ON FORMULATING NONLINEAR DYNAMICS IN LPV FORM

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Abstract

The shortcomings of a popular LPV gain-scheduling design approach are demonstrated by a simple counter-example. It is shown that, for a very general class of nonlinear systems, such an ad hoc design approach is unnecessary since soundly-based methods exist for transforming the plant dynamics into LPV/quasi-LPV form.

1. Popular Design Approach & Counter-Example

Combining ideas from conventional and LPV/quasi-LPV gain-scheduling, the following hybrid control design procedure is similar to *ad hoc* approaches proposed in the literature (see, for example, Apkarian *et al.* 1995, Spillman *et al.* 1996, Fialho *et al.* 1997, Lee & Spillman 1997). Consider the nonlinear system,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}),\ \ \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}) \qquad (1)$$

where $\mathbf{r} \in \Re^m$, $\mathbf{y} \in \Re^p$, $\mathbf{x} \in \Re^n$, $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are differentiable. The series expansion linearisation of (1) about an equilibrium point $(\mathbf{x_o},\mathbf{r_o})$ is

$$\begin{aligned} \delta\dot{\mathbf{x}} &= \nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}_o,\mathbf{r}_o)\delta\mathbf{x} + \nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x}_o,\mathbf{r}_o)\delta\mathbf{r} \\ \delta\mathbf{y} &= \nabla_{\mathbf{x}}\mathbf{G}(\mathbf{x}_o,\mathbf{r}_o)\delta\mathbf{x} + \nabla_{\mathbf{r}}\mathbf{G}(\mathbf{x}_o,\mathbf{r}_o)\delta\mathbf{r} \end{aligned} \qquad (2)$$

where $\delta\mathbf{x} = \mathbf{x}\text{-}\mathbf{x_0}$, $\delta\mathbf{r} = \mathbf{r}\text{-}\mathbf{r_0}$, $\delta\mathbf{y} = \mathbf{y}\text{-}\ \mathbf{G}(\mathbf{x_0}, \mathbf{r_o})$ (3)

Assume that the locus of equilibrium operating points is parameterised by some quantity, $\theta(\mathbf{x},\mathbf{r})$. In order to transform the control design task into a form amenable to LPV/quasi-LPV methods, the following quasi-LPV system associated with the linearisation family defined by (2)-(3) is considered

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{A}(\theta)\mathbf{z} + \mathbf{B}(\theta)\mathbf{r} \\ \mathbf{y} &= \mathbf{C}(\theta)\mathbf{z} + \mathbf{D}(\theta)\mathbf{r} \end{aligned} \qquad (4)$$

where $\begin{aligned} \mathbf{A}(\theta) &= \nabla_{\mathbf{x}}\mathbf{F}(\theta(\mathbf{z},\mathbf{r})), \mathbf{B}(\theta) = \nabla_{\mathbf{r}}\mathbf{F}(\theta(\mathbf{z},\mathbf{r})) \\ \mathbf{C}(\theta) &= \nabla_{\mathbf{x}}\mathbf{G}(\theta(\mathbf{z},\mathbf{r})), \mathbf{D}(\theta) = \nabla_{\mathbf{r}}\mathbf{G}(\theta(\mathbf{z},\mathbf{r})) \end{aligned}$ (5)

Standard LPV/quasi-LPV design methods can be applied to obtain a controller for the dynamics, (4)-(5). The controller obtained may then applied to the original nonlinear plant, (1).

Example

Consider the nonlinear system with dynamics described by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}r + \begin{bmatrix}0\\ -|x_2|x_2 - 10\end{bmatrix},\quad y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \quad (6)$$

(It should be noted that systems with similar types of nonlinearity are frequently encountered in practice). The requirement is to design an output-feedback controller which ensures a step response settling time of less than 2 seconds with zero steady-state error (this is, of course, not a complete performance specification but is sufficient in the present context). The series expansion linearisation of (6) about an equilibrium point $(x_{10}, x_{20}, r_o, y_o)$ is

$$\begin{bmatrix}\delta\dot{x}_1\\ \delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 1 & -2|x_{2o}|\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}\delta r,\quad \delta y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} \quad (7)$$

where

$\delta x_1 = x_1\text{-}x_{10}$, $\delta x_2 = x_2\text{-}x_{20}$, $\delta r = r\text{-}r_0$, $\delta y = y\text{-}y_0$ (8)

$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}),\ \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r})$ (1)

The associated quasi-LPV system is

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \mathbf{A}(\theta)\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \mathbf{B}r,\quad y = \mathbf{C}\begin{bmatrix}z_1\\ z_2\end{bmatrix} \qquad (9)$$

where $\mathbf{A}(\theta) = \begin{bmatrix}-1 & 0\\ 1 & -2\theta\end{bmatrix}$, $\mathbf{B} = \begin{bmatrix}1\\ 0\end{bmatrix}$, $\mathbf{C} = \begin{bmatrix}0 & 1\end{bmatrix}$ (10)

and θ equals $|z_2|$. Assume, for the moment, that $0 \le \theta \le 10$ and θ may vary arbitrarily within this range (this assumption amounts to a restriction on the class of allowable initial conditions and inputs to the system such that $|z_2| \le 10$). Using standard software from the MATLAB LMI toolbox and a conventional L_2 objective function with performance weighting, w_1, and control weighting, w_2, transfer functions[1]

$$w_1(s) = \frac{0.5}{s + 0.002},\quad w_2(s) = \frac{2.0\text{x}10^{-2}s}{s + 1000} \qquad (11)$$

the controller obtained for this system is

$$\dot{\mathbf{x}}_c = \mathbf{A}_c(\theta)\mathbf{x}_c + \mathbf{B}_c(y_{ref} - y),\quad r = \mathbf{C}_c\mathbf{x} \qquad (12)$$

where $\mathbf{A_c}(\theta) = \alpha\mathbf{A}_o + (1\text{-}\alpha)\mathbf{A}_1$, $\alpha = (10\text{-}\theta)/10$,

