Stabilizing Receding Horizon H_∞ Controls for Linear Continuous Time-varying Systems

Ki Baek Kim and Tae-Woong Yoon†

School of Electrical Engineering, Korea University
1, 5-ka, Anam-dong Sungbuk-ku, Seoul 136-701, Korea
Fax: +82-2-928-8909, Tel: +82-2-3290-3240
E-mail: kkb@classic.korea.ac.kr, twy@cello.korea.ac.kr

Abstract

In this paper, new matrix inequality conditions on the terminal weighting matrices are proposed for linear continuous time-varying systems. Under these conditions, nonincreasing and nondecreasing monotonicities of the saddle point value of a dynamic game are shown to be guaranteed. It is proved that the proposed terminal inequality conditions ensure the closed-loop stability of the receding horizon H_∞ control (RHHC). The stabilizing RHHC guarantees an H_∞ norm bound of the close-loop system. The proposed terminal inequality conditions for the monotonicity of the saddle point value and the closed-loop stability include most well-known existing terminal conditions as special cases. The results for time-invariant systems are obtained correspondingly from those in the time-varying case.

1 Introduction

Receding horizon control (RHC) is a closed-loop strategy, where an optimal decision of the control is taken at each sample, and thus finite horizons can be considered. This contrasts with the steady-state linear quadratic (LQ) control, where the control law is determined at the initial time and only infinite horizons can be taken into account. For these reasons, the RHC has been widely investigated as a successful feedback strategy in [1]-[3].

For the closed-loop stability of the RHC, it is well known that conditions on terminal weighting matrices as well as state weighting matrices are important. For finite horizons, one approach to achieve the closed-loop stability is to impose infinite terminal weighting which is equivalent to setting a zero terminal weighting matrix for the inverse Riccati equation [1]. This is referred to as the terminal equality condition. Since imposing infinite terminal weighting is demanding, use of finite terminal weighting matrices has been investigated in [3]-[5] for continuous-time systems. Finite terminal weighting matrices have often been represented by matrix inequality conditions. As an alternative approach for finite horizons, infinite horizon formulations have been explored in [2], [3]. However, infinite horizon formulations in [2], [3] can also be expressed as finite horizon formulations with appropriate finite terminal weighting matrices for linear systems, as shown in [3], [5].

In the proofs of the closed-loop stability of the RHC, the monotonicity of the optimal cost has been widely used in recent years since it does not only imply the monotonicity of the Riccati equations for linear systems but can also be used for the closed-loop stability of nonlinear systems. For this reason, the nonincreasing monotonicity of the optimal cost has often been investigated [2], [3]. However, the nondecreasing monotonicity of the optimal cost has been very recently discussed in [5] for continuous time-invariant systems.

The terminal inequality conditions in [5] seem to be the weakest for the cost monotonicity, which include most previous results of the RHC in [1], [3], [4] for continuous-time systems. In [5], the well-known terminal equality condition has been extended to the terminal inequality condition for the nondecreasing cost monotonicity, and the condition on the state weighting matrix has been weakened so as to include the zero matrix.

This RHC has been applied to H_∞ problems recently in order to combine the practical advantage of the RHC with the robustness of the H_∞ control. For the closed-loop stability of the receding horizon H_∞ control (RHHC), the terminal equality condition is introduced in [6] for continuous time-varying systems. The terminal inequality condition for the nonincreasing monotonicity of the Riccati equations is proposed in [7] for continuous time-varying systems. The saddle point value in H_∞ problems corresponds to the optimal cost in the LQ problems. The terminal inequality conditions for the nonincreasing monotonicity of the saddle point value are proposed in [8] for discrete time-varying systems. However, the nonincreasing monotonicity of the saddle point value is not fully discussed in [8]. To authors knowledge, in H_∞ problems, there are no results corresponding to the inequality conditions for the cost monotonicity in [5]. Thus, it will be interesting to investigate the monotonicity of the saddle point value in detail. In [8], it is not clear if the closed-loop stability is guaranteed for positive semidefinite state weighing matrices. Thus, in H_∞ problems, it will also be interesting to investigate whether the condition on the state weighting matrix can be weakened so as to include zero matrices.

In this paper, new terminal inequality conditions are proposed for linear conditions time-varying systems under which nonincreasing and nondecreasing monotonicities of the saddle point value of a dynamic game hold. It is shown that the closed-loop stability of the RHHC is guaranteed under the proposed terminal inequality conditions. The well known terminal equality condition is extended to a wider class of matrix inequality condition under which the closed-loop stability holds. It is also shown that the state weighting matrix can be even zero, still guaranteeing the closed-loop stability of the RHHC.

