Rank-one LMIs and Lyapunov's Inequality

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Abstract

We describe a new proof of the well-known Lyapunov's matrix inequality about the location of the eigenvalues of a matrix in some region of the complex plane. The proof makes use of standard facts from quadratic and semi-definite programming. Links are established between the Lyapunov matrix, rank-one LMIs and the Lagrange multiplier arising in duality theory.

1 Introduction

Let $A \in \mathbb{C}^{n\times n}$ be a given complex matrix and let

$$\mathcal{D} = \{s \in \mathbb{C} : \left[\begin{array}{c} 1 \\ s \end{array}\right]^\star \left[\begin{array}{cc} a & b \\ b^\star & c \end{array}\right] \left[\begin{array}{c} 1 \\ s \end{array}\right] < 0\}$$

denote a given open region of the complex plane, where Hermitian matrix

$$\left[\begin{array}{cc} a & b \\ b^\star & c \end{array}\right] \in \mathbb{C}^{2\times 2}$$

has one strictly negative eigenvalue and one strictly positive eigenvalue, and the star denotes transpose conjugate. In the sequel, the notation $P \succ 0$ or $-P \prec 0$ (resp. $P \succeq 0$ or $-P \preceq 0$) means that matrix P is positive definite (resp. semi-definite). The location of the eigenvalues of A can be characterized as follows.

Theorem 1 (Lyapunov's Inequality) *Matrix A has all its eigenvalues in region $\mathcal{D}$ if and only if there is a matrix $P = P^\star \succ 0 \in \mathbb{C}^{n\times n}$ such that*

$$\left[\begin{array}{c} I \\ A \end{array}\right]^\star \left[\begin{array}{cc} aP & bP \\ b^\star P & cP \end{array}\right] \left[\begin{array}{c} I \\ A \end{array}\right] \prec 0. \qquad (1)$$

Matrix inequality (1) is referred to as Lyapunov's inequality.

Lyapunov's proof of the above theorem – originally developed in the case that $\mathcal{D}$ is the open left half-plane, i.e. $a = c = 0$, $b = 1$ and inequality (1) becomes $A^\star P + PA \prec 0$ – relies on the construction of a positive quadratic function whose derivative is negative along the trajectories of an associated dynamical system, see e.g. [5]. It can be extended to arbitrary regions $\mathcal{D}$ via a conformal mapping. Another proof of Theorem 1 can be found in [1, Theorem 3.19]. Eigenvectors of matrix A are used to show that existence of P implies stability of A, whereas the converse statement is shown via properties of the matrix exponential function.

The aim of this note is to give a new, alternative proof of Lyapunov's inequality without referring to stability of the trajectories of a dynamical system or to matrix exponentials. We use elementary concepts from linear algebra, quadratic and semi-definite programming. Links are established between the Lyapunov matrix and the Lagrange multiplier arising in duality theory. Relationships with rank-one LMIs, the Kalman-Yakubovich-Popov Lemma and (D, G)-scaling in μ-analysis are also pointed out.

The proof relies on the following important result, proved e.g. as in [9, Lemma 3].

Lemma 1 *Two column vectors $p, q \in \mathbb{C}^n$ with q non-zero satisfy*

$$[q \quad p] \left[\begin{array}{cc} a & b^\star \\ b & c \end{array}\right] [q \quad p]^\star \succeq 0 \qquad (2)$$

if and only if $p = sq$ for some $s \in \mathcal{D}^C$.

2 Rank-one LMI Problem

First we show the equivalence between location of the eigenvalues of A in region $\mathcal{D}$ and a rank-one LMI optimization problem or a rank-one LMI feasibility problem.

If $s \in \mathbb{C}$ is an eigenvalue of A, then there exists a non-zero vector $q \in \mathbb{C}^n$ such that

$$(A - sI)q = 0. \qquad (3)$$

Pursuing an idea proposed in [3, Chapter 1], it follows that all the eigenvalues of A belong to $\mathcal{D}$ if and only if the optimal value μ of the quadratic optimization problem

$$\begin{array}{rcll} \mu & = & \min & q^\star(A-sI)^\star(A-sI)q \\ & & \text{s.t.} & s \in \mathcal{D}^C \\ & & & q^\star q = 1 \end{array} \qquad (4)$$

