Extension of control Lyapunov functions to time-delay systems

Mrdjan Jankovic
Ford Research Laboratory
P.O. Box 2053, MD 2036 SRL
Dearborn, MI 48121
e-mail: mjankov1@ford.com

Abstract

The concept of control Lyapunov function has been proven a useful tool for designing robust control laws for nonlinear systems. Recently, this concept has been extended to time-delay systems in the form of Control Lyapunov Razumikhin functions. In this paper we further develop this extension by introducing Control Lyapunov Krasovsky functionals and show their use for robust stabilization of time-delay nonlinear systems. To further motivate the new concept we establish robustness properties for several control laws based on control Lyapunov Krasovsky functionals.

1. Introduction

Motivated by the concept of control Lyapunov functions (CLF) and the role it played in robust stabilization of nonlinear systems, the concept of control Lyapunov-Razumikhin functions (CLRF) for time delay systems has been introduced in [5]. The basis for the definition of CLRF is the stability theory due to Razumikhin, which relies on Lyapunov-like functions to prove stability of a time delay system. If a CLRF for the time delay system is known, the so called "domination redesign" control law can be applied to achieve global asymptotic stability [5]. Other universal formulas, including Sontag's and Freeman's formulas, developed for non-delay nonlinear systems cannot be used for time-delay systems and CLRF's.

In this paper we extend the CLF concept to time-delay systems using the stability theory that relies on Lyapunov Krasovsky functionals. An immediate problem is that, in general, the time derivative of a functional may be a complex function of the control input and its past values. In addition, the functionals have to be restricted to assure that the resulting control laws are well defined and bounded when the past trajectory is bounded. A particular form of the Lyapunov Krasovsky functionals under consideration is motivated by [10, 11]. While the form appears restrictive, in situations when delay terms are relatively simple CLFK's are more flexible and easier to construct than CLRF's.

One advantage of a CLKF over a CLRF is that the universal formulas like Sontag's formula [14] and Freeman's formula [2] apply and achieve global asymptotic stability of the closed loop system. On the other hand, the domination redesign formula [13] that provides global stability for CLRF's, guarantees only semiglobal stability in the case of CLKF's. These universal formulas have been studied in [7, 13] in the context of robust (inverse optimal) designs for non-delay systems.

The connection between optimality and passivity [8, 12] has provided the robustness of optimal control systems to static (more precisely, memoryless) and dynamic input uncertainties. In the literature, this robustness has been interpreted as the $(\frac{1}{2}, \infty)$ gain margin and $\pm 60^\circ$ phase margin of linear quadratic regulators [1]. The robustness guarantees of control laws derived from a CLKF for time delay systems resemble closely those derived from a CLF for non-delay systems: Sontag's and Freeman's formula guarantee robustness to memoryless sector input nonlinearities, while the domination redesign formula guarantees robustness to dynamic input uncertainties equivalent to that of optimal systems. Putting the region of attraction issue aside, the CLKF based domination redesign control provides a stronger robustness property than the one established for the CLRF based one [6].

The paper is organized as follows. Review of the concept of CLRF, definition of CLKF, and the relationship between the two are given in Section 2. Section 3 contains results on global stabilization and robustness of the closed loop system to input uncertainties. Concluding remarks can be found in Section 4.

Notation

We consider two types of objects that describe the state of the time delay system: $x(t) \in \mathbb{R}^n$ a time dependent vector and $x_d(t) : [-r, 0] \mapsto \mathbb{R}^n$ a time dependent function defined by $x_d(t)(\theta) = x_d(t + \theta)$. We have found this notation, introduced in [15] more convenient than the more conventional $x_t(\theta)$. For the sake of simplicity, we shall often omit the dependence on t in the notation. For example, we shall write $\dot{x} = f(x, x_d)$ instead of $\dot{x}(t) = f(x(t), x_d(t))$ and $x_d(\theta)$ instead of $x_d(t)(\theta)$. The notation $|\cdot|$ is used to denote the Euclidean 2-norm of a vector, while $||\cdot||$ denotes the norm of uniform convergence of functions, that is, for $\phi_d : [-r, 0] \mapsto \mathbb{R}^n$, $||\phi_d|| = sup_{\theta\in[-r,0]}|\phi_d(\theta)|$. By $C([-r, 0], \mathbb{R})$ we denote the space of continuous functions from $[-r, 0]$ into $\mathbb{R}$ and by $CP([-r, 0], \mathbb{R})$ the space of piecewise continuous functions from $[-r, 0]$ into $\mathbb{R}$. A continuous scalar function α is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$; α is said to belong to class $\mathcal{K}_\infty$ if, in addition, $\alpha(s) \to \infty$ as $s \to \infty$.

