The category of affine connection control systems

Andrew D. Lewis
Department of Mathematics and Statistics
Queen's University
Kingston, ON K7L 3N6, Canada
andrew@mast.queensu.ca

Abstract

The category of affine connection control systems is one whose objects are control systems whose drift vector field is the geodesic spray of an affine connection, and whose control vector fields are vertical lifts to the tangent bundle of vector fields on configuration space. We initiate an investigation of morphisms (feedback transformations) in this category, including the study of subsystems and factor systems.

1 Introduction

It is apparent that the study of what we will in this paper call "affine connection control systems" has a significant rôle to play in the field of mechanical control systems. In a series of papers, e.g., [11, 12, 10, 4], the author and various coauthors have shown how the affine connection framework is useful in looking at mechanical systems whose Lagrangian is the kinetic energy with respect to a Riemannian metric, possibly in the presence of constraints linear in velocity e.g., [10, 9]. In such an investigation, there appears to be no particular advantage to work with affine connections which come from physics, i.e., from the Riemannian metric and the constraints. Therefore, in this paper we deal with general affine connections.

The emphasis here is to lay a groundwork for the investigation of ways in which one can simplify or alter affine connection control systems by using feedback transformations. That simplification of these systems is important can be seen in the work of the author [9] where even simple physical systems yield rather complicated expressions for the system's affine connection. Apart from the matter of simplification, one might also wish to use feedback to change the system into one whose characteristics are more desirable. The idea of restricting the types of feedback so that one remains in a certain class of systems is not new. Bloch, et al., [1] retain the Hamiltonian structure of their system through feedback, and in work initiated in [2] (see also [6]), the emphasis is on retaining the Riemannian structure through "kinetic shaping." Our focus in this work is on equivalence which maintains the affine connection structure. For general control affine systems, the issues we address here are reviewed in the paper of Elkin [5].

Besides introducing the basic notion of equivalence, we also look at how one may investigate subsystems and factor systems. In the former case, one wishes to determine when the dynamics of a given affine connection control systems are "contained in" the controlled dynamics of another. For factor systems, one wishes to project the dynamics of an affine connection control system onto another affine connection control system. Scenarios such as this arise, for example, when one is dealing with systems with symmetry and can perform a reduction of some sort.

The matters we address in this paper are technically challenging ones in practice. For example, the matter of equivalence typically produces a set of overdetermined nonlinear partial differential equations [6] which one must solve. However, we hope that by illuminating some of the special structure in the class of affine connection control systems, we can point the way for certain profitable lines of investigation. An unfortunate byproduct of publishing a paper which is technical in nature in such a restricted length format is that one cannot, without seriously impairing the paper's scientific content, include examples which illustrate the theory. Indeed, to get the paper within page limits, many interesting existing results have also been omitted. In a subsequent paper of greater length, examples will be provided which might better illustrate the concepts we discuss here, and further results will be proved.

2 Relevant affine differential geometry

We refer the reader to [7] for details of our following brief discussion of affine differential geometry. The intent here is mainly to introduce notation. The reader is expected to be familiar with basic concepts related to affine connections.

Let Q be a finite-dimensional manifold and let ∇ be an affine connection on Q. The curves for which $\nabla_{c'(t)}c'(t) = 0$ are called ***geodesics***. The geodesic equations are second-order and so give rise to a second-order vector field on TQ which we denote by Z and which is called the ***geodesic spray***. An affine connection ∇ is ***complete*** if its corresponding geodesic spray is a complete vector field.

A mapping $\phi\colon Q \to \tilde{Q}$ is ***totally geodesic*** if

$$T_q\phi(\nabla_X X)_q = (\tilde{\nabla}_{\tilde{X}}\tilde{X})_{\phi(q)}$$

where $\tilde{X}$ is a vector field on $\tilde{Q}$ which is ϕ-related to X. Clearly a totally geodesic mapping has the property that it maps geodesics of ∇ to geodesics of $\tilde{\nabla}$. The converse is also true.

The interaction of submanifolds and distributions with affine connections will arise when we talk about restricting affine connection control systems. Let us introduce here the necessary terminology. We let Q be a manifold with an affine connection ∇. A submanifold $N \subset Q$ is ***totally geodesic*** if for a geodesic $c\colon I \to Q$, $c'(t_0) \in T_{c(t_0)}N$ for some $t_0 \in I$ implies that $c'(t) \in T_{c(t)}N$ for every $t \in I$. Thus a submanifold N is totally geodesic if geodesics which start tangent to N remain tangent to N. In like manner, an integrable distribution D is ***totally geodesic*** if for a geodesic $c\colon I \to Q$, $c'(t_0) \in D_{c(t_0)}$ for some $t_0 \in I$ implies that $c'(t) \in D_{c(t)}$ for every $t \in I$. Lewis [10] proposes the related, but weaker notion of a ***geodesically invariant*** distribution, whose definition reads just like that for a totally geodesic distribution, but the condition on integrability is not present. Lewis proves the following result.

2.1 Proposition: *A distribution D is geodesically invariant under an affine connection ∇ if and only if it is closed under the* ***symmetric product*** *which is defined by $\langle X : Y\rangle = \nabla_X Y + \nabla_Y X$.*

Our final preliminary matter to deal with concerns what we will have to do to "factor" affine connection control systems. Here we consider affine connections on a manifold Q in the presence of a totally geodesic surjective submersion $\phi\colon Q \to \tilde{Q}$. Although this subject has been one of considerable research energy when ∇ is the Levi-Civita connection associated with a Riemannian metric e.g., [14], the situation for an arbitrary connection is not well studied. Nevertheless, we have the following result which gives a property of affine connections in the current context.