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[1] The performance requirement specifies zero steady-state error which implies that the magnitude of the transfer function of w_1 should be infinite at d.c. ; for example, by including an integrator term. However, the transfer functions here specify the actual values used in the numerical calculations, with approximate rather than exact integral action present in w_1.

$$\mathbf{A}_o = \begin{bmatrix} 5.6949e+003 & -2.7145e+000 & 2.7060e+001 & 7.6000e+004 \\ -3.5506e+004 & -1.5530e+001 & 2.4283e+002 & -4.7350e+005 \\ -1.9130e+005 & 2.8303e-001 & -5.6208e+000 & -2.5521e+006 \\ -2.0873e+004 & 3.1121e-001 & -1.6133e+000 & -2.7847e+005 \end{bmatrix}$$

$$\mathbf{A}_1 = \begin{bmatrix} 5.6777e+003 & -5.4523e-001 & 2.6029e+001 & 7.6032e+004 \\ -3.5504e+004 & -1.5777e+001 & 2.4295e+002 & -4.7350e+005 \\ -1.9130e+005 & 4.0036e-001 & -5.6766e+000 & -2.5521e+006 \\ -2.0869e+004 & -1.5741e-001 & -1.3905e+000 & -2.7848e+005 \end{bmatrix} \quad (13)$$

$$\mathbf{B}_c = \begin{bmatrix} 3.3565e-002 \\ -3.1706e-003 \\ -1.5648e-002 \\ 1.5715e+000 \end{bmatrix}, \mathbf{C}_c = \begin{bmatrix} 4.7249e+004 \\ 1.4691e+001 \\ -2.4353e+002 \\ 6.3016e+005 \end{bmatrix}^T$$

The response to step change in demand from –3.16 units to 0 units of the closed-loop system, consisting of the Jacobean-based quasi-LPV plant, (9)-(10), and controller, (12)-(13), is shown by the dashed-line in figure 1. (Note that θ (*i.e.* $|y|$) lies in the required range [0,10]). The settling time requirement is evidently satisfied. Nevertheless, when the same controller, (12)-(13), is applied to the original nonlinear system, (6), the corresponding response is as shown by the solid line in figure 1. It can be seen that the performance requirement is clearly not met and, indeed, that the nonlinear closed-loop system appears to be unstable.

Remark It is interesting to note that the performance requirement is in fact met for larger step demands; for example, a step change from –3.16 units to 6 units. This is perhaps unexpected, since larger steps are associated with excursions into operating regions further from equilibrium and with faster parameter variations, and clearly indicates that the behaviour observed is not associated with any restriction to near equilibrium operation arising from the use of equilibrium linearisations for the controller design. Indeed, the system in this example is intentionally selected to be benign in the sense that it satisfies the extended local linear equivalence condition of Leith & Leithead (1996,1998c); that is, the neighbourhood of validity of each equilibrium linearisation is unbounded and the union of these neighbourhoods covers the entire operating space. Hence, control design approaches based on the equilibrium linearisations are *not* a priori restricted to near equilibrium operation. Further loss of performance associated with deviations from equilibrium operation can, of course, be anticipated for systems which do not satisfy such a condition. ■

The poor performance achieved in the foregoing example is perhaps unsurprising since no direct relationship is established between the quasi-LPV system, (9), used for control design and the nonlinear system which is actually of interest, (6). It is emphasised that the family of linear systems defined by the equilibrium linearisations of (6), being a *collection* of individual dynamic systems (each with its own distinct state, input and output defined by the transformations (8)) rather than a *single* dynamic system, is conceptually quite different from the quasi-LPV system, (9). Of course, controllers designed by approaches similar to that here may sometimes inadvertently achieve acceptable performance. Nevertheless, the foregoing example indicates that this is certainly not the case in general.

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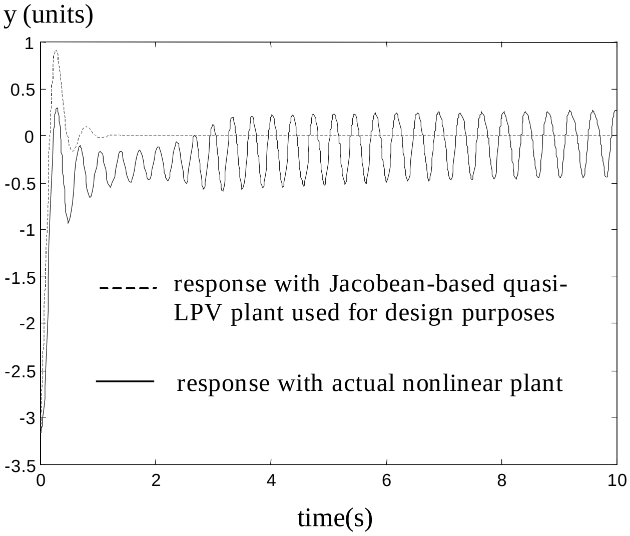

Figure 1 Nonlinear step responses for Jacobian-based quasi-LPV controller