The rest of this paper is organized as follows. In Section 2, terminal inequality conditions are proposed which guarantee the monotonicity of the saddle point value. In Section 3, under the proposed terminal inequality conditions, the closed-loop stability of the RHHC is shown. Finally, conclusions are presented in Section 4.

2 Monotonicity of the saddle point value

Consider a linear continuous time-varying system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \quad (1)\\ z(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ the control, $w(t) \in R^l$ the disturbance, $z(t) \in R^p$ the controlled output, $C^T(t)D(t) = 0$, and $D^T(t)D(t) = I$. Throughout the rest of this paper, the time index (t) is often omitted for simplicity. For this system, consider the following dynamic game problem with a finite cost horizon

$$\min_u \max_w \quad J(t_0, t_f) \qquad (2)$$

where $J(t_0,t_f) = \int_{t_0}^{t_f}[x^TQx + u^Tu - \gamma^2 w^T w]dt + x^T(t_f)Q_f(t_f)x(t_f)$. Here, $Q = C^TC \geq 0$, $Q_f \geq 0$, and γ is the disturbance attenuation level. The matrices A, B_1, B_2, Q, and Q_f are assumed to be bounded.

Define B_γ as $B_\gamma = \gamma^{-1}B_1$, $\hat{Q}$ as $\hat{Q} = B_2\ B_2^T - B_\gamma\ B_\gamma^T$ and the threshold value $\hat{\gamma}^{CL}$ as

$$\begin{aligned} \hat{\gamma}^{CL} &= inf\{\gamma > 0 : K(\tau,\sigma) \text{ does not have a} \\ &\quad \text{conjugate point for all} \tau \leq \sigma\} \qquad (3) \end{aligned}$$

where $K(\tau,\sigma)$ satisfies

$$\begin{aligned} -\frac{\partial K(\tau,\sigma)}{\partial \tau} &= A^T(\tau)K(\tau,\sigma) + K(\tau,\sigma)A(\tau) + Q(\tau) \\ &\quad -K(\tau,\sigma)\hat{Q}(\tau)K(\tau,\sigma),\ \tau \leq \sigma \qquad (4) \end{aligned}$$

with the boundary condition

$$K(\sigma,\sigma) = Q_f(\sigma). \qquad (5)$$

Then, the dynamic game theory described by (1) and (2) [9] admits a unique feedback saddle-point solution, if and only if $\gamma > \hat{\gamma}^{CL}$ with $\sigma = t_f$. In this case, the unique feedback saddle point solution for the dynamic game problem (2) is given by

$$\begin{aligned} u^*(t) &= -B_2^T(t)K(t,t_f)x(t) \qquad (6)\\ w^*(t) &= \gamma^{-1}B_\gamma^T(t)K(t,t_f)x(t), \quad t_0 \leq t \leq t_f \quad (7) \end{aligned}$$

and the saddle point value of the dynamic game of (2) with (6) is given by

$$J^*(t,t_f) = x^T(t)K(t,t_f)x(t). \qquad (8)$$

For simplicity, $K(\tau,\sigma)$ is often denoted as $K(\tau)$ with $K(\sigma) = Q_f(\sigma)$.

Now, we give a sufficient condition for the nonincreasing monotonicity of the saddle point value.

THEOREM 1 *Assume that $Q_f(\sigma)$ satisfies*

$$\begin{aligned} &(A(\sigma) - B_2(\sigma)H(\sigma))^T Q_f(\sigma) + Q_f(\sigma)(A(\sigma) - B_2(\sigma) \\ &H(\sigma)) + Q(\sigma) + H^T(\sigma)H(\sigma) + Q_f(\sigma)B_\gamma(\sigma)B_\gamma^T(\sigma) \\ &Q_f(\sigma) + \dot{Q}_f(\sigma) \leq 0 \quad \text{for some } H(\sigma) \in R^{m\times n}. \quad (9) \end{aligned}$$

The saddle point value $J^*(t,t_f)$ in (8) then satisfies:

$$\frac{\partial J^*(\tau,\sigma)}{\partial \sigma} \leq 0, \quad \tau \leq \sigma \qquad (10)$$

and thus $\frac{\partial K(\tau,\sigma)}{\partial \sigma} \leq 0$, $\tau \leq \sigma$.