2.2 Proposition: *Let ∇ and $\tilde{\nabla}$ be affine connections on manifolds Q and $\tilde{Q}$. If $\phi\colon Q \to \tilde{Q}$ is a totally geodesic surjective submersion, then each of the submanifolds $\phi^{-1}(\tilde{q})$, $\tilde{q} \in \tilde{Q}$, is totally geodesic.*

Proof: If ϕ is totally geodesic, the geodesic of ∇ with initial condition $v_q \in TQ$ is mapped to the geodesic of $\tilde{\nabla}$ with initial condition $T_q\phi(v_q)$. In particular, if $T_q\phi(v_q) = 0$, the geodesic with initial condition v_q is mapped to the trivial geodesic fixing the point $\tilde{q} = \phi(q) \in \tilde{Q}$. But this is precisely the statement that geodesics of ∇ with initial velocities tangent to the submanifold $\phi^{-1}(\tilde{q})$ will evolve in that same submanifold. ■

Finally, we say that an affine connection ∇ on Q is ***geodesically ϕ-projectable*** if for geodesics $c_1, c_2\colon I \to Q$ with initial conditions $v_1 = c_1'(0)$ and $v_2 = c_2'(0)$, the condition $T_{c_1(0)}\phi(v_1) = T_{c_2(0)}\phi(v_2)$ implies that $\phi \circ c_1 = \phi \circ c_2$. One may verify that the projected geodesics for ∇ are then the geodesics of an affine connection $\tilde{\nabla}$ on $\tilde{Q}$, and $\tilde{\nabla}$, if specified to have zero torsion, is uniquely defined. Furthermore, with $\tilde{\nabla}$ so defined, the mapping ϕ is a totally geodesic mapping from ∇ to $\tilde{\nabla}$.

3 The category of affine connection control systems

Now we can properly discuss the actual subject of the paper. What we consider in this section is a special class of control affine systems. We use category theoretic language as it effectively organises what we wish to say. Let us begin with a discussion of the objects in our category, and note that it is precisely the systems described here which form the basis for the work of the author and coauthors on "simple mechanical control systems."

3.1 Objects in ACCS

We shall denote by ACCS the ***category of affine connection control systems***. An ***object*** in this category is a triple $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ where Q is a finite-dimensional manifold, ∇ is an affine connection on Q, and $\mathscr{Y} = \{Y_1, \ldots, Y_m\}$ is a collection of vector fields on Q.

To an affine connection control system $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ we associate a control system given by

$$\nabla_{c'(t)} c'(t) = u^a(t) Y_a(c(t)). \tag{1}$$

A ***controlled trajectory*** for $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ is a pair (c, u) with $c\colon I \to Q$ having the property that its derivative $t \mapsto c'(t)$ is an absolutely continuous curve on TQ, and $u\colon I \to \mathbb{R}^m$ is a measurable essentially bounded control such that together c and u satisfy (1). If $U \subset Q$ is an open submanifold, we may define the ***restricted object*** $\Sigma_{\mathrm{aff}}|U = (U, \nabla|U, \mathscr{Y}|U)$.

3.2 Morphisms in ACCS

Now let us look at morphisms in the category ACCS. Thus we need to specify a way to send an affine connection control system to another affine connection control system. We let $\Sigma_2(TQ)$ denote the bundle of symmetric $(0,2)$ tensors on Q, and we denote by $\mathbb{R}^m_Q$ the trivial vector bundle $Q \times \mathbb{R}^m$ over Q. If $S \in \mathbb{R}^m_Q \otimes \Sigma_2(TQ)$ then for $a = 1, \ldots, m$ we define $S^a \in \Sigma_2(TQ)$ by $S^a(X,Y) = S(e_a \otimes (X,Y))$ where e_a is the ath standard basis vector for $\mathbb{R}^m$. If U and V are $\mathbb{R}$-vector spaces, $L(U;V)$ denotes the set of linear maps from U to V.

We consider affine connection control systems denoted $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ with $\mathscr{Y} = \{Y_1, \ldots, Y_m\}$ and $\tilde{\mathscr{Y}} = \{\tilde{Y}_1, \ldots, \tilde{Y}_{\tilde{m}}\}$. An **ACCS *morphism*** sending Σ_{aff} to $\tilde{\Sigma}_{\mathrm{aff}}$ is a triple (ϕ, S, Λ) with the following properties:

1. $\phi\colon Q \to \tilde{Q}$ is a smooth mapping;
2. S is a smooth section of $\mathbb{R}^{\tilde{m}}_Q \otimes \Sigma_2(TQ)$ and $\Lambda\colon Q \to L(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ is a smooth map which together satisfy the following conditions:
   (a) $T_q\phi(Y_a(q)) = \Lambda^\alpha_a(q) \tilde{Y}_\alpha(\phi(q))$;
   (b) $T_q\phi(\nabla_X X)_q = (\tilde{\nabla}_{\tilde{X}} \tilde{X})_{\phi(q)} + S^\alpha(X(q), X(q)) \tilde{Y}_\alpha(\phi(q))$ where $\tilde{X}$ is a vector field on $\tilde{Q}$ which is ϕ-related to the vector field X on Q.

The ***identity morphism*** which sends $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ to itself is defined by $(\mathrm{id}_Q, 0, q \mapsto \mathrm{id}_{\mathbb{R}^m})$. If

$$T_q\phi(\mathrm{span}\{Y_1(q), \ldots, Y_m(q)\}) = \mathrm{span}\{\tilde{Y}_1(\phi(q)), \ldots, \tilde{Y}_{\tilde{m}}(\phi(q))\}$$

for all $q \in Q$, the ACCS morphism (ϕ, S, Λ) is called ***complete***. An **ACCS *isomorphism*** is a morphism (ϕ, S, Λ) for which ϕ is a diffeomorphism.

