frac{\partial^2}{\partial U \partial W}J(U(t),x(t),W(t))$.

With this notation, Lemma 1 gives the solution of this optimisation problem.

*Lemma 1.* (Explicit solution of unconstrained problem). When the system (within the moving horizon) is given by Equation 2, the cost function $J$ has a global minimum with respect to $U(t)$ if $J_{UU}$ is not singular. The corresponding minimising $U(t)$ is then given by:

$$U(t) = K_w W(t) - K_x x(t) \tag{6}$$

where

$$K_w = J_{UU}^{-1} J_{UW} \tag{7}$$

$$K_x = J_{UU}^{-1} J_{Ux} \tag{8}$$

and the derivatives are given by

$$J_{UU} = \int_{\tau_1}^{\tau_2} y_U^\star(\tau)^T Q(\tau) y_U^\star(\tau) d\tau \qquad (9)$$

$$J_{Ux} = \int_{\tau_1}^{\tau_2} y_U^\star(\tau)^T Q(\tau) y_x^\star(\tau) d\tau \qquad (10)$$

$$J_{UW} = \int_{\tau_1}^{\tau_2} y_U^\star(\tau)^T Q(\tau) W^\star(\tau) d\tau \qquad (11)$$

where the $i$th column $y_{Ui}^\star(\tau)$ of $y_U^\star(\tau)$ is the solution of the ode:

$$\begin{cases} \frac{d}{d\tau} x_{Ui}^\star(\tau) & = A x_{Ui}^\star(\tau) + B U_i^\star(\tau) \\ y_{Ui}^\star(\tau) & = C x_{Ui}^\star(\tau) \\ x_{Ui}^\star(0) & = 0_{n_x} \end{cases} \qquad (12)$$

where $0_{n_x}$ is a column vector with all of its $n_x$ elements zero and $y_x^\star(\tau)$ is the solution of:

$$\begin{cases} \frac{d}{d\tau} x_x^\star(\tau) & = A x_x^\star(\tau) \\ y_x^\star(\tau) & = C x_x^\star(\tau) \\ x_x^\star(0) & = 1_{n_x} \end{cases} \qquad (13)$$

where $1_{n_x}$ is a column vector with all of its $n_x$ elements unity.

The closed loop control is given by

$$u(t) = k_w w(t) - k_x x(t) \qquad (14)$$

where:

$$k_x = U^\star(0) K_x \qquad (15)$$
$$k_w = U^\star(0) K_w \qquad (16)$$

**PROOF.** See Gawthrop and Ronco (1999 submitted). □

*Remark 1.* Equation 6, together with Equation 4, gives the *open-loop control signal* as a function of $\tau$, the initial state $x(t)$ and the setpoint at time $t$ and the basis functions $U^\star(\tau)$ as:

$$u^\star(t,\tau) = U^\star(\tau) K_w W(t) - U^\star(\tau) K_x x(t) \qquad (17)$$

*Remark 2.* The open-loop control signal has two independent terms: a *tracking* term $U^\star(\tau) K_w W(t)$ driven by the setpoint and a *regulation* term $-U^\star(\tau) K_x x(t)$ driven by the initial state $x(0)$.

*Remark 3.* Equation 13 has the explicit solution:

$$y_x^\star(\tau) = C e^{A\tau} \qquad (18)$$

It is convenient to choose the following special form of $U^\star(\tau)$:

$$U^\star(\tau) = \begin{pmatrix} U_1^\star(\tau)^T & 0_{1\times n_2} & \dots & 0_{1\times n_n} \\ 0_{1\times n_1} & U_2^\star(\tau)^T & \dots & 0_{1\times n_n} \\ 0_{1\times n_1} & \dots & \dots & 0_{1\times n_n} \\ 0_{1\times n_1} & 0_{1\times n_2} & \dots & U_{n_u}^\star(\tau)^T \end{pmatrix} \qquad (19)$$

Where $U_i^\star(\tau) \in R^{n_i}$ (the set of functions defining the $i$th system input) is generated as the state of the autonomous system:

$$\begin{cases} \dot{U}_i^\star(\tau) & = A_i U_i^\star(\tau) \\ U_i^\star(0) & = U_{i\,0}^\star \end{cases} \qquad (20)$$

which has the explicit solution:

$$U_i^\star(\tau) = e^{A_i\tau} {U^\star}_0 \qquad (21)$$

## 3. PROPERTIES OF UNCONSTRAINED PPP

This section looks at the basic properties of the PPP algorithm and gives the fundamental result on the relationship between the *open*-loop control $u^\star(t,\tau)$ and the *closed*-loop control $u(t)$.