proof:

$$\begin{aligned} \frac{\partial J^*(\tau,\sigma)}{\partial \sigma} &= \lim_{\Delta\to 0}\frac{1}{\Delta}\{J^*(\tau,\sigma+\Delta) - J^*(\tau,\sigma)\} \\ &= \lim_{\Delta\to 0}\frac{1}{\Delta}\{\int_\tau^\sigma [x_1^TQx_1 + u_1^Tu_1 \\ &\quad -\gamma^2 w_1^Tw_1]dt + J^*(\sigma,\sigma+\Delta) \\ &\quad -\int_\tau^\sigma [x_2^TQx_2 + u_2^Tu_2 - \gamma^2 w_2^Tw_2]dt \\ &\quad -x_2^T(\sigma)Q_f(\sigma)x_2(\sigma) \end{aligned}$$

where the pair (u_1, w_1) is a saddle point solution for $J(\tau, \sigma+\Delta)$ and the pair (u_2, w_2) is for $J(\tau,\sigma)$. If we replace u_1 and w_2 by u_2 and w_1 up to σ, then

$$\frac{\partial J^*(\tau,\sigma)}{\partial \sigma} \leq \lim_{\Delta\to 0}\frac{1}{\Delta}\{J(\sigma,\sigma+\Delta) - x^T(\sigma)Q_f(\sigma)x(\sigma)\}$$

where $x(\sigma)$ is the state at σ due to $x(\tau)$, $u_2(t)$, and $w_1(t)$ for $t \in [\tau, \sigma)$.

Here, if we replace $u_1(t)$ by $u_1(t) = -H(t)x(t)$ for $t \geq \sigma$, then $w^*(t)$ for $t \in [\sigma, \sigma+\Delta)$, which maximizes the cost $J(\sigma, \sigma+\Delta)$, is given by

$$w^*(t) \quad = \quad \gamma^{-1}B_\gamma^T(t)K_H(t,\sigma+\Delta)x(t) \qquad (11)$$

where $K_H(t, \sigma+\Delta)(= K_H(t))$ satisfies

$$\begin{aligned} -\frac{\partial K_H(t)}{\partial t} \quad &= \quad (A - B_2H)^T K_H(t) + K_H(t)(A - B_2H) \\ &\quad +Q + H^T H + K_H(t)B_\gamma B_\gamma^T K_H(t) \quad (12) \end{aligned}$$

with $K_H(\sigma+\Delta, \sigma+\Delta) = Q_f(\sigma+\Delta)$.

Note that by classical analysis, there exists a bounded solution to this equation for $\sigma \leq t \leq \sigma+\Delta$ if $\Delta > 0$ is sufficiently small.

In this case $\frac{\partial J^*(\tau,\sigma)}{\partial\sigma} \leq 0$ by (9). From (8), we have $\frac{\partial K(\tau,\sigma)}{\partial\sigma} \leq 0$. □

In (9), if $H(\sigma)$ is replaced by an optimal gain $H(\sigma)$=$B_2^T(\sigma)Q_f(\sigma)$, then we have

$$A^T Q_f + Q_f A + Q - Q_f\hat{Q}Q_f + \dot{Q}_f \quad \leq \quad 0. \quad (13)$$

Note that under this well-known inequality condition, the monotonicity of Riccati equation is shown in [7] where $Q_f(\sigma) > 0$.

Consider a cost function with an infinite horizon. If the pair (A, B_2) is uniformly stabilizable and the system is uniformly asymptotically stable with $u(t) = -H(t)x(t)$ and $w(t) = \gamma^{-1}B_\gamma^T(t)K(t,\infty)x(t)$ for $t \geq \sigma \geq \tau$, then $J = \int_\tau^\infty [x^TQx + u^Tu - \gamma^2 w^T w]dt = \int_\tau^\sigma [x^TQx + u^Tu - \gamma^2 w^T w]dt + x^T(\sigma)Q_f(\sigma)x(\sigma)$ where $Q_f(\sigma)$ is bounded and satisfies

$$\begin{aligned} -\dot{Q}_f \quad &= \quad (A - B_2H)^T Q_f + Q_f(A - B_2H) + Q \\ &\quad +H^T H + Q_f B_\gamma B_\gamma^T Q_f \qquad (14) \end{aligned}$$

which is a special case of (9).

Consider another cost function with an infinite horizon. If the system is uniformly asymptotically stable with $u(t) = 0$ and $w(t) = \gamma^{-2}B_1^T(t)K(t,\infty)x(t)$ for $t \geq \sigma \geq \tau$, then $Q_f(\sigma)$ is bounded and satisfies

$$-\dot{Q}_f \quad = \quad A^T Q_f + Q_f A + Q + Q_f B_\gamma B_\gamma^T Q_f \quad (15)$$

which is also a special case of (9).

In the following, the nondecreasing monotonicity of the saddle point value is studied.