Let us now give an essential property for ACCS morphisms. We introduce the notation $c_\phi = \phi \circ c$ where c is a curve on Q, and $\phi\colon Q \to \tilde{Q}$.

3.1 Proposition: *If (ϕ, S, Λ) is an ACCS morphism sending $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ to $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ and if (c, u) is a controlled trajectory for Σ_{aff}, then $(c_\phi, \tilde{u})$ is a controlled trajectory for $\tilde{\Sigma}_{\text{aff}}$ where $\tilde{u}(t) = \Lambda(c(t))u(t) - S^\alpha(c'(t), c'(t))\tilde{Y}_\alpha(c_\phi(t))$.*

Conversely, suppose that $\phi\colon Q \to \tilde{Q}$ is a smooth mapping with the property that if (c, u) is a controlled trajectory for Σ_{aff}, then there exists an admissible input $\tilde{u}$ for $\tilde{\Sigma}_{\text{aff}}$ so that $(c_\phi, \tilde{u})$ is a controlled trajectory for $\tilde{\Sigma}_{\text{aff}}$. Then there exists a smooth section S of $\mathbb{R}^{\tilde{m}}_Q \otimes \Sigma_2(TQ)$ and a smooth mapping $\Lambda\colon Q \to L(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ so that (ϕ, S, Λ) is an ACCS morphism sending Σ_{aff} to $\tilde{\Sigma}_{\text{aff}}$.

Proof: We are given $\nabla_{c'(t)}c'(t) = u^a(t)Y_a(c(t))$. Using the properties of ACCS morphisms we compute

$$T_{c(t)}\phi\big(\nabla_{c'(t)}c'(t) - u^a(t)Y_a(c(t))\big) = \tilde{\nabla}_{c'_\phi(t)}c'_\phi(t) - \tilde{u}^\alpha \tilde{Y}_\alpha(c_\phi(t))$$

where $\tilde{u}(t) = \Lambda(c(t))u(t) - S^\alpha(c'(t), c'(t))\tilde{Y}_\alpha(c_\phi(t))$, as desired.

For the converse, we first look at the case of the zero input for Σ_{aff}. In this case, given $v_q \in TQ$, we let c be the geodesic which passes through v_q at $t = 0$. Our hypotheses guarantee the existence of an admissible input $\tilde{u}_0$ for $\tilde{\Sigma}_{\text{aff}}$ so that

$$\tilde{\nabla}_{c'_\phi(t)}c'_\phi(t) = \tilde{u}_0^\alpha(t)\tilde{Y}_\alpha(c_\phi(t)).$$

Since c is a geodesic, we may write

$$\tilde{\nabla}_{c'_\phi(t)}c'_\phi(t) - T_{c(t)}\phi(\nabla_{c'(t)}c'(t)) = \tilde{u}_0^\alpha(t)\tilde{Y}_\alpha(c_\phi(t)). \tag{2}$$

The expression on the left-hand side is bilinear in $(c'(t), c'(t))$, being essentially the difference of two covariant derivatives. Therefore, since the right-hand side is a vector field along c_ϕ taking values in $\text{span}_{C^\infty(\tilde{Q})}\{\tilde{Y}_1, \ldots, \tilde{Y}_{\tilde{m}}\}$, there exists a tensor $S_{c(t)} \in \mathbb{R}^{\tilde{m}} \otimes \Sigma_2(T_{c(t)}Q)$ so that

$$S^\alpha_{c(t)}(c'(t), c'(t))\tilde{Y}_\alpha(c_\phi(t)) = \tilde{u}_0^\alpha(t)\tilde{Y}_\alpha(c_\phi(t)).$$

In particular, at $t = 0$ we have

$$S^\alpha_q(v_q, v_q)\tilde{Y}_\alpha(\phi(q)) = \tilde{u}_0^\alpha(0)\tilde{Y}_\alpha(\phi(q)).$$

Since the terms in (2) vary smoothly as we vary v_q, the mapping $q \mapsto S_q$ defines a smooth section of $\mathbb{R}^{\tilde{m}}_Q \otimes \Sigma_2(TQ)$. Next we look at the situation when the input for Σ_{aff} is the constant input $u(t) = e_a$. In this case we have a curve c through $v_q \in TQ$ satisfying $\nabla_{c'(t)}c'(t) = Y_a(c(t))$. Our hypotheses assert the existence of an admissible input $\tilde{u}_a$ for $\tilde{\Sigma}_{\text{aff}}$ so that

$$\begin{aligned}
&\tilde{\nabla}_{c'_\phi(t)}c'_\phi(t) = \tilde{u}_a^\alpha(t)\tilde{Y}_\alpha(c_\phi(t)) \\
\implies\quad &S^\alpha(c'(t), c'(t))\tilde{Y}_\alpha(c_\phi(t)) + T_{c(t)}\phi(Y_a(c(t))) = \\
&\tilde{u}_a^\alpha(t)\tilde{Y}_\alpha(c_\phi(t)).
\end{aligned}$$

Evaluating this at $t = 0$ gives

$$S^\alpha(v_q, v_q)\tilde{Y}_\alpha(\phi(q)) + T_q\phi(Y_a(q)) = \tilde{u}_a^\alpha(0)\tilde{Y}_\alpha(\phi(q)).$$