Lemma 1 presents a general algorithm parameterised by the choice of input functions $U^\star(\tau)$ and setpoint functions $W^\star(\tau)$. A key idea in this paper is to choose the input functions $U^\star(\tau)$ to be the solutions of the autonomous system of Equation 20. It turns out that a suitable combination of such functions yields an open-loop system that has the same input, state and output as the corresponding closed-loop system. This relationship between open and closed loop systems is given in the following lemma:

*Lemma 2.* (Open and closed-loop response). If

(1) An open-loop controller of the form $u^\star(t,\tau) = k_w w(t) - U^\star(\tau) \mathfrak{K}_x x(t)$ is applied to the system of Equation 2 where:
   (a) $U^\star(\tau)$ is generated from Equation 19 where $A_{ui} = A_u \in R^{n_x \times n_x} \forall i$, ${U^\star}_0 = u_0 \forall i$.
   (b) $[A_u u_0]$ is controllable.
(2) A closed-loop controller of the form of Equation 14 is applied to the system of Equation 1
(3) The open-loop system matrix $A_u$ has the same $n_x$ eigenvalues as the closed-loop system matrix $A_c$ where $A_c$ is

$$A_c = A - B k_x \qquad (22)$$

Then there exists a unique $\mathfrak{K}_x$ such that the solution of the open loop system 2 is the same as the solution of the the closed loop system 1 in the sense that:

$$x(t+\tau) = x^\star(t,\tau) \qquad (23)$$
$$y(t+\tau) = y^\star(t,\tau) \qquad (24)$$
$$u(t+\tau) = u^\star(t,\tau) \qquad (25)$$

**PROOF.** See Gawthrop and Ronco (1999 submitted). □

*Remark 4.* This result is independent of the optimisation based design of Lemma 1.

Lemma 2 paves the way for showing the properties of the algorithm contained in the following Theorem.

*Theorem 1.* If

(1) The pair $[AB]$ is controllable.
(2) The assumptions of Lemma 2 hold.
(3) The matrix $A_u$ is chosen such that $A_u - A < 0$
(4) The upper horizon $\tau_2$ is given by

$$\tau_2 = \tau_1 + \Delta\tau \tag{26}$$

where $\Delta\tau > 0$

then in the limit as $\tau_1 \to \infty$

(1) The solution of the open loop system Equations 2 and 4 is the same as the solution of the the closed loop system Equation 1
(2) The closed-loop system poles (eigenvalues of $A_c = A - Bk_x$) are the $n_x$ eigenvalues of $A_u$.
(3) $K_x = \mathfrak{K}_x$

**PROOF.** See Gawthrop and Ronco (1999 submitted). □

*Remark 5.* Choosing this particular horizon to weight the output is similar to the standard approach of using terminal state weighting.

*Remark 6.* To avoid numerical problems in the solution of Equation 8, $\tau_1$ and $\tau_2$ should be chosen so that $e^{A_c}$ is small compared to $e^{A}$ for $\tau_1 \leq \tau \leq \tau_2$.

*Remark 7.* If $A_u$ is chosen to to have strictly negative-real eigenvalues, this result also implies stability.

*Remark 8.* This result can be extended in a number of ways.

(1) If condition 3 is reversed ($A - A_u < 0$) then there is no feedback $k_x = 0$.
(2) If $U^\star(\tau)$ contains more elements than necessary (that is the dimension of $A_u$ is greater than that of $A$), the elements with the *slowest* time-constants are discarded.
(3) As suggested in Remark 8, and unlike standard pole-placement, PPP never slows down a system. Thus desired poles *slower* than open-loop poles are discarded.
(4) As suggested in Remark 6, numerical problems are possible for large control horizon $\tau_1$.

## 4. INVESTIGATION OF THE ASYMPTOTIC PROPERTIES

Section 3 gives *asymptotic* properties of the linear PPP algorithm. In particular, it gives results in the limit as $\tau_1 \to \infty$. This Section investigates the properties of the linear PPP algorithm when $\tau_1$ is finite. In particular the following features are investigated:

(1) The *closed-loop* poles converge toward those implied by Theorem 1.

In each of the following examples, closed-loop poles are determined by computing $k_x$ from Equations 15 and 8 and finding the eigenvalues of $A - Bk_x$.