THEOREM 2 *Assume that $Q_f(\sigma)$ in (5) satisfies*

$$A^T Q_f + Q_f A + Q - Q_f\hat{Q}Q_f + \dot{Q}_f \quad \geq \quad 0. \qquad (16)$$

The saddle point value $J^*(t, t_f)$ in (8) then satisfies

$$\frac{\partial J^*(\tau,\sigma)}{\partial\sigma} \quad \geq \quad 0, \quad \tau \leq \sigma \qquad (17)$$

and thus $\frac{\partial K(\tau,\sigma)}{\partial\sigma} \geq 0, \quad \tau \leq \sigma$.

proof: In the same way as the proof of Theorem 1, if we replace u_2 and w_1 by u_1 and w_2 up to σ, then $\frac{\partial J^*(\tau,\sigma)}{\partial\sigma} \geq 0$ where $x(\sigma)$ is the trajectory at σ due to $x(\tau)$, $u_1(t)$, and $w_2(t)$ for $t \in [\tau, \sigma)$. The monotonicity of the Riccati equations follows from (8). □

The following result, which is based on the optimality, shows that when the monotonicity of the saddle point value or the Riccati equations holds once, it holds for all prior times.

THEOREM 3 *1) If $\frac{\partial J^*(\tau',\sigma)}{\partial\sigma} \leq 0$ (or ≥ 0) for some τ' then $\frac{\partial J^*(\tau'',\sigma)}{\partial\sigma} \leq 0$ (or ≥ 0) where $\tau_0 \leq \tau'' \leq \tau'$. 2) If $\frac{\partial K(\tau',\sigma)}{\partial\sigma} \leq 0$ (or ≥ 0) for some τ' then $\frac{\partial K(\tau'',\sigma)}{\partial\sigma} \leq 0$ (or ≥ 0) where $\tau_0 \leq \tau'' \leq \tau'$.*

proof: It can be easily proved in the same way as in Theorems 1 and 2. □

Now, we introduce the inverse form of (4), which is used in the next section. If $K(\tau,\sigma)$ is nonsingular at $\tau > t_0$, there exists $K^{-1}(\tau,\sigma) = P(\tau,\sigma)$ which satisfies

$$\begin{aligned} \frac{\partial P(\tau,\sigma)}{\partial\tau} \quad &= \quad P(\tau,\sigma)A^T(\tau) + A(\tau)P(\tau,\sigma) + P(\tau,\sigma) \\ &\quad Q(\tau)P(\tau,\sigma) - \hat{Q}(\tau). \qquad (18) \end{aligned}$$

In the next section, we also consider a little different approach. We assume that $P(\tau,\sigma)$ in (18) is given from the beginning with a terminal constraint $P(\sigma,\sigma) = P_f(\sigma)$ and some $\gamma > 0$ rather than the one obtained from inverting (4). Note that $P_f(\sigma) = 0$ corresponds to the case $Q_f(\sigma) = \infty I$ (i.e., $Q_f^{-1}(\sigma) = 0$) even if it is mathematically not rigorous. In fact, the Riccati equation (18) with the condition $P_f(\sigma) \geq 0$ can be obtained from the following problem. Consider the following adjoint system of the system (1)

$$\dot{\hat{x}}(t) = -A^T(t)\hat{x}(t) + C^T(t)\hat{u}(t) \qquad (19)$$

with the cost function

$$\hat{J}(t_0, t_f) = \int_{t_0}^{t_f} [\hat{x}^T \hat{Q}\hat{x} + \hat{u}^T\hat{u}]dt + \hat{x}^T(t_f)P_f(t_f)\hat{x}(t_f)$$

where $\hat{Q}(t) \geq 0$ for all t and some $\gamma > 0$.

Corollary 1 *Assume that* $\hat{Q}(t) \geq 0$ *for all* t *and* $P_f(t_f)$ *satisfies*

$$-AP_f - P_fA^T - P_fQP_f + \hat{Q} + \dot{P}_f \leq 0. \qquad (20)$$

The optimal cost $\hat{J}^*(t, t_f)$ *for (19) then satisfies* $\frac{\partial \hat{J}^*(\tau,\sigma)}{\partial \sigma} \geq 0$, $\tau \leq \sigma$, *and thus*

$$\frac{\partial P(\tau, \sigma)}{\partial \sigma} \geq 0, \quad \tau \leq \sigma. \qquad (21)$$

The proof procedure follows that of Theorem 9. Note that $P_f(\sigma)$ can be singular.

Since $P_f(\sigma) = 0$ satisfies (20), the well-known terminal equality condition in [6] is a special case of (20) for guaranteeing (21).

In the following section, stabilizing receding horizon H_∞ controls are proposed by using the monotonicity of the saddle point value or the Riccati equations.

