On stabilization performance

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Abstract

Recently, significant interest has been raised in the study of hybrid systems, [2,3,9,13,15]. In this paper we analyze the performance of various stabilizers, including discontinuos and hybrid controls, to stabilize two model problems, namely a linerized pendulum with observed position and the Brockett system. In relation to this study we faced the presence of periodic orbits in hybrid stabilizers that are responsible for low performance of these.

1 Introduction

The aim of this paper is the study of the performance of various stabilizers for nonlinear problems that can not be solved by means of classical linear feedbacks (or even continuous ones). We focus our attention on the definition of hybrid system given in [3]. The examples examined in details are the following: a linearized pendulum with controlled external force and observed position and the well known Brockett system, also named nonholonomic integrator [6]. Let us start describing the first example. In this case, due to the restriction on the observed variables, the system can not be stabilized even by discontinuous feedbacks (but by dynamical stabilizer). We analyze in details the dependence of the stabilization on the location times (i.e. the time between two switchings, see Definitions 1 - 2) The natural appearance of periodic orbits for hybrid controls leads to the problem of existence and stability of periodic orbits for general hybrid dynamics. The second example needs no introduction, since it is the most famous system that is controllable but for which there is no smooth stabilizer. Various discontinuous feedbacks as well as hybrid stabilizers are considered. In particular, we analyze the discontinuous feedbacks proposed by Block and Drakunov in [5] and by Astolfi in [4] (two different feedbacks). Moreover, we consider also the stabilizer for an augmented system proposed by Brockett and Khaneja in [10] and some hybrid stabilizers proposed in [14].

2 Basic Definitions

We consider the control system

$$\dot{x} = f(x,u), \quad y = h(x), \tag{1}$$

where $f : R^n \times U \to R$, $h : R^n \to R^m$ and $u \in U \subseteq R^l$. Our aim is to asymptotically stabilize the system to the origin with a control depending only on the observed variable y. Obviously, one needs some assumptions on h; for observability theory we refer to [8].

Definition 1 *A hybrid control for the system (1) is a 6-ple $\Sigma = (Q, \Delta, I, M, \Xi, u)$, where*

1. *$Q = \{q_1, ..., q_k\}$ is a finite set representing the states of an automaton, called locations;*
2. *$\Delta : Q \longrightarrow R^+$ gives location times;*
3. *$I = \{\iota_1, ..., \iota_p\}$ is a finite set called the input alphabet;*
4. *$\Xi = \{\xi_1, ..., \xi_k\}$ is the set of input maps, $\xi_i : R^m \longrightarrow I, \quad i = 1, ..., k$;*
5. *$M : Q \times I \longrightarrow Q$ is the transition map among locations;*
6. *$u : Q \times R^m \longrightarrow U$ is the feedback map.*

A hybrid control describes the evolution of a state called the hybrid state:

Definition 2 *A hybrid state is a triplet (x, q, τ), where $x \in R^n$ is the state of the control system, $q \in Q$ is the location and $\tau \in]0, \Delta(q)]$ is the time before next switching.*

The state x evolves in location q for time τ when a location switching occurs according to the switching map. We require the stabilization to be uniform over locations and location times. For practical purpose we consider the concept of approximate stabilization corresponding to an ϵ-ball.

3 Simulations

Example 1. We consider the following system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{2}$$

where $x = (\eta, \theta)$, A,B,C are given by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

and $U = R$. Thus the system represents a linearized pendulum whose position is η and u is the external force.