Since v_q can be selected as an arbitrary vector in T_qQ, this implies that $\tilde{u}_a(0)$ is a sum of two components, one which is bilinear in (v_q, v_q) (let us denote this by $\tilde{v}_a(v_q)$) and another which is independent of the velocity v_q, and only depends on the configuration q (let us denote this by $\tilde{w}_a(q)$). The bilinear component must then be $\tilde{v}_a(v_q) = S^\alpha(v_q, v_q)\tilde{Y}_\alpha(\phi(q))$, leaving the term independent of velocity to satisfy

$$T_q\phi(Y_a(q)) = \tilde{w}_a^\alpha(q)\tilde{Y}_\alpha(\phi(q)).$$

Let us define $\Lambda(q) \in L(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ by $\Lambda_a^\alpha(q) = \tilde{w}_a^\alpha(q)$. As usual, we may choose Λ so that $q \mapsto \Lambda(q)$ is smooth. With the S and Λ we have defined, one then easily verifies that (ϕ, S, Λ) is an ACCS morphism sending Σ_{aff} to $\tilde{\Sigma}_{\text{aff}}$. ■

As we mentioned at the beginning of this section, affine connection control systems can be regarded as control affine systems, and the corresponding first-order equations evolve on the tangent bundle TQ. Elkin [5] studies morphisms of control affine systems. It may be shown that there are control affine morphisms but which are *not* ACCS morphisms. Nevertheless, Proposition 3.1 suggests that ACCS morphisms are the natural ones to study as they arise from maps between configuration spaces.

3.3 Compositions and special classes of ACCS morphisms

One often wishes to write an ACCS morphism as a product of two simpler ACCS morphisms. The ***product*** of ACCS morphisms (ϕ_1, S_1, Λ_1) from $\Sigma_{\text{aff},1} = (Q^1, \nabla^1, \mathscr{Y}^1)$ to $\Sigma_{\text{aff},2} = (Q^2, \nabla^2, \mathscr{Y}^2)$ and (ϕ_2, S_2, Λ_2) from $\Sigma_{\text{aff},2}$ to $\Sigma_{\text{aff},3} = (Q^3, \nabla^3, \mathscr{Y}^3)$ is given by

1. $\phi_{21} = \phi_2 \circ \phi_1$,
2. $S^\sigma_{21}(X(q), Y(q)) = S^\sigma_2(T_q\phi_1(X(q)), T_q\phi_1(Y(q))) + S^\alpha_1(X(q), Y(q))(\Lambda_2)^\sigma_\alpha(\phi_1(q))$, and
3. $(\Lambda_{21})^\sigma_a(q) = (\Lambda_1)^\sigma_\alpha(q)(\Lambda_2)^\sigma_\alpha(\phi_1(q))$.

Now let us define the special classes of ACCS morphisms one may consider. An ACCS morphism (ϕ, S, Λ) which maps $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ to $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ is a ***morphism over controls*** if $Q \subset \tilde{Q}$ and if $\phi\colon Q \to \tilde{Q}$ is the inclusion map. The category whose objects are affine connection control systems and whose morphisms are ACCS morphisms which are morphisms over controls we denote by CACCS. The idea is that a morphism over controls does essentially nothing to the system's states, and alters only the controls, and these in an algebraic manner.

An ACCS morphism (ϕ, S, Λ) is a ***morphism over configurations*** if $S_q = 0$ and $\Lambda(q) = \text{id}_{\mathbb{R}^m}$ for each $q \in Q$. We denote by QACCS the category whose objects are affine connection control systems and whose morphisms are ACCS morphisms which are morphisms over configurations. The idea here is that one leaves the controls alone, and alters only the configuration spaces. The following result is clear.

3.2 Proposition: *A triple (ϕ, S, Λ) is a QACCS morphism mapping $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ to $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ if and only if the following two conditions hold:*

(i) $\phi\colon Q \to \tilde{Q}$ is a totally geodesic mapping between ∇ and $\tilde{\nabla}$;

(ii) each control vector field $\tilde{Y}_a$ on $\tilde{Q}$ is ϕ-related to the control vector field Y_a on Q.

3.3 Remark: Often one wishes to decompose a given ACCS morphism into the product of a CACCS morphism and a QACCS morphism. In this way, one can isolate the "hard" part of the problem which typically involves the coordinate changes involved with a QACCS morphism. We shall see in the remaining two sections how this is done for special classes of ACCS morphisms. □

4 Restricted systems

Let us now turn to the question of describing ACCS morphisms (ϕ, S, Λ) for which the map ϕ has certain properties. We begin with a description of the situation when the dynamics of one affine connection control system are "contained in" the dynamics of another.

An affine connection control system $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ is a ***subsystem*** in the category ACCS of another affine connection control system $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ if there is an ACCS morphism (ϕ, S, Λ) for which $\phi\colon Q \to \tilde{Q}$ is an embedding. We say the affine connection control system $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ ***restricts*** in the category ACCS to a submanifold $N \subset \tilde{Q}$ if there exists an affine connection control system $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ which is an ACCS subsystem of $\tilde{\Sigma}_{\text{aff}}$ and where $N = \text{image}(\phi)$. The idea is that the controlled dynamics of a subsystem can be contained in that of the full system.

In the category QACCS, subsystems have a very particular structure.

4.1 Proposition: *An affine connection control system $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ restricts in the category QACCS to a submanifold $N \subset \tilde{Q}$ if and only if the following two conditions are satisfied:*

(i) N is a totally geodesic submanifold of $\tilde{Q}$;

(ii) the vector fields in $\tilde{\mathscr{Y}}$ are all tangent to N.