is strictly positive, where

$$\mathcal{D}^C = \{s \in \mathbb{C} : \left[\begin{array}{c} 1 \\ s \end{array}\right]^\star \left[\begin{array}{cc} a & b \\ b^\star & c \end{array}\right] \left[\begin{array}{c} 1 \\ s \end{array}\right] \geq 0\}$$

is the closed region complementary to $\mathcal{D}$ in $\mathbb{C}$. Define the rank-one positive semidefinite matrix

$$X = xx^\star = \left[\begin{array}{c} q \\ p \end{array}\right] \left[\begin{array}{c} q \\ p \end{array}\right]^\star \succeq 0$$

and use the notations

$$\begin{array}{rclcl} A(s)q & = & [\,A \quad -I\,]\,x & = & \mathcal{A}x \\ q & = & [\,I \quad 0\,]\,x & = & \mathcal{Q}x \\ p & = & [\,0 \quad I\,]\,x & = & \mathcal{P}x \end{array}$$

to write inequality (2) as an LMI in rank-one matrix X, namely

$$F(X) = \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array}\right] \left[\begin{array}{cc} aX & b^\star X \\ bX & cX \end{array}\right] \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array}\right]^\star \succeq 0 \qquad (5)$$

where F is a linear map from $\mathbb{C}^{2n\times 2n}$ to $\mathbb{C}^{n\times n}$. Using these notations, an alternative formulation of quadratic optimization problem (4) is given by the following lemma.

Lemma 2 *The eigenvalues of matrix A belong to region $\mathcal{D}$ if and only if $\mu > 0$ in rank-one LMI optimization problem*

$$\begin{array}{rcll} \mu & = & \min & \text{Trace}\,\mathcal{A}^\star\mathcal{A}X \\ & & \text{s.t.} & F(X) \succeq 0 \\ & & & X = X^\star \succeq 0 \\ & & & \text{Trace}\,\mathcal{Q}^\star\mathcal{Q}X = 1 \\ & & & \text{Rank}\,X = 1. \end{array} \qquad (6)$$

The above rank-one LMI problem is an optimization problem. It turns out that we can equivalently state this result via a feasibility problem, following an idea exposed in [7]. To see this, note that Lemma 1 and equation (3) imply that A has no eigenvalue in $\mathcal{D}^C$ if and only if there is no non-zero vector q for which

$$[q \quad Aq] \left[\begin{array}{cc} a & b^\star \\ b & c \end{array}\right] [q \quad Aq]^\star \succeq 0. \qquad (7)$$

The left-hand side of inequality (7) can alternatively be expressed as

$$\begin{array}{l} aqq^\star + bAqq^\star + b^\star qq^\star A^\star + cAqq^\star A^\star \\ = \left[\begin{array}{cc} I & A \end{array}\right] \left[\begin{array}{cc} aqq^\star & b^\star qq^\star \\ bqq^\star & cqq^\star \end{array}\right] \left[\begin{array}{cc} I & A \end{array}\right]^\star. \end{array}$$

Now define the linear map

$$G(Q) = \left[\begin{array}{cc} I & A \end{array}\right] \left[\begin{array}{cc} aQ & b^\star Q \\ bQ & cQ \end{array}\right] \left[\begin{array}{cc} I & A \end{array}\right]^\star \succeq 0 \qquad (8)$$

from $\mathbb{C}^{n\times n}$ to $\mathbb{C}^{n\times n}$. With Q denoting the non-zero rank-one matrix $Q = qq^\star$ we arrive at the following result which is equivalent to Lemma 2.

Lemma 3 *The eigenvalues of matrix A belong to region $\mathcal{D}$ if and only if there is no solution to the rank-one LMI feasibility problem*

$$\begin{array}{l} G(Q) \succeq 0 \\ Q = Q^\star \succeq 0 \\ \text{Trace}\,Q = 1 \\ \text{Rank}\,Q = 1. \end{array} \qquad (9)$$