3 Stability of the receding horizon H_∞ control

The receding horizon H_∞ control (RHHC) is obtained by replacing t_f with $t + T$ for $0 < T < \infty$ in (6):

$$u^*(t) = -B_2^T(t)K(t, t+T)x(t) \qquad (22)$$

where $K(t, t+T)$ is computed from (4) with $K(t+T, t+T) = Q_f(t+T)$. Assume that the selected $Q_f(t+T)$ is positive semidefinite and bounded for all t, and T is nonzero and finite.

In this Section, assume that there exists a unique saddle point solution (6). Definitions of uniform controllability and uniform observability with positive integers δ_c and δ_o in [1] are used. Let $\delta = max\{\delta_c, \delta_o\}$.

The following result is needed to show the stabilizing properties of (22). Denote the transition matrix of $f(\sigma)$ as $\Phi_f(\sigma, \tau)$ satisfying $\frac{\partial \Phi_f(\sigma,\tau)}{\partial \sigma} = f(\sigma)\ \Phi_f(\sigma, \tau)$, $\sigma \geq \tau$ with $\Phi_f(\tau, \tau) = I$.

Lemma 1 *Assume that the system* $\dot{x}(t) = A(t)x(t)$ *is uniformly attractive where* $A(t)$ *is bounded for all* t. *Then the system is uniformly asymptotically stable.*

proof: If the system $\dot{x}(t) = A(t)x(t)$ is uniformly attractive over $[t_0, \infty)$ for all $\epsilon > 0$, there exist positive constants γ and $N = N(\gamma, \epsilon)$ such that $\|x(t_1)\| \leq \gamma$ implies $\|x(t_2)\| \leq \epsilon$ for all $t_2 \geq t_1 + N$, independently of t_1. Let us define $\Gamma = \max_{\tau \in [t_1, t_1+N)} \|\Phi_A(\tau, t_1)\|$ and $\delta = min\{\gamma, \frac{\epsilon}{\Gamma}\}$. Since $A(t)$ is bounded, Γ is finite and thus there exists a positive constant δ. For all $t_2 \geq t_1 + N$, $\|x(t_2)\| \leq \epsilon$. For $t_1 \leq t_2 < t_1 + N$, $\|x(t_2)\| \leq \epsilon$ since $\delta \leq \frac{\epsilon}{\Gamma}$ and $\|x(t_2)\| \leq \|\Phi_A(t_2, t_1)\|\|x(t_1)\|$. Therefore, we have $\|x(t_1)\| \leq \delta$ implies $\|x(\sigma)\| \leq \epsilon$ for all $\sigma \geq t_1$. □

THEOREM 4 *Assume that the pair* (A, C) *is uniformly observable. If* $\frac{\partial J^*(t,\sigma)}{\partial \sigma}|_{\sigma=t+T} \leq 0$ *for all* t, *then the system (1) with the RHHC (22) is uniformly asymptotically stable for* $0 < T < \infty$.

proof: We show that the zero state is uniformly attractive. Since $\frac{\partial J^*(t,\sigma)}{\partial \sigma}|_{\sigma=t+T} \leq 0$ and $J^*(u^*, w) \leq J^*(u^*, w^*) \leq J^*(u, w^*)$), if $\Delta > 0$ is sufficiently small $J^*(t, t+T) = \int_t^{t+\Delta}[x^TQx + u^{*T}u^* - \gamma^2 w^{*T}w^*]d\tau + J^*(t+\Delta, t+T) \geq \int_t^{t+\Delta}[x^TQx + u^{*T}u^*]d\tau + J^*(t + \Delta, t+T)$ where $x_2(t)$ is the state trajectory from $x_2(t) = x(t)$ when $w(t+i) = 0$ for $i \in [0, \Delta)$ and $w(t+i) = w_2^*(t+i)$ for $i \in [\Delta, T)$. Hence, $\dot{J}^*(t, t+T) \leq$ -$[x^T(t)\ Q(t)\ x(t) + u^{*T}(t)\ u^*(t)]$ = -$[x^T(t)\ Q(t)\ x(t) + u^T(t)\ u(t)]$ where $u(t)$ is the RHHC (22) and $w(t) = 0$ for all t. From this and $J^*(t, t+T) \geq 0$, $J^*(t, t+T) \to c$ for some nonnegative constant c as $t \to \infty$. Thus, as $t \to \infty$, $u^*(t) \to 0$ and $\int_t^{t+\theta} x^TQxd\tau = x^T(t)\int_t^{t+\theta}\Phi_A^T(\tau, t)Q(\tau)\Phi_A(\tau, t)d\tau\ x(t) = x^T(t)G_o(t, t+\theta)x(t) \to 0$ where $G_o(t, t+\theta)$ is an observability Grammian [1]. However, since the pair (A, C) is uniformly observable, there exists a positive constant α_1 satisfying $G_o(t, t+\theta) \geq \alpha_1 I$ for $\theta \geq \delta_o$. This means that $x(t) \to 0$ as $t \to \infty$, independently of t_0. Therefore, the closed-loop system is uniformly attractive when $w(t) = 0$ for all t. Since $Q_f(t+T)$ is bounded, the closed-loop matrix $[A(t) - B_2(t)B_2^T(t)K(t, t+T)]$ with (22) is bounded. Therefore, the closed-loop system with $w(\cdot) = 0$ is uniformly asymptotically stable from Lemma 1. □