### 4.1 *Simple first-order system*

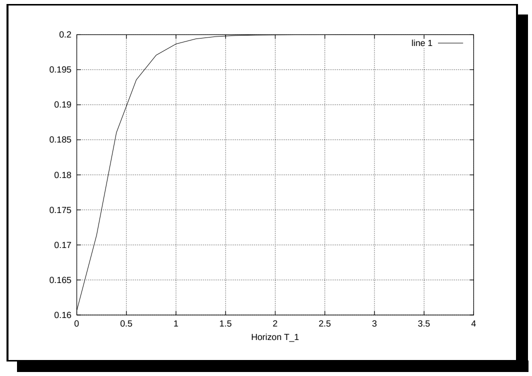



Fig. 1. *Simple system: The* actual *closed-loop time-constant against the lower horizon $\tau_1$. The* actual *time-constant is asymptotic to the* desired *closed-loop time-constant of $\tau_d = 0.2$*

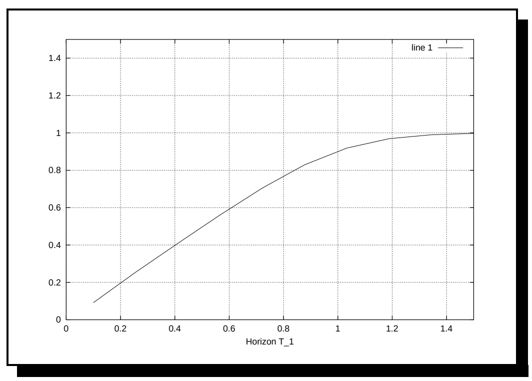



Fig. 2. *Simple system:* actual *closed-loop time-constant against* desired *time-constant. The* actual *time-constant is approximately equal to the* desired *time constant except when the desired time-constant is* longer *than the* open-loop *time-constant $\tau_o = 1$*

Consider the first order system with state-space representation

$$A = -1;\ B = 1;\ C = 1 \tag{27}$$

with corresponding transfer function

$$G(s) = \frac{1}{s+1} \tag{28}$$

This has a single open loop pole at $s = 1$ with open-loop time-constant $\tau_o = 1$. The input basis functions are a unit step function and $e^{-5\tau}$. The latter function corresponds to a desired closed-loop polynomial of

$$p(s) = s + 5 \tag{29}$$

in other words to a *desired* closed-loop time-constant of $\tau_d = \frac{1}{5} = 0.2$.

Figure 1 shows the effect of the lower time horizon $\tau_1$ on the resultant closed-loop time-constant. The upper horizon $\tau_2 = \tau_1 + 1$. As can be seen, the closed loop time-constant is close to the desired time-constant when $\tau_1 > 2$. This suggests choosing $\tau_1 = 10\tau_d$ is a good rule-of-thumb for first-order systems.

Figure 2 shows the effect of the desired closed-loop time-constant $\tau_d$ on the actual time constant $\tau_c$. The horizon parameters are $\tau_1 = 10\tau_d$ and $\tau_2 = \tau_1 + 1$. This demonstrates that

$$\tau_c \approx \begin{cases} \tau_d & \text{if } \tau_d < \tau_o \\ \tau_o & \text{if } \tau_d \geq \tau_o \end{cases} \tag{30}$$

### 4.2 *Unstable and inverse-unstable system*

A third order unstable system with unstable inverse is described by the transfer function

$$G(s) = \frac{1 - 0.5s}{(s-1)^3} \tag{31}$$

A state-space representation is:

$$A = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; C = \begin{pmatrix} 0 & -0.5 & 1 \end{pmatrix} \tag{32}$$

The controller is designed to have three closed loop poles at a Butterworth configuration with radius 5. That is the three desired closed-loop poles are the roots of the polynomial

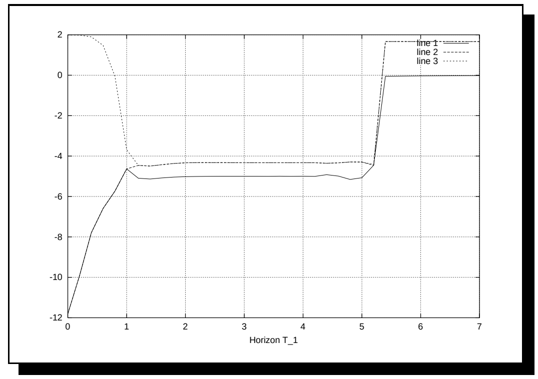
$$p(s) = s^3 + 13.66s^2 + 68.30s + 125.00 \tag{33}$$

which are at $s = -5$ and $s = -4.33 \pm 2.5j$.

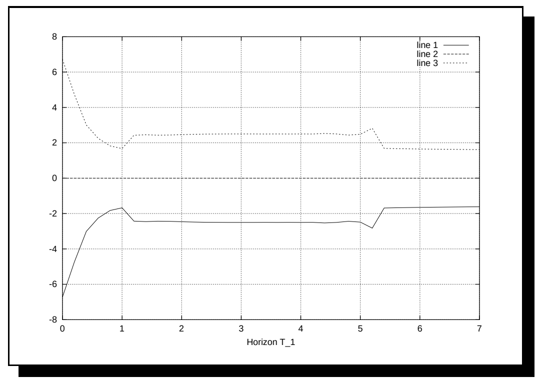
Figure 3 shows how the lower horizon $\tau_1$ affects the closed loop poles.