Proof: First suppose that $\tilde{\Sigma}_{\text{aff}}$ restricts in QACCS to a subsystem Σ_{aff} via the QACCS morphism $(\phi, 0, q \mapsto \text{id}_{\mathbb{R}^m})$. Then all geodesics of ∇ must be mapped to geodesics of $\tilde{\nabla}$ by ϕ. Since ϕ is a diffeomorphism onto its image, this implies that all geodesics of $\tilde{\nabla}$ which are somewhere tangent to N are everywhere tangent to N. Thus (i) holds. By the definition of a QACCS morphism we must also have $\tilde{Y}_a(\tilde{q}) = T_{\phi^{-1}(\tilde{q})}\phi(Y_a(\phi^{-1}(\tilde{q})))$, from which follows (ii).

For the converse, suppose that $\tilde{\Sigma}_{\text{aff}}$ satisfies (i) and (ii). In this case we can define $Q = N$ and we note that (i) and (ii) imply that $\nabla = \tilde{\nabla}|Q$ and $\mathscr{Y} = \tilde{\mathscr{Y}}|N$ are well-defined. Then it is easy to see that if one takes $\phi\colon N \to \tilde{Q}$ to be inclusion, the QACCS morphism $(\phi, 0, q \mapsto \text{id}_{\mathbb{R}^m})$ renders $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ a subsystem of $\tilde{\Sigma}_{\text{aff}}$. ■

Note that if an affine connection control system $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ admits a restriction to $N \subset \tilde{Q}$ in either of the categories ACCS or CACCS, this does not imply that the control system $\tilde{\Sigma}_{\text{aff}}$ leaves invariant the submanifold $\text{image}(\phi) \subset \tilde{Q}$. Indeed, it is easy to construct examples of systems $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ which possess ACCS or QACCS subsystems on a submanifold of $N \subset \tilde{Q}$, but where the dynamics of the affine connection control system $\tilde{\Sigma}_{\text{aff}}$ do not leave N invariant. Thus we introduce the notion of invariance. For an affine connection control system $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$, a submanifold $N \subset \tilde{Q}$ is ***invariant*** if the properties (i) and (ii) of Proposition 4.1 are satisfied.

The following result indicates that the term "invariant" is justified as we have used it.

4.2 Proposition: *A manifold N is an invariant manifold for $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ if and only if any controlled trajectory (c, u) for $\tilde{\Sigma}_{\text{aff}}$ which has the property that $c'(t_0) \in T_{c(t_0)}N$ for some t_0 on the domain of definition of c also has the property that $c'(t) \in T_{c(t)}N$ for every t in the domain of definition of c.*

Proof: If (i) holds, then this implies that $TN \subset TQ$ is an invariant manifold for the geodesic spray for $\tilde{\nabla}$. Condition (ii) on the other hand implies that the vertical lift, $\text{verlift}(\tilde{Y}_\alpha)$, of $\tilde{Y}_\alpha$ is tangent to TN. Thus TN is an invariant manifold for the first-order control affine system on TQ associated with $\tilde{\Sigma}_{\text{aff}}$. This means that N is invariant under all curves which are projections from TN to N of controlled trajectories for $\tilde{\Sigma}|TN$. However, these projected curves are precisely the controlled trajectories for $\tilde{\Sigma}_{\text{aff}}$ whose initial conditions are tangent to N. Thus we have shown that any controlled trajectory of $\tilde{\Sigma}_{\text{aff}}$ which starts tangent to N remains tangent to N.

Now suppose that every controlled trajectory which starts tangent to N remains tangent to N. In particular, every geodesic of $\tilde{\nabla}$ which starts tangent to N remains tangent to N. Thus N is totally geodesic. By Proposition 2.1 this implies that $\nabla_{c'(t)}c'(t) \in T_{c(t)}N$ for every curve c which is tangent to N. This also means that for $a = 1, \ldots, m$, $\tilde{\nabla}_{c'(t)}c'(t) - \tilde{Y}_\alpha(c(t)) \in T_{c(t)}N$ for any curve c which is tangent to N, which implies that $Y_a(c(t)) \in T_{c(t)}N$ for any curve c which is tangent to N. This means that for $a = 1, \ldots, m$ the vector field $\tilde{Y}_a$ is tangent to N, and so N is then an invariant manifold for $\tilde{\Sigma}_{\text{aff}}$. ■

Let us determine the manner in which we can factor morphisms which give rise to restrictions in ACCS.

4.3 Proposition: *Let $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ be affine connection control systems with Σ_{aff} a subsystem of $\tilde{\Sigma}_{\text{aff}}$ via the ACCS morphism (ϕ, S, Λ). Then there exists a neighbourhood $\tilde{U}$ of $N = \text{image}(\phi)$ in $\tilde{Q}$ and a CACCS subsystem $\Sigma'_{\text{aff}} = (Q', \nabla', \mathscr{Y}')$ of $\tilde{\Sigma}_{\text{aff}}|\tilde{U}$ with the property that there is a QACCS morphism $(\phi', 0, q \mapsto \text{id}_{\mathbb{R}^m})$ which renders Σ_{aff} a QACCS subsystem of Σ'_{aff}.*

Proof: If we let $\tilde{U}$ be a tubular neighbourhood of N, we may regard $\tilde{U}$ as an open subset of the zero section of a vector bundle $\pi\colon E \to N$. We make this identification, and write points in $\tilde{U}$ as $e_{\tilde{q}}$ for some $\tilde{q} \in N$. We also let $HE \subset TE$ be a linear connection on $\pi\colon E \to N$ which allows us to define a complement to $\ker(T_{e_{\tilde{q}}}\pi)$ for each $e_{\tilde{q}} \in E$. Recall that such connections always exist, and that they have the property that $HE|N = TN$ [8, §11.10]. We call vectors ***vertical*** which are in $\ker(T_{e_{\tilde{q}}}\pi)$ and ***horizontal*** which are in $H_{e_{\tilde{q}}}E$.