Next, we suggest a sufficient condition for Theorem 4

Corollary 2 *Assume that the pair* (A, C) *is uniformly observable. If* $Q_f(t+T)$ *satisfies (9) for all* t, *the system (1) with (22) is uniformly asymptotically stable for* $0 < T < \infty$.

Note that there exists a bounded $Q_f(t+T)$ satisfying (9) if the pair (A, B_2) is uniformly stabilizable. The condition (13) is shown in [7] for time-varying systems where it requires $Q_f(t+T) \geq \alpha_2 I$ and $\alpha_3 I \leq K(t, t+T) \leq \alpha_4 I$ for positive constants α_3 and α_4. Therefore, the results in [7] are special cases of Corollary 2.

In the following, we consider an extended boundary condition $P(t+T, t+T) = P_f(t+T) \geq 0$ which includes the well-known terminal equality condition $P_f(t+T) = 0$ in [6]. Let us consider another receding horizon H_∞ control (RHHC)

$$u^*(t) \quad = \quad -B_2^T(t) P^{-1}(t, t+T) x(t) \qquad (23)$$

where $P(t, t+T)$ is computed from (18) with $P(t+T, t+T) = P_f(t+T)$.

From here, we assume that the selected $P_f(t+T) \geq 0$ is bounded and $\hat{Q}(t) \geq 0$ for all t unless otherwise specified. The following results are needed to show the stabilizing properties of (23).

Lemma 2 *There exists a positive constant α_5 satisfying $P(t, t+T) \geq \alpha_5 I$ for $T \geq \delta_c$ if the pair (A, B) is uniformly controllable where B satisfies $BB^T = \hat{Q}$.*

proof: It is proved by showing $\hat{J}^*(t, t+T) \geq \alpha_5 \|\hat{x}(t)\|^2$. Consider the adjoint system (19) from which $P(t, t_1)$ stems. Assume that $\hat{J}^*(t, t+T) = 0$ for $\hat{x}(t) \neq 0$. Since $\hat{x}^T(\tau)\ \hat{Q}(\tau)\ \hat{x}(\tau) = \hat{u}^{*T}(\tau)\ \hat{u}^*(\tau) = \hat{x}^T(t+T)\ P_f(t+T)\ \hat{x}(t+T) = 0$ for $\tau \in [t, t+T)$, $\hat{J}^*(t, t+T) \geq \int_t^{t+T} \hat{x}^T \hat{Q} \hat{x} d\tau = \hat{x}^T(t) \int_t^{t+T} \Psi_{-A}(\tau, t)\ \hat{Q}\ \Psi_{-A^T}(\tau, t)\ d\tau\ \hat{x}(t) = \hat{x}^T(t) G_c(t, t+T) \hat{x}(t)$ where $G_c(t, t+T)$ is the controllability Grammian [1]. Since the pair (A, B) is uniformly controllable, there exists a positive constant α_5 satisfying $\hat{J}^*(t, t+T) \geq \alpha_5 \|\hat{x}(t)\|^2$ for $T \geq \delta_c$. This contradicts $\hat{J}^*(t, t+T) = 0$. Note that $P_f(t+T) = 0$ is included. □

In [6], they assume that there exists a positive constant α_5 satisfying $P(t, t+T) \geq \alpha_5 I$.

Lemma 3 *If the pair (A, B) is uniformly controllable with B as in Lemma 2, then the pair (A, B_2) is uniformly controllable.*

proof: It can be proved directly from [10]. □

Now, we are ready to state the following result.