3 LMI Problem

Now we show that the non-convex rank constraints in LMI problems (6) and (9) are actually irrelevant. Let

$$N = \left[\begin{array}{c} I \\ A \end{array}\right]$$

denote a matrix whose columns span the n-dimensional right null-space of full row-rank matrix $\mathcal{A}$. If $s_k \in \mathbb{C}$ is a non-defective eigenvalue of A (i.e. its algebraic multiplicity is equal to its geometric multiplicity) and $q_k \in \mathbb{C}^n$ is the corresponding eigenvector, then the vector

$$x_k = \left[\begin{array}{c} q_k \\ s_k q_k \end{array}\right]$$

belongs to the right null-space of matrix $\mathcal{A}$. Similarly, if s_k is a defective eigenvalue of A (i.e. its algebraic multiplicity is greater than its geometric multiplicity), then the corresponding chain of linearly independent generalized eigenvectors $q_k, q_{k+1}, q_{k+2}, \ldots$ gives rise to vectors

$$\begin{array}{ll} x_k = \left[\begin{array}{c} q_k \\ s_k q_k \end{array}\right] & x_{k+1} = \left[\begin{array}{c} q_{k+1} \\ s_k q_{k+1} + q_k \end{array}\right] \\ x_{k+2} = \left[\begin{array}{c} q_{k+2} \\ s_k q_{k+2} + q_{k+1} \end{array}\right] & \cdots \end{array} \qquad (10)$$

also belonging to the right null-space of $\mathcal{A}$. Let

$$V = [x_1 \quad \cdots \quad x_n]$$

denote a matrix built up from all the vectors x_i associated with all the eigenvalues s_i of A. It follows from the above discussion that the columns of N and V span the same vector space. By definition, vectors q_i are linearly independent, thus we can define linearly independent vectors $\bar{q}_i \in \mathbb{C}^n$ such that

$$[\bar{q}_1 \quad \cdots \quad \bar{q}_n]^\star [q_1 \quad \cdots \quad q_n] = I. \tag{11}$$

Following these preliminaries, consider now the following relaxation of rank-one LMI problem (6)

$$\begin{array}{rcll} \nu & = & \min & \operatorname{Trace} \mathcal{A}^\star \mathcal{A} X \\ & & \text{s.t.} & F(X) \succeq 0 \\ & & & X = X^\star \succeq 0 \\ & & & \operatorname{Trace} \mathcal{Q}^\star \mathcal{Q} X = 1 \end{array} \tag{12}$$

where the non-convex rank constraint has been dropped. Since the non-convex feasible set in problem (6) is a subset of the convex feasible set in problem (12), LMI optimization problem (12) is referred to as a convex relaxation of the non-convex rank-one LMI problem (6). In relation to the above problem, we can state the following central result.

Lemma 4 *$\mu > 0$ in rank-one LMI optimization problem (6) if and only if $\nu > 0$ in LMI optimization problem (12).*

Proof The inner product of positive semi-definite matrices $\mathcal{A}^\star \mathcal{A}$ and X is always non-negative, hence $\nu \geq 0$. Moreover, the fact that $\nu > 0$ implies $\mu > 0$ is trivial since the feasible set in problem (6) is a subset of the feasible set in problem (12), i.e. it holds $\mu \geq \nu$. Consequently, in order to show that $\mu > 0$ implies $\nu > 0$, the remainder of the proof will consist in proving that $\nu = 0$ implies $\mu = 0$. So suppose that X is a positive semi-definite matrix such that $\nu = 0$ in problem (12). Let W be a $2n \times r$ full column rank matrix such that $X = WW^\star$. By putting matrix $\mathcal{A}^\star \mathcal{A}$ into Schur form, it can easily be shown that $\operatorname{Trace} \mathcal{A}^\star \mathcal{A} W W^\star = 0$ implies $\mathcal{A}^\star \mathcal{A} W W^\star = 0$. Consequently, the columns of W span a subspace that belongs to the right null-space of $\mathcal{A}$. In view of the above definition of matrix V, there exists a matrix M such that $W = VM$. Let m_{ij} denote the entries of positive semi-definite matrix $MM^\star \in \mathbb{C}^{n \times n}$. For a given index k, it holds either $m_{kk} > 0$ or $m_{ik} = m_{ki} = 0$ for all $i = 1, \ldots, n$. Matrix X is feasible for problem (12) thus

$$F(X) = F(VMM^\star V^\star) = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} F(x_i x_j^\star) \succeq 0. \tag{13}$$

Since matrix X cannot be zero by assumption, matrix $MM^\star$ is also non-zero and there exists at least one index k such that $m_{kk} > 0$. Let x_{k+l} be the last eigenvector in the chain of generalized eigenvectors with eigenvalue s_k for which $m_{(k+l)(k+l)}$ is non-zero (note that $l = 0$ if s_k is non-defective). From relations (5), (10), (11) and (13) it follows that

$$\bar{q}_{k+l}^\star F(X) \bar{q}_{k+l} = m_{(k+l)(k+l)} (a + b s_k + b^\star s_k^\star + c s_k s_k^\star) \geq 0.$$