2. Control Lyapunov-Krasovsky functionals

In this section we introduce the concept of control Lyapunov-Krasovsky functionals (CLKF) for a class of time-delay systems. As the first step, we briefly review the concept of CLRF's introduced in [5] for input affine, time delay nonlinear systems described by

$$\dot{x}(t) = f(x_d) + g(x_d)u \tag{2.1}$$

with the initial condition given by $x_d(0)(\cdot) = \phi_d$, where $\phi_d : [-r, 0] \mapsto \mathbb{R}^n$ is a continuous vector valued function. Vector fields f and g are smooth functionals mapping piecewise continuous functions into $\mathbb{R}^n$.

The stability theory underlying our definition of control Lyapunov-Razumikhin functions is provided by Razumikhin theorems [4, 9], which state that the equilibrium at the origin for the system

$$\dot{x} = f(x_d)$$

is globally stable[1] if there exists a positive definite, radially unbounded function $V(x)$ such that

$$\dot{V} = \frac{\partial V}{\partial x} f = L_f V \leq 0$$

whenever $V(x) \geq V(x(t+\theta))$, $\theta \in [-r, 0)$, and globally asymptotically stable if there exists a function α, $\alpha(s) > 0$ for $s > 0$, such that

$$\dot{V} = L_f V \leq -\alpha(|x|)$$

whenever $\pi(V(x)) \geq V(x(t+\theta))$, $\theta \in [-r, 0)$, with the continuous nondecreasing function $\pi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ satisfying $\pi(s) > s$ for all $s > 0$.

Definition 1 *(Control Lyapunov-Razumikhin Functions)*
A smooth, positive definite, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}^+$, is a control Lyapunov-Razumikhin function (CLRF) for the system (2.1) if there exists a continuous nondecreasing function $\pi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\pi(s) > s,\ s > 0$, and a function $\alpha : \mathbb{R}^+ \mapsto \mathbb{R}^+$, $\alpha(s) > 0$ for $s > 0$, such that

$$L_f V(\chi_d) \leq -\alpha(|x|)$$

for all piecewise continuous functions $\chi_d : [-r, 0] \mapsto \mathbb{R}^n$ for which $\chi_d(0) = x$, $L_g V(\chi_d) = 0$, and

$$\pi(V(x)) \geq V(\chi_d(\theta)), \quad \forall \theta \in [-r, 0) \tag{2.2}$$

The condition (2.2) is referred to as the Razumikhin condition. □

To design an asymptotically stabilizing controller based on a CLRF, the class of time-delay system under consideration has been restricted in [5] to the form

$$\begin{aligned}\dot{x} = f(x_d) + g(x_d)\, u &= f_0(x, x(t-\tau_1), \ldots, x(t-\tau_l)) \\ &+ \int_{-r}^{0} \Gamma(\theta) F(x, x(t-\tau_1), \ldots, x(t-\tau_l), x(t+\theta))\, d\theta \\ &+ g(x, x(t-\tau_1), \ldots, x(t-\tau_l))\, u\end{aligned} \tag{2.3}$$

where f_0, g, and $F : \mathbb{R}^{(l+2)n} \mapsto \mathbb{R}^{r_\Gamma}$ are smooth functions of their arguments. Without loss of generality we assume that $F(x, x(t-\tau_1), \ldots, x(t-\tau_l), 0)) = 0$. The matrix $\Gamma : [-r, 0] \mapsto \mathbb{R}^{n\times r_\Gamma}$ is assumed to be piecewise continuous (hence, integrable) and bounded. This restriction is needed to avoid the problems that arise due to non-compactness of closed bounded sets in the space $(C([-r, 0], \mathbb{R}^n), ||\cdot||)$.