THEOREM 5 *Assume that the pair (A, B) is uniformly controllable. If $\frac{\partial P(t,\sigma)}{\partial \sigma}|_{\sigma = t+T} \geq 0$ for all t, then the system (1) with (23) is uniformly asymptotically stable for $\delta_c \leq T < \infty$.*

proof: Consider the adjoint system of (1) with the RHHC (23) when $w(t) = 0$:

$$\dot{\hat{x}} \quad = \quad -[A - B_2 B_2^T P^{-1}(t)]^T \hat{x} \quad = \quad \Psi \hat{x}. \qquad (24)$$

Then, we define the associated scalar valued function $V(\hat{x}, t) = \hat{x}^T(t) P(t, t+T) \hat{x}(t)$. $P(t, t+T)$ is bounded since the system matrices and $P_f(t+T)$ are bounded, and T is finite. Therefore, from Lemma 2, there exist positive constants α_5 and α_6 satisfying $\alpha_5 \|\hat{x}(t)\|^2 \leq V(\hat{x}, t) = \hat{x}^T(t) P(t, t+T) \hat{x}(t) \leq \alpha_6 \|\hat{x}(t)\|^2$.

$$\dot{V}(\hat{x}, t) \quad \geq \quad 0. \qquad (25)$$

For $t_s \geq t_0 + \delta_c$, there exists a positive constant α_7 satisfying $V(\hat{x}(t_s), t_s) - V(\hat{x}(t_0), t_0) = \int_{t_0}^{t_s} \dot{V}(t) dt \geq \hat{x}^T(t_0) G_c(t_0, t_s)\ \hat{x}(t_0) \geq \alpha_7\ \|\hat{x}(t_0)\|^2$ where $G_c(t_0, t_s)$ is the controllability Grammian [1].This implies that the closed-loop system (24) is exponentially increasing, i.e., the closed-loop system (1) with (22) is exponentially decreasing when $w(t) = 0$ for all t. □

Next, we suggest a sufficient condition for Theorem 5.

Corollary 3 *Assume that the pair (A, B) is uniformly controllable. If $P_f(t+T)$ satisfies (20), or if $P_f(t+T) = 0$ for all t, then the system (1) with (23) is uniformly asymptotically stable for $\delta_c \leq T < \infty$.*

Note that there exists a bounded $P_f(t+T)$ satisfying (20) if the pair (A, C) is uniformly detectable. The results in Corollary 2 and 3 are different in that the former is applied to the condition (9), while the latter to (20).

Remark 1 *Note that if $P_f(t+T) = 0$, Corollary 3 holds for arbitrary $Q(t) \geq 0$ including the zero matrix which is different from Corollary 2. Corollary 3 is an extension from the well-known results in [6] which require $P_f(t+T) = 0$. When $Q(t) = 0$ for all t, $P(t, t+T)$ can be expressed as $P(t, t+T) = \int_t^{t+T} \Phi_A(t, \tau)\ \hat{Q}\ \Phi_A^T(t, \tau)\ d\tau + \Phi_A(t, t+T)\ P_f(t+T)\ \Phi_A^T(t, t+T)$. Note that $P(t, t+T) \geq \alpha_5 I$ for $\delta_c \leq T < \infty$ if the pair (A, B) is uniformly controllable. In the above equation, $P_f(t+T)$ can be zero. This may be the simplest RHHC.*

In the following, the state weighting matrix is also weakened so as to include even the zero matrix for guaranteeing the closed-loop stability of the RHHC.

Lemma 4 *If there exists a positive constant α_8 satisfying $Q_f(t+T) \geq \alpha_8 I$, then $J^*(t, t+T) \geq \alpha_8 \|x(t)\|^2$, and thus $K(t, t+T) \geq \alpha_8 I$ for all t and $T \geq 0$.*

proof: Assume that $J^*(t, t+T) = 0$ for $x(t) \neq 0$. Since $J^*(u^*, w^*) \geq J(u^*, w = 0)$, $x^T(\tau) Q(\tau) x(\tau) = u^{*T}(\tau) u^*(\tau) = x^T(t+T) Q_f(t+T) x(t+T) = 0$ for $\tau \in [t, t+T)$. Since $x^T(t+T) Q_f(t+T) x(t+T) = x^T(t) \Phi_A^T(t+T, t) Q_f(t+T) \Phi_A(t+T, t) x(t)$ and $\Phi_A(t+$

$T, t)$ is nonsingular, $x(t) = 0$. This contradicts $x(t) \neq 0$. Note that T can be 0. $\square$

Now, we introduce our last main result.

THEOREM 6 *Assume that the pair (A, B) is uniformly controllable. If $Q_f(t+T)$ satisfies $Q_f(t+T) \geq \alpha_8 I$ for a positive constant α_8 and $\frac{\partial K(t,\sigma)}{\partial \sigma}|_{\sigma=t+T} \leq 0$ for all t, then the system (1) with (22) is uniformly asymptotically stable for $\delta_c \leq T < \infty$.*

proof: From Lemma 4, there exist positive constants α_8 and α_9 satisfying $\alpha_8 I \leq K(t, t+T) \leq \alpha_9 I$. Therefore, it can be proved in the same way as Theorem 5 by replacing $P(t, t+T)$ by $K^{-1}(t, t+T)$. $\square$