Since $m_{(k+l)(k+l)} > 0$ we see that $s_k \in \mathcal{D}^C$, hence vector x_k in equation (10) is such that $\operatorname{Trace} \mathcal{A}^\star \mathcal{A} x_k x_k^\star = 0$ and $F(x_k x_k^\star) \succeq 0$ in virtue of Lemma 1. Consequently, matrix $x_k x_k^\star$ is a solution to rank-one LMI problem (6) such that $\mu = 0$ and the lemma is proved. ■

The following result is then a straightforward corollary to Lemma 4:

Lemma 5 *The eigenvalues of matrix A belong to region $\mathcal{D}$ if and only if $\nu > 0$ in LMI optimization problem (12).*

Now consider the following relaxation to rank-one LMI feasibility problem (9):

$$\begin{array}{l} G(Q) \succeq 0 \\ Q = Q^\star \succeq 0 \\ \operatorname{Trace}, Q = 1 \end{array} \tag{14}$$

where the rank constraint has been dropped. Using the same kind of arguments as above, we can show the following counterpart to Lemma 5:

Lemma 6 *The eigenvalues of matrix A belong to region $\mathcal{D}$ if and only if there is no solution to LMI feasibility problem (14).*

4 Dual LMI Problem

Now we use standard semidefinite programming duality results [10] to come up with a more compact formulation of the stability conditions of Lemmas 5 and 6 and prove the Lyapunov's inequality of Theorem 1.

Define the linear map

$$F^D(P) = \begin{bmatrix} \mathcal{Q} \\ \mathcal{P} \end{bmatrix}^\star \begin{bmatrix} aP & bP \\ b^\star P & cP \end{bmatrix} \begin{bmatrix} \mathcal{Q} \\ \mathcal{P} \end{bmatrix} = \begin{bmatrix} aP & bP \\ b^\star P & cP \end{bmatrix} \tag{15}$$

dual to the map introduced in (5). It is easy to show that

$$\operatorname{Trace} F^D(P) X = \operatorname{Trace} F(X) P.$$

Using standard duality arguments, we now prove that the LMI feasibility problem

$$\begin{array}{l} \mathcal{A}^\star \mathcal{A} \succ F^D(P) \\ P = P^\star \succ 0 \end{array} \tag{16}$$

is dual to LMI optimization problem (12). To see this, build the Lagrangian

$$\begin{array}{rcl} L(P, X, Y) & = & -\operatorname{Trace} (\mathcal{A}^\star \mathcal{A} - F^D(P)) X - \operatorname{Trace} PY \\ & = & -\operatorname{Trace} \mathcal{A}^\star \mathcal{A} X + \operatorname{Trace} (F(X) - Y) P \end{array}$$

of problem (16) where $X = X^\star \succeq 0$ and $Y = Y^\star \succeq 0$ are Lagrange multiplier matrices. The dual function associated with the Lagrangian reads

$$\begin{array}{rcl} g(X,Y) & = & \min_P L(P,X,Y) \\ & = & \left\{ \begin{array}{ll} -\mathrm{Trace}\,\mathcal{A}^\star\mathcal{A}X & \mathrm{if} F(X) = Y \succeq 0 \\ -\infty & \mathrm{otherwise.} \end{array} \right. \end{array}$$

The dual optimization problem, obtained by maximizing dual function $g(X,Y)$ is therefore LMI optimization problem (12), where the equality constraint $\mathrm{Trace}\,\mathcal{Q}^\star\mathcal{Q}X = 1$ ensures compactness of the feasible set. The matrix inequalities in problem (16) are strict, hence there is no duality gap and $\nu > 0$ in LMI optimization problem (12) if and only if LMI problem (16) is feasible. Recall that N denotes a matrix whose columns span the right null-space of $\mathcal{A}$. Then it follows from the Elimination Lemma [2] that feasibility problem (16) can equivalently be written as

$$\begin{array}{l} N^\star F^D(P)N \prec 0 \\ P = P^\star \succ 0. \end{array} \qquad (17)$$

This is exactly the statement of Theorem 1.