To prove the result, it suffices to find the following objects: (1) an affine connection ∇' on $\tilde{U}$; (2) a family $\mathscr{Y}' = \{Y'_1, \ldots, Y'_m\}$ of vector fields on $\tilde{U}$; (3) a smooth section $\tilde{S}$ of $\mathbb{R}^{\tilde{m}}_{\tilde{U}} \otimes \Sigma_2(T\tilde{U})$; (4) a smooth map $\tilde{\Lambda}\colon \tilde{U} \to L(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ with the properties: (5) $\Sigma'_{\rm aff} = (\tilde{U}, \nabla', \mathscr{Y}')$ is a CACCS subsystem of $\tilde{\Sigma}_{\rm aff}|\tilde{U}$ via the CACCS morphism $(\tilde{\phi}, \tilde{S}, \tilde{\Lambda})$ where $\tilde{\phi}$ is the inclusion of $\tilde{U}$ in $\tilde{Q}$; (6) N is an invariant manifold for $\Sigma'_{\rm aff}$. To this end, for $e_q \in \tilde{U}$ we define $\tilde{S}^\alpha_{e_{\tilde{q}}}$ to be $T^*_{e_{\tilde{q}}}\pi S^\alpha_{\phi^{-1}(\tilde{q})}$ on pairs of horizontal vectors, and zero otherwise. Since $\tilde{S}^\alpha$ is symmetric, this suffices to define it for general vectors. We also define $\tilde{\Lambda}(e_{\tilde{q}}) = \Lambda(\phi^{-1}(\tilde{q})$ and $Y'_a(e_{\tilde{q}}) = Y_a(\phi^{-1}(\tilde{q}))$. The affine connection ∇' we define by

$$(\nabla'_{\tilde{X}}\tilde{Y})_{e_{\tilde{q}}} = (\tilde{\nabla}_{\tilde{X}}\tilde{Y})_{e_{\tilde{q}}} + \tilde{S}^\alpha(\tilde{X}(e_{\tilde{q}}), \tilde{Y}(e_{\tilde{q}}))\tilde{Y}_\alpha(e_{\tilde{q}}).$$

(In writing this equation we are identifying points in U with their image in $\tilde{Q}$ under $\tilde{\phi}$.)

With these definitions, let us check that condition (5) is satisfied. Since $\tilde{\phi}$ is the inclusion of $\tilde{U}$ in $\tilde{Q}$, it is obvious that

$$T_{e_{\tilde{q}}}\tilde{\phi}(\nabla'_{\tilde{X}}\tilde{Y})_{e_{\tilde{q}}} = (\tilde{\nabla}_{\tilde{X}}\tilde{Y})_{e_{\tilde{q}}} + \tilde{S}^\alpha(\tilde{X}(e_{\tilde{q}}), \tilde{Y}(e_{\tilde{q}}))\tilde{Y}_\alpha(e_{\tilde{q}}).$$

We also can verify that

$$T_{e_{\tilde{q}}}\tilde{\phi}(Y'_a(e_{\tilde{q}})) = \tilde{\Lambda}^\alpha_a(e_{\tilde{q}})\tilde{Y}_\alpha(e_{\tilde{q}})$$

using the definition of $\tilde{\Lambda}$ and the fact that (ϕ, S, Λ) maps $\Sigma_{\rm aff}$ to $\tilde{\Sigma}_{\rm aff}$. Thus (5) holds.

Now we verify that condition (6) holds with the definitions we have given. By Propositions 2.1 and 4.1 we first need to show that the vector field $\nabla'_{\tilde{X}}\tilde{X}$ is tangent to N for any vector field $\tilde{X}$ which is tangent to N. This follows from the definition of ∇', the definition of $\tilde{S}$, and the fact that (ϕ, S, Λ) renders $\Sigma_{\rm aff}$ a subsystem of $\tilde{\Sigma}_{\rm aff}|\tilde{U}$. To complete the proof we note that the vector fields Y'_a, $a = 1, \ldots, m$, are tangent to N. ■

The idea here is that by a change of controls one arrives at the system $\Sigma'_{\rm aff}$ which possesses $\phi(Q)$ as an invariant manifold.

5 Factor systems

In the previous section we looked at how the controlled dynamics of an affine connection control system can be embedded into the dynamics of another affine connection control system. Now we project the controlled dynamics of an affine connection onto those of another. Scenarios such as this arise, for example, when talking about reduction of affine connection control system. This is something for which a completely satisfactory theory does not yet exist, but we refer to the work of Ostrowski [13] for a discussion of reduction for control systems with nonholonomic constraints.

If $\Sigma_{\rm aff} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\rm aff} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ are affine connection control systems, $\tilde{\Sigma}_{\rm aff}$ is a ***factor system*** of $\Sigma_{\rm aff}$ if there exists a complete ACCS morphism (ϕ, S, Λ) for which $\phi\colon Q \to \tilde{Q}$ is a surjective submersion. As usual, we may talk about ACCS, CACCS, or QACCS factor systems, depending on the character of the morphism (ϕ, S, Λ). We say that an affine connection control system $\Sigma_{\rm aff} = (Q, \nabla, \mathscr{Y})$ admits a ***factorisation to*** $\tilde{\rm Q}$ ***via*** ϕ in the category ACCS (resp. CACCS or QACCS) if there exists an ACCS (resp. CACCS or QACCS) factor system $\tilde{\Sigma}_{\rm aff} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ of $\Sigma_{\rm aff}$ with a morphism of the form (ϕ, S, Λ).