Consider first $\tau_1 < 5$. As predicted, the closed loop poles are asymptotic to the desired values. In this case, the poles have converged after about $\tau_1 = 2$. Taking the desired time-constant to be the inverse pole-radius: $\tau_d = \frac{1}{5} = 0.2$ This corresponds to $\tau_1 \approx 12.5\tau_d$. Notice that the closed-loop system is *unstable* for $\tau_1 < 0.8$

Consider next $\tau_1 > 5$. Here, the closed-loop poles deviate substantially from their desired values. This is due to the *numerical* problems associated with the solution of Equation 8 mentioned in Remark 6. However, this problem only occurs if $\tau_1 > 50\tau_d$ and is therefore easily avoidable. Once




(a) Real parts




(b) Imaginary parts

Fig. 3. *Non-minimum-phase system: The* actual *closed-loop poles against the lower horizon* $\tau_1$*. The* actual *poles are asymptotic to the* desired *closed-loop poles at* $s = -5, s = -4.33 \pm 2.50j$*. Numerical problems occur for* $\tau_1 > 5$

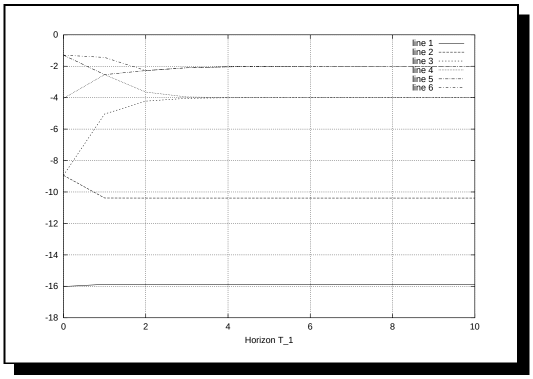
again, this suggests choosing $\tau_1 = 10\tau_d$ is a good rule-of-thumb.

### 4.3 *Turbogenerator system*

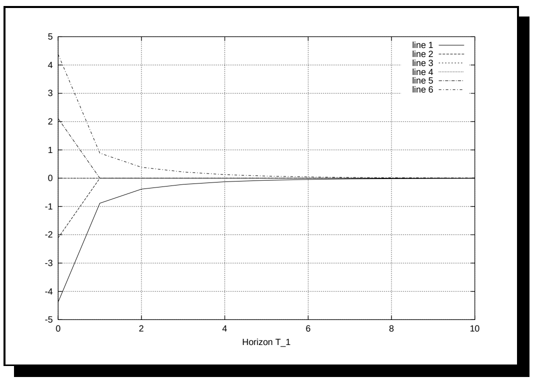
This example is taken from Maciejowski (1989) and represents a turbogenerator as a linear, two-input, two-output, six-state system with the six open loop poles at:

$$s = -0.234 \tag{34}$$
$$s = -0.349 \pm 6.34j \tag{35}$$
$$s = -1.04 \tag{36}$$
$$s = -10.5 \tag{37}$$
$$s = -15.9 \tag{38}$$

This system has two inputs. The basis functions of each input are chosen to be second-order Laguerre polynomials. The first input has parameter 2.0 and the second has parameter 4.0. Thus the desired closed-loop poles are the roots of




(a) Real parts




(b) Imaginary parts

Fig. 4. *Turbogenerator system: The* actual *closed-loop poles against the lower horizon* $\tau_1$. *The* actual *poles are asymptotic to the* desired *closed-loop poles of* $s = -2$ *and* $s = -4$ *and the "fast" open-loop poles at* $s = -10.4$ *and* $s = -15.9$

$$p(s) = (s^2 + 4s + 4)(s^2 + 8s + 16) \tag{39}$$

two of which are at $s = -2$ and two of which are at $s = -4$. These poles are *faster* than the four open-loop poles of Equations 34–36 but slower than the two open-loop poles of Equations 37 and 38.

Figure 4 one again demonstrates that the closed-loop poles are close to their asymptotic value for $\tau_1 > 10\tau_d$.

Moreover, the "fast" open-loop poles of Equations 37 and 38 become closed-loop poles: they are ignored by the PPP algorithm.

## 5. CONCLUSION

The asymptotic properties of the linear PPP algorithm have been verified using three example systems, and a rule-of-thumb for choosing the horizon parameter $\tau_1$ has been established. Possible numerical problems have been investigated and shown to be easily avoidable for the three systems investigated.

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