The set of the locations of the system is $Q = \{q_1, q_2, q_3\}$ and the location times are given by $\Delta : Q \longrightarrow R^+$ s.t. $\Delta(q_i) = \delta$ for $i = 1, 2, 3$. The input alphabet is $I = \{+, -\}$ and the set of input maps is $\Xi = \{\xi_1, \xi_2, \xi_3\}$

where $\xi_i : R \longrightarrow I$ s.t. $\xi_i(\eta) = +$ if $\eta \geq 0$, $\xi_i(\eta) = -$ otherwise. The transition map among locations is $M : Q \times I \longrightarrow Q$ s.t. $M(q_1, \eta) = q_2$, if $\eta \geq 0$, $M(q_1, \eta) = q_1$ otherwise, $M(q_2, \eta) = q_3$, if $\eta \geq 0$, $M(q_2, \eta) = q_1$ otherwise, $M(q_3, \eta) = q_3$, if $\eta \geq 0$, $M(q_3, \eta) = q_2$ otherwise. The feedback map is $u : Q \times R \longrightarrow U$ s.t. $u(q_2, \eta) = -\eta$, $u(q_i, \eta) = 0$ for $i = 1, 3$. Some more hybrid stabilers are considered in [1].

First, we investigate the ϵ-stabilization time for an initial point in the first orthant and initial location q_3. It is easy to check that the conclusions hold also for the other possible initial hybrid states. For $\delta = \frac{\pi}{4}$ the hybrid trajectory has the following switching strategy. First the state x rotates in the first and fourth orthant and the location is constantly equal to q_3. When the trajectory crosses the negative θ-axis, the location switches to q_2 and the norm of x contracts. Then the trajectory rotates in the third and second orthant with location constantly equal to q_1. After crossing the positive θ-axis the location switches to q_2 and the norm of x contracts. Finally the location goes back to q_3 and repeats the same path.

We study the behaviour of the exit time as a function of the location times. There is a peak for a value close to 0.82. This is due to the presence of a periodic orbit for the hybrid control. The behaviour of the corresponding trajectory is described by the following picture (different grey levels correspond to different locations). The same phenomenon happens for integral costs.

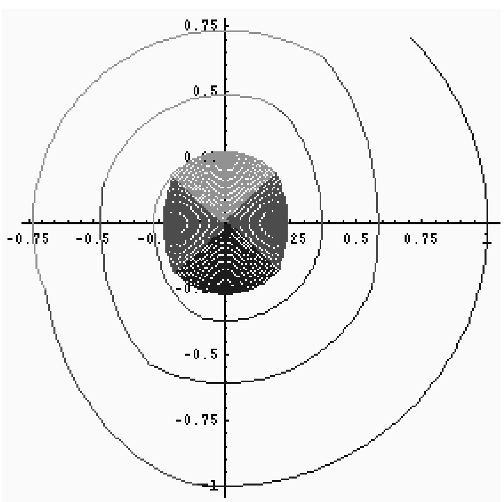


Figure 3.1: initial point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$; $\epsilon = 0.01$; $j = 3$.

4 Analytic study

Example 1 The peak in convergence time is due to the presence of a family of periodic orbits. Let us describe the switching strategy of the periodic orbit. We start from a point $(0, a)$, $a > 0$, location q_1. At time δ we switch to location q_2 and contract the norm of the state reaching a point in the fourth orthant. Then we switch to location q_3 and arrive at the point $(0, -a)$. After using again location q_3, we switch to location q_2 in the second orthant. Finally we switch to location q_1 and go back to the initial point. The value $\bar{\delta}$ for which there is a periodic orbit satisfies the equation

$$\bar{\delta} + \arctan\left(\sqrt{2}\tan\frac{\bar{\delta}}{\sqrt{2}}\right) = \frac{\pi}{2}. \quad (3)$$

This equation can not be solved explicitly but an approximate value of $\bar{\delta}$ is 0.82209. We state the following two results which are proved in [1].

Proposition 1 *In the framework of example 1 if δ is smaller than $\overline{\delta}$ (see equation (3)), all the hybrid dynamics are contractive. If $\delta = \overline{\delta}$, the only periodic orbit is the one determined in example 1.*

Proposition 2 *If $\delta \neq \overline{\delta}$ is in the interval $[0, 1]$, no periodic orbit exists having the same sequence of locations as the one for $\delta = \overline{\delta}$, i.e. $\{q_1, q_2, q_3, q_3, q_2, q_1\}$*

5 Brockett system

In this section we discuss some stabilizers for the Brockett system

$$\begin{cases} \dot{x} = u \\ \dot{y} = v \\ \dot{z} = yu - xv \end{cases} . \quad (4)$$

We start considering the discontinuos feedback proposed by Bloch and Drakunov in [5].