We have the usual terminology associated with a surjective submersion. That is, a vector $v_q \in T_qQ$ is ***vertical*** if $T_q\phi(v_q) = 0$. We denote the subbundle of vertical vectors by VQ. The set $\phi^{-1}(\tilde{q})$ is called the ***fibre*** over $\tilde{q} \in \tilde{Q}$. A vector field X on Q is ϕ-***projectable*** if $\phi(q_1) = \phi(q_2)$ implies that $T_{q_1}\phi(X(q_1)) = T_{q_2}\phi(X(q_2))$.

Let us begin our discussion of factor systems by indicating that ACCS morphisms which factor can indeed be thought of as epimorphisms in ACCS. The following result relies on a result of Blumenthal [3] which states that a surjective submersion $\phi\colon Q \to \tilde{Q}$ has the path lifting property provided that Q and $\tilde{Q}$ are connected and that Q possesses a complete affine connection.

5.1 Proposition: *Let $\Sigma_{\rm aff} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\rm aff} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ be affine connection control systems with Q and $\tilde{Q}$ connected and ∇ complete. If (ϕ, S, Λ) is an ACCS morphism which makes $\tilde{\Sigma}_{\rm aff}$ a factor system of $\Sigma_{\rm aff}$, then for every controlled trajectory $(\tilde{c}, \tilde{u})$ for $\tilde{\Sigma}_{\rm aff}$ there exists a controlled trajectory (c, u) for $\Sigma_{\rm aff}$ so that $\tilde{c} = c_\phi$.*

Proof: Let $(\tilde{c}, \tilde{u})$ be a controlled trajectory for $\tilde{\Sigma}_{\rm aff}$ defined on $I \subset \mathbb{R}$, and let $\bar{c}$ be a lift of $\tilde{c}$—thus c is a curve with the property that $\tilde{c} = \bar{c}_\phi$. Since (ϕ, S, Λ) is complete, we can define a bounded, essentially measurable map $u\colon I \to \mathbb{R}^m$ with the property

$$u^a(t)T_q\phi(Y_a(\bar{c}(t))) = \tilde{u}^\alpha\tilde{Y}_\alpha(\tilde{c}(t)),$$

for all $t \in \mathbb{R}$. To obtain a controlled trajectory (c, u) for $\Sigma_{\rm aff}$ with the property that $\tilde{c} = c_\phi$, we use the time-dependent second-order differential equation

$$\nabla_{c'(t)}c'(t) = u^a(t)Y_a(c(t))$$

with initial condition $c'(0) = \bar{c}'(0)$ to derive the motion of the system in the direction of the fibres of ϕ. Since the time-dependence is through u *and* $\bar{c}$, it follows that the curve c so obtained will have the property that $\tilde{c} = c_\phi$. ■

Let us begin by providing a description of QACCS factor systems.

5.2 Proposition: *An affine connection control system $\Sigma_{\rm aff} = (Q, \nabla, \mathscr{Y})$ admits a QACCS factorisation if and only if there exists a manifold $\tilde{Q}$ and a surjective submersion $\phi\colon Q \to \tilde{Q}$ so that the two conditions*

(i) ∇ is geodesically ϕ-projectable and

(ii) the vector fields in $\mathscr{Y}$ are ϕ-projectable

are satisfied.

Proof: Suppose that $\tilde{\Sigma}_{\rm aff}$ is a QACCS factor system of $\Sigma_{\rm aff}$ via $(\phi, 0, q \mapsto {\rm id}_{\mathbb{R}^m})$. Since ϕ maps geodesics of ∇ to geodesics of $\tilde{\nabla}$, it is true that $T_{c(t)}\phi(\nabla_{c'(t)}c'(t)) = \tilde{\nabla}_{c'_\phi(t)}c'_\phi(t)$ for any curve c on Q. Thus ∇ is geodesically ϕ-projectable. Also, if $\tilde{\Sigma}_{\rm aff}$ is a QACCS factor system of $\Sigma_{\rm aff}$, this implies that $T_q\phi(Y_a(q)) = \tilde{Y}_a(\phi(q))$, $a = 1, \ldots, m$, for all $q \in Q$. This clearly implies that Y_a is ϕ-projectable for $a = 1, \ldots, m$.

For the converse, suppose that we have $\tilde{Q}$ and a surjective submersion $\phi\colon Q \to \tilde{Q}$ so that (i) and (ii) hold. Since ϕ-projectability of the vector fields Y_a implies that $T_q\phi(Y_a(q)) = \tilde{Y}_a(\phi(q))$, $a = 1, \ldots, m$, for all $q \in Q$, we need only show that ϕ maps geodesics of ∇ to geodesics of some affine connection $\tilde{\nabla}$ on $\tilde{Q}$. However, this follows directly from the fact that ∇ is geodesically ϕ-projectable, and that the projected geodesics of ∇ are geodesics of some affine connection. ■

Let us investigate the manner in which we can decompose morphisms which give rise to factor objects in the category ACCS. As was the case with our factorisation result for subsystems, the result here is local.