Corollary 4 *Assume that the pair (A, B) is uniformly controllable. If $Q_f(t+T)$ satisfies (9) for all t and the pair $(A - B_2 H, H)$ is uniformly observable, then the system (1) with (22) is uniformly asymptotically stable for $\delta_c \leq T < \infty$.*

proof: From (9), $Q_f(t+T) \geq \int_{\tau=t+T}^{t+T+s} \Phi_{A-B_2H}^T(t+T, \tau)$ $[Q + H^T\ H + Q_f\ B_\gamma\ B_\gamma^T\ Q_f]\ \Phi_{A-B_2H}(t+T, \tau)\ d\tau$ $+ \Phi_{A-B_2H}(t+T+s, t+T)\ Q_f(t+T)\ \Phi_{A-B_2H}^T(t+T+s, t+T)$. Therefore, $Q_f(t+T) \geq \alpha_8 I$ for $s \geq \delta_o$ if $Q_f(t+T)$ satisfies (9) and the pair $(A - B_2H, H)$ is uniformly observable. $\frac{\partial K(t,\sigma)}{\partial \sigma}|_{\sigma=t+T} \leq 0$ follows from Theorem 1. $\square$

For any $Q(t) \geq 0$, $Q_f(t+T) \geq \alpha_8 I$ if $Q_f(t+T)$ satisfies (9) and the pair $(A - B_2H, H)$ is uniformly observable. In this case, $Q(t)$ can be zero. This case is different from Corollary 2 since uniform observability in Corollary 2 is not satisfied when $Q(t)$ is zero.

By using the method of [7], we can show that the stabilizing receding horizon H_∞ controls guarantee the H_∞ norm bound of the closed-loop system.

It is obvious that the whole results obtained here apply to the case of time-invariant systems. In this case, the controllability and observability can be replaced by the stabilizability as in Proposition 3.1 of [1] and by detectability as in Remark 8 of [5], respectively.

4 Conclusion

In this paper, new matrix inequality conditions on terminal weighting matrices are proposed which guarantee the monotonicity of the saddle point value for linear continuous time-varying systems. Under these conditions, it is shown that the closed-loop stability of the receding horizon H_∞ control (RHHC) is guaranteed for continuous time-varying systems. Compared to the existing requirements available in the literature, this paper presents the weakest conditions, still guaranteeing the closed-loop stability of the RHHC for continuous time-varying systems.

The proposed terminal inequality conditions extend the existing class of terminal weighting matrices in [6], [7]. Thus, previous results for the stability of the RHHC [6], [7] are obtained as special cases of the results in this paper. The monotonicity of the saddle point value obtained in this paper will be useful for the closed-loop stability of constrained, delayed linear systems, and/or nonlinear systems with the RHHC.

References

[1] W. H. Kwon and A. E. Pearson, “A modified quadratic cost problem and feedback stabilization of a linear system,” *IEEE Trans. Automat. Contr.*, vol. 22, no. 5, pp. 838 – 842, 1977.

[2] J. B. Rawlings and K. R. Muske, “The stability of constrained receding horizon control,” *IEEE Trans. Automat. Contr.*, vol. 38, no. 10, pp. 1512 – 1516, 1993.

[3] H. Chen and F. Allgower, “A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability,” *Automatica*, vol. 34, pp. 1205–1217, 1998.

[4] W. H. Kwon, A. M. Bruckstein, and T. Kailath, “Stabilizing state-feedback design via the moving horizon method,” *Int. J. Contr.*, vol. 37, pp. 631 – 643, 1983.

[5] W. H. Kwon and K. B. Kim, “On stabilizing receding horizon controls for linear continuous time-invariant systems,” *IEEE Trans. Automat. Contr.*, vol. 45, no. 8, 2000.

[6] G. Tadmor, “Receding horizon revisited: An easy way to robustly stabilize an LTV system,” *Systems and Control Letters*, vol. 18, pp. 285 – 294, 1992.

[7] S. Lall and K. Glover, “A game theoretic approach to moving horizon control,” *Advances in Model-Based Predictive Control*, pp. 131 – 144, 1994.

[8] J. W. Lee, W. H. Kwon, and J. H. Lee, “Receding horizon H_∞ tracking control for time-varying discrete linear systems,” *Int. J. Contr.*, vol. 68, pp. 385–399, 1999.

[9] T. Basar and P. Bernhard, *H_∞ - Optimal Control and Related Minimax Design Problems : A Dynamic Game Approach.* Boston: Birkhauser, 1995.

[10] B. D. O. Anderson and J. B. Moore, “New results in linear system stability,” *SIAM Journal on Control and Optimization*, vol. 7, pp. 398 – 414, 1969.