The control is given by $u = -\alpha x + \beta y \,\mathrm{Sgn}(z)$, $v = -\alpha y + \beta x \,\mathrm{Sgn}(z)$ and stabilizes the system under the condition $\frac{\beta}{2\alpha}\left(x^2(0), y^2(0)\right) \geq |z(0)|$; a constant control is used when the condition is not satisfied (similarly to [14]).

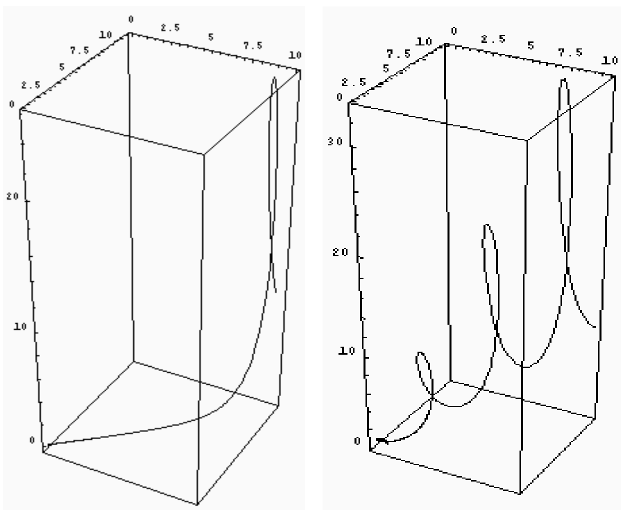
Through computer simulations it is easy to see that the convergence time is big for data near the boundary of the region corresponding to constant control. This boundary consists of a paraboloid that passes through the origin, so the problem of slow convergence happens in every neighborhood of the origin. Moreover the value of the control oscillates causing implementation problems. Finally also the use of the xy-plane as sliding mode introduce some oscillations. In this case the parameters α and β should be taken as big as possible to have fast convergence, obviuosly this reflects in high physical cost of the control.

Let us now consider the discontinuos feedback introduced by Astolfi in [4]. The first control stabilizes for initial data outside the surface where $ax^n + by^n = 0$. For initial data near this surface, the trajectories are not even bounded. The second control stabilizes for initial data outside the z-axis. This control has better performance, however there are still some oscillations when starting near the z-axis and near the xy-plane (sliding mode).

Brockett and Khaneja in [10] used an augmented system introducing the extra variable θ that satisfies the equation $\dot{\theta} = z$. They use the control

$$u = -x + z\cos\theta, \; v = -y - z\sin\theta. \quad (5)$$

This feedback was designed to include in a general framework time varying controls [7] and to stabilize with minimal amount of information. The corresponding trajectories depend in sensitive (non chaotic) way on the initial value of the extra variable θ. The figures below show the two trajectories starting from (x, y, z, θ) with $x = y = z = 10$ and respectively $\theta = 10$ and $\theta = 15$. An interesting problem is the one of optimizing the initial value of θ for given initial value of x, y, z. The convergence is exponential.



Finally in [14] some hybrid stabilizers were proposed. The convergence happens to be exponential and the amount of information is smaller than that used by a time varying feedback [7,12]. The controls used are bounded. However the first hybrid control stabilizes only outside the z-axis while the last bang-bang hybrid control has chattering problems, see [14] for details.

6 Conclusions

We considered various stabilizing controls for two model problems that are a linearized pendulum with observed position and external force control, and the Brockett nonhonolomic integrator. Through computer simulations we studied the velocity of convergence and the integral cost of each control. An interesting phenomenon is the presence of periodic orbits for the hybrid dynamics that are responsible of low performances. These can be studied analytically in our case and are linked to the interesting problem of aymptotic dynamics for hybrid systems [11].

For Brockett system various discontinuos controls are analyzed. All of them have good stabilization performance outside some critical regions and some make use of sliding modes so presenting chattering problems. From the point of view of minimal use of information of particular interest are the controls used in [10] and [14].

7 References

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