Similarly, we can define

$$G^D(P) = N^\star F^D(P)N = \left[\begin{array}{c} I \\ A \end{array}\right]^\star \left[\begin{array}{cc} aP & bP \\ b^\star P & cP \end{array}\right] \left[\begin{array}{c} I \\ A \end{array}\right]$$

as the linear map from $\mathbb{C}^{n\times n}$ to $\mathbb{C}^{n\times n}$ dual to the linear map $G(Q)$ introduced in (8). It is easy to show that

$$\mathrm{Trace}\,G^D(P)Q = \mathrm{Trace}\,G(Q)P.$$

It now follows that non-existence of a non-zero $Q = Q^\star \succeq 0$ for which $G(Q) \succeq 0$ is equivalent to the existence of $P = P^\star \succ 0$ for which $G^D(P) \prec 0$. In other words we proved Theorem 1.

5 Numerical Examples

5.1 First Example

Let

$$A = \left[\begin{array}{cc} 1 & -2 \\ 3 & -4 \end{array}\right]$$

be a constant matrix with eigenvalues -1 and -2 and let

$$\mathcal{D} = \{s \in \mathbb{C} : s + s^\star < 0\}$$

be the stability region.

Primal LMI problem (12) reads

$$\begin{array}{rcl} \nu & = & \min \quad \mathrm{Trace} \left[\begin{array}{cccc} 10 & -14 & -1 & -3 \\ -14 & 20 & 2 & 4 \\ -1 & 2 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array}\right] X \\ & & \mathrm{s.t.} \quad \left[\begin{array}{cc} 0&0\\0&0\\1&0\\0&1 \end{array}\right]^\star X \left[\begin{array}{cc} 1&0\\0&1\\0&0\\0&0 \end{array}\right] + \left[\begin{array}{cc} 1&0\\0&1\\0&0\\0&0 \end{array}\right]^\star X \left[\begin{array}{cc} 0&0\\0&0\\1&0\\0&1 \end{array}\right] \succeq 0 \\ & & X = X^\star \succeq 0 \\ & & \mathrm{Trace} \left[\begin{array}{cccc} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{array}\right] X = 1. \end{array}$$

With a relative accuracy of 10^{-8}, the LMI Control Toolbox 1.0.5 for Matlab 5.3 [6] returns

$$\nu = 0.1339312527$$

and

$$X = \left[\begin{array}{cccc} 0.6681681985 & 0.4708709558 & 0.0000000000 & 0.0000000000 \\ 0.4708709558 & 0.3318318015 & 0.0000000000 & 0.0000000000 \\ 0.0000000000 & 0.0000000000 & 0.0000000000 & 0.0000000000 \\ 0.0000000000 & 0.0000000000 & 0.0000000000 & 0.0000000000 \end{array}\right]$$

as the optimum of the above problem. In virtue of Theorem 5, ν is strictly positive hence all the eigenvalues of matrix A belong to region $\mathcal{D}$.

Dual LMI problem (17) reads

$$\left[\begin{array}{cc} 1&0\\0&1\\1&-2\\3&-4 \end{array}\right]^\star \left(\left[\begin{array}{cc} 0&0\\0&0\\1&0\\0&1 \end{array}\right] P \left[\begin{array}{cc} 1&0\\0&1\\0&0\\0&0 \end{array}\right]^\star + \left[\begin{array}{cc} 1&0\\0&1\\0&0\\0&0 \end{array}\right] P \left[\begin{array}{cc} 0&0\\0&0\\1&0\\0&1 \end{array}\right]^\star \right) \left[\begin{array}{cc} 1&0\\0&1\\1&-2\\3&-4 \end{array}\right] \prec 0$$

$$P = P^\star \succ 0.$$

With the help of the LMI Toolbox, we obtained the matrix

$$P = \left[\begin{array}{cc} 1.2825871062 & -0.5410485223 \\ -0.5410485223 & 0.4092194407 \end{array}\right]$$

as a feasible solution for the above problem. On the other hand, LMI problem (14) reads

$$\left[\begin{array}{cccc} 1&0&1&-2\\0&1&3&-4 \end{array}\right] \left[\begin{array}{cc} 0 & Q \\ Q & 0 \end{array}\right] \left[\begin{array}{cccc} 1&0&1&-2\\0&1&3&-4 \end{array}\right]^\star \succeq 0$$

$$Q = Q^\star \succeq 0, \quad \mathrm{Trace}\,Q = 1.$$

This problem is infeasible, which is consistent with the above results and Theorem 1.