The CLF concept can also be extended to Lyapunov-Krasovsky functionals $V : C([-r, 0], \mathbb{R}^n) \mapsto \mathbb{R}^+$. The underlying stability result, provided by the Krasovsky theorem [4, 9] states that the equilibrium at the origin for the system

$$\dot{x} = f(x_d)$$

is globally stable if there exist a functional $V(t, x_d)$ and two class $\mathcal{K}_\infty$ functions β_1 and β_2 such that

$$\beta_1(|x_d(0)|) \leq V(x_d) \leq \beta_2(||x_d||)$$

$$\dot{V} = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x(t+h)) - V(t, x(t))] \leq 0 \ .$$

The origin is globally asymptotically sable if, in addition,

$$\dot{V} \leq -\alpha(|x_d(0)|)$$

where α is a scalar function such that $\alpha(s) > 0$ for $s > 0$.

From the definition of $\dot{V}$ one can immediately see that, if the derivative is taken along the trajectories of the controlled system, it may depend on the past values of the control input u preventing an obvious extension

[1] We say that an equilibrium is globally stable if it is locally Lyapunov stable and all the trajectories of the system are bounded.

of the CLF concept to Lyapunov-Krasovsky functionals. Moreover, design of stabilizing control laws will require a restriction on the type of functional that appear in $\dot{V}$ similar to the CLRF case [5]. Motivated by these requirements and by the Lyapunov-Krasovsky functionals proposed in [10, 11] for systems with discrete delays and distributed delays in the integral form, we propose the following form for CLKF's:

$$V(x_d) = V_1(x) + V_2(x_d) + V_3(x_d) \ , \qquad (2.4)$$

where $V_1(x)$ is the positive definite radially unbounded function of the current state x, V_2 is a nonnegative functional due to the discrete delays in (2.3),

$$V_2(x_d) = \sum_{i=1}^{l} \int_{-\tau_i}^{0} S_i(x(t-\varsigma)) \ d\varsigma \ , \qquad (2.5)$$

and V_3 is a nonnegative functional due to the distributed delay in (2.3),

$$V_3(x_d) = \int_{-r}^{0} \int_{t+\theta}^{t} L(\theta, x(\varsigma)) \ d\varsigma \ d\theta \ . \qquad (2.6)$$

$S_i : \mathbb{R}^n \mapsto \mathbb{R}$ and $L : \mathbb{R}^n \times \mathbb{R}^+ \mapsto \mathbb{R}$ are nonnegative functions. The time derivative of the functional V along trajectories of the system (2.3) is given by

$$\begin{aligned} \dot{V} \ = \ & \frac{\partial V_1}{\partial x}\left(f_0 + \int_{-r}^{0} \Gamma(\theta) F \ d\theta\right) + \frac{\partial V_1}{\partial x} g u \\ & + \sum_{i=1}^{l} S_i(x) - S_i(x(t-\tau_i)) \\ & + \int_{-r}^{0} L(\theta, x) - L(\theta, x(t+\theta)) \ d\theta \end{aligned} \ . \qquad (2.7)$$

By using the Lie derivative notation $L_g V_1 = \frac{\partial V_1}{\partial x} g$ and defining an extended Lie derivative for functionals of the form (2.4), (2.5), (2.6) via

$$\begin{aligned} L_f^* V(x_d) = & L_{f_0 + \int_{-r}^{0} \Gamma(\theta) F d\theta} V + \sum_{i=1}^{l} S_i(x) - S_i(x(t-\tau_i)) \\ & + \int_{-r}^{0} L(\theta, x) - L(\theta, x(t+\theta)) \ d\theta \end{aligned}$$

we can define control Lyapunov Krasovsky functionals in analogy with the CLF's.

Definition 2 *(Control Lyapunov Krasovsky Functionals)*
A smooth functional of the form (2.4) is called a control Lyapunov Krasovsky functional for the system (2.3) if there exist a function α, $\alpha(s) > 0$ for $s > 0$, and two class $\mathcal{K}_\infty$ functions β_1 and β_2 such that

$$\beta_1(|\chi_d(0)|) \le V(\chi_d) \le \beta_2(\|\chi_d\|)$$

and

$$L_g V_1(\chi_d) = 0 \Rightarrow L_f^* V(\chi_d) \le -\alpha(|\chi_d(0)|) \qquad (2.8)$$

for all piecewise continuous functions $\chi_d : [-r, 0] \mapsto \mathbb{R}^n$. □

As in the case of CLRF's, we could not restrict our consideration to trajectories of the systems (2.3). Before the control input u is chosen, the trajectories are not defined.