5.3 Proposition: *Let $\phi\colon Q \to \tilde{Q}$ be a surjective submersion, and let $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ be affine connection control systems with $\tilde{\Sigma}_{\text{aff}}$ a factor system of Σ_{aff} via an ACCS morphism (ϕ, S, Λ). Then for each $q \in Q$ there exists a neighbourhood U of q and a CACCS factor system $\Sigma'_{\text{aff}} = (Q, \nabla', \mathscr{Y}')$ of $\Sigma_{\text{aff}}|U$ with the property that there is a QACCS morphism $(\tilde{\phi}, 0, q \mapsto \text{id}_{\mathbb{R}^{\tilde{m}}})$ which makes $\tilde{\Sigma}_{\text{aff}}$ a QACCS factor system of Σ'_{aff}.*

Proof: Since the morphism (ϕ, S, Λ) is complete, in a neighbourhood U of each $q_0 \in Q$ it is possible to define a map $\Theta\colon U \to L(\mathbb{R}^{\tilde{m}}; \mathbb{R}^m)$ with the property that

$$\Theta^a_\alpha(q)\Lambda^\beta_a(q) = \delta^\beta_\alpha, \quad q \in U.$$

So for $q_0 \in Q$, let U be such a neighbourhood. On U define an affine connection ∇' by

$$(\nabla'_X Y)_q = (\nabla_X Y)_q - S^\alpha(X(q), Y(q))\Theta^a_\alpha(q)Y_a(q),$$

and define a family of vector fields $\mathscr{Y}' = \{Y'_1, \dots, Y'_{\tilde{m}}\}$ on U by

$$Y'_\alpha(q) = \Theta^a_\alpha(q)Y_a(q).$$

Let $\tilde{\phi} = \phi|U$. One then verifies that

$$T_q\tilde{\phi}(\nabla'_X X)_q = (\tilde{\nabla}_{\tilde{X}}\tilde{X})_q, \quad q \in U,$$

where $\tilde{X}$ is $\tilde{\phi}$-related to the $\tilde{\phi}$-projectable vector field X. In particular, it follows from this that ∇' is geodesically $\tilde{\phi}$-projectable. We also note that

$$T_q\tilde{\phi}(Y'_\alpha(q)) = \tilde{Y}_\alpha(\tilde{\phi}(q)), \quad q \in U.$$

Therefore the vector fields from $\mathscr{Y}'$ are $\tilde{\phi}$-projectable which shows, by Proposition 5.2 that $\tilde{\Sigma}_{\text{aff}}$ is a QACCS factor system of $\Sigma_{\text{aff}}|U$. ■

Note that the idea here is quite similar to the decomposition we saw for subsystems.

6 Closing remarks

In this paper we have given a brief glimpse at some of the basic results one can obtain when thinking about the "category" of affine connection control systems. There is a great deal of work to do in this area, and we can only scratch the surface here. Examples of problems which one might study include the characterisation of "normal forms" in the category of affine connection control systems. Thus, given a system with m inputs on an n-dimensional configuration manifold, is it possible to classify the possible type of control systems one can encounter. One might also wish to attempt a simplification of systems based upon their possession of certain types of subsystems or factor systems.

In any case, there is clearly a great deal of room for further work in this area, and we hope here to have provided an organising framework from which to launch such work.

Acknowledgements

The author has enjoyed the financial support of a research grant from the Natural Sciences and Engineering Research Council of Canada.

References

[1] Anthony M. Bloch, P. S. Krishnaprasad, Jerrold E. Marsden, and G. Sánchez de Alvarez. Stabilization of rigid body dynamics by internal and external torques. *Automatica—J. IFAC*, 28(4):745–756, 1992.

[2] Anthony M. Bloch, Naomi Ehrich Leonard, and Jerrold E. Marsden. Stabilization of mechanical systems using controlled Lagrangians. In *Proceedings of the 36th IEEE CDC*, pages 2356–61, San Diego, CA, December 1997. IEEE.

[3] Robert A. Blumenthal. Affine submersions. *Ann. Global Anal. Geom.*, 3(3):275–287, 1985.

[4] Francesco Bullo, Naomi E. Leonard, and Andrew D. Lewis. Controllability and motion algorithms for underactuated Lagrangian systems on Lie groups. To appear in *IEEE Transactions on Automatic Control*, March 1997.

[5] V. I. Elkin. Affine control systems: Their equivalence, classification, quotient systems, and subsystems. *J. Math. Sci. (New York)*, 88(5):675–721, 1998.

[6] Johan Hamberg. General matching conditions in the theory of controlled lagrangians. In *Proceedings of the 38th IEEE CDC*, pages 2519–2523, Phoenix, AZ, December 1999. IEEE.

[7] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry*, Volume I. Number 15 in Interscience Tracts in Pure and Applied Mathematics. Interscience Publishers, New York, 1963.

[8] Ivan Kolář, Peter W. Michor, and Jan Slovák. *Natural Operations in Differential Geometry*. Springer-Verlag, New York-Heidelberg-Berlin, 1993.

[9] Andrew D. Lewis. Simple mechanical control systems with constraints. To appear in *IEEE Transactions on Automatic Control*, March 1997.

[10] Andrew D. Lewis. Affine connections and distributions with applications to nonholonomic mechanics. *Rep. Math. Phys.*, 42(1/2):135–164, 1998.

[11] Andrew D. Lewis and Richard M. Murray. Controllability of simple mechanical control systems. *SIAM J. Control Optim.*, 35(3):766–790, May 1997.

[12] Andrew D. Lewis and Richard M. Murray. Decompositions of control systems on manifolds with an affine connection. *Systems Control Lett.*, 31(4):199–205, 1997.

[13] James Patrick Ostrowski. *The Mechanics and Control of Undulatory Robotic Locomotion*. PhD thesis, California Institute of Technology, Pasadena, California, USA, September 1995.

[14] Jaak Vilms. Totally geodesic maps. *J. Differential Geom.*, 4:73–79, 1970.