5.2 Second Example

Now let

$$A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$$

be a constant matrix with eigenvalues -2 and 5 and let

$$\mathcal{D} = \{s \in \mathbb{C} : s + s^\star < 0\}$$

be the stability region.

With a relative accuracy of 10^{-8}, the LMI Toolbox returns

$$\nu = 0.0000000000$$

and

$$X = \begin{bmatrix} 0.1000000000 & 0.3000000000 & 0.5000000000 & 1.5000000000 \\ 0.3000000000 & 0.9000000000 & 1.5000000000 & 4.5000000000 \\ 0.5000000000 & 1.5000000000 & 2.5000000000 & 7.5000000000 \\ 1.5000000000 & 4.5000000000 & 7.5000000000 & 22.5000000000 \end{bmatrix}$$

as the optimum of primal problem (12). In virtue of Theorem 5, some eigenvalues of matrix A do not belong to region $\mathcal{D}$. One can check that

$$X = xx^\star = \begin{bmatrix} 0.3162277660 \\ 0.9486832981 \\ 1.5811388301 \\ 4.7434164903 \end{bmatrix} \begin{bmatrix} 0.3162277660 \\ 0.9486832981 \\ 1.5811388301 \\ 4.7434164903 \end{bmatrix}^\star$$

is actually a rank-one solution to LMI problem (12). Vector x can be written as

$$x = \begin{bmatrix} q \\ sq \end{bmatrix}$$

where q is an eigenvector of matrix A corresponding to the eigenvalue $s = 5 \in \mathcal{D}^C$. One can check that positive semidefinite matrix

$$Q = qq^\star = \begin{bmatrix} 0.1000000000 & 0.3000000000 \\ 0.3000000000 & 0.9000000000 \end{bmatrix}$$

is a feasible solution for LMI problems (9) or (14). On the other hand, dual LMI problem (17) is found infeasible, which is consistent with the above results and Theorem 1.

6 Conclusion

We have proposed a new proof of Lyapunov's matrix inequality that relies on elementary optimization techniques and linear algebra. Following ideas proposed in [7] and [3, Chapter 1], we consider the eigenvalue location problem as a mere quadratic optimization problem. Then, the quadratic problem can be formulated as an LMI problem with a non-convex rank constraint. The Lyapunov matrix can be viewed as a Lagrange multiplier matrix arising when dualizing this problem.

In [3, Chapter 1, §1.4.6], it is shown that removing the non-convex rank-one constraint leads to a sufficient LMI stability condition. Our contribution is in showing in Lemmas 4 and 6 that the LMI conditions are also necessary. In other words, the rank constraint in problems (6) and (9) are irrelevant as far as eigenvalue location is concerned.

In a similar fashion, the eigenvalue location problem can be viewed as a frequency-dependent μ-analysis problem with one repeated scalar block sI corresponding to the Laplace variable s. The Lyapunov matrix P plays the role of a D-scaling matrix associated with the repeated scalar block, and the irrelevance of the non-convex rank constraint readily follows from the losslessness of the (D, G)-scaling as pointed out in [8].

Equivalence of primal problem (12) and dual problem (16) can also be shown via geometric arguments similar to that used in the proof of the Kalman-Yakubovich-Popov (KYP) Lemma in [9, Theorem 1], in the proof of losslessness of (D, G)-scaling [7, Lemma 3.1], in the $\mathcal{S}$-procedure [11] or also in the generalized $\mathcal{S}$-procedure proposed in [4, Theorem 1].

Our approach is also very similar in spirit to the one pursued in [9] to provide an alternative proof of the KYP Lemma. Note however that in this reference the author considers a version of the KYP Lemma where the Laplace variable s varies on the imaginary axis or the unit circle. This result has been extended to other one-dimensional curves of the complex plane such as the real axis [7] or a segment on the imaginary axis [4]. These curves are boundaries of the two-dimensional stability regions $\mathcal{D}$ considered in the present note. It is therefore expected that we can similarly derive more general versions of the KYP Lemma in two-dimensional stability regions.

Finally, we are currently investigating the application of these techniques to the study of stability of polynomial matrices, two-indeterminate polynomial matrices and uncertain polynomial matrices. Related results will be reported elsewhere.

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