Even for the class of time-delay systems (2.3), the form of the CLKF appears restrictive because the dependence of S_i and L on a single delay prevents cross-delay terms in $\dot{V}_2$ and $\dot{V}_3$. Still, in some cases a CLKF may be easier to construct than a CLRF as illustrated by the following example.

Example 1 For the time-delay system

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= -x_2 + x_2(t-\tau) + x_1^3 + u \end{aligned} \qquad (2.9)$$

it may be difficult or even impossible to construct a CLRF. On the other hand, the $V_2 = \frac{1}{2}\int_{-\tau}^{0} x_2^2(\theta) \ d\theta$ term in the functional

$$V = \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{2}\int_{-\tau}^{0} x_2^2(\theta) \ d\theta$$

provides the precision needed to dominate the $xx(t-\tau)$ term in $\dot{V}$. Setting $\alpha(|x|) = \frac{1}{4}|x|^4$ we obtain

$$\begin{aligned} & L_g V = x_1 + x_2 = 0 \ \Rightarrow x_1 = -x_2 \ \Rightarrow \\ & L_f^* V = x_2(-x_2 + x_2(t-\tau) + x_1^3) + \tfrac{1}{2}(x_2^2 - x_2^2(t-\tau)) \\ & \quad = -\tfrac{1}{2}(x_2 - x_2(t-\tau))^2 - x_2^4 \le -x_2^4 \le -\alpha(|x|) \end{aligned}$$

which proves that V is a CLKF.

3. Robust control design

Several different "universal formulas" exist for global stabilization of a nonlinear system with a known CLF including Sontag's formula, Freeman's formula, and the "domination redesign" formula [2, 13, 14]. For a time delay system with a CLRF, conditions are such that only the domination redesign control law applies. An advantage of a CLKF is that other formulas apply as well, while a disadvantage is that the domination redesign can provide only semiglobal stability. Most properties of these control laws established in this section rely on the following version of the fundamental result from [5].

Lemma 1 Define $\Omega = \{x_d \in CP([-r, 0], \mathbb{R}^n) : d \le \|x_d\| \le D\}$. If V is a CLKF for the system of the form (2.3) then for all $0 < \delta < D$ there exists $\varepsilon > 0$ such that $L_f^* V(\chi_d) \ge -\frac{1}{2}\alpha(|\chi_d(0)|) \ \Rightarrow |L_g V(\chi_d)| \ge \varepsilon$ for $\chi_d \in \Omega$.

As in the case of CLF's, a globally stabilizing control law for the nonlinear system (2.3) with a CLKF of the form (2.4) is given by the Sontag's formula [14]:

$$u_S(x) = -p_S(x_d)\left(L_gV_1(x, x(t-\tau_1), \ldots, x(t-\tau_l))\right)^T \tag{3.1}$$

with

$$p_S(x_d) := \begin{cases} \frac{L_f^*V + \sqrt{(L_f^*V)^2(x) + |L_gV_1|^4}}{|L_gV_1|^2} & , \quad L_gV_1(x) \neq 0 \\ 0 & , \quad L_gV_1(x) = 0 \end{cases} \tag{3.2}$$

Because $u_S(x)$ achieves

$$\dot{V} = L_f^*V - p_S|L_gV_1|^2 = -\sqrt{(L_f^*V)^2 + |L_gV_1|^4} \leq -\epsilon(|x|) \tag{3.3}$$

by the stability theorem of Krasovsky, we conclude that Sontag's formula provides a globally stabilizing control law. The last inequality in (3.3) follows from Lemma 1.

If we slightly strengthen the CLKF conditions to require that $\alpha(|x|)$ be a smooth function of x (to assure smoothness of the resulting control law away from the origin) one can use a stabilizing control law with pointwise minimum norm called the Freeman's formula [2]:

$$u_F(x_d) = -p_F(x_d)\left(L_gV_1(x, \ldots, x(t-\tau_l))\right)^T \tag{3.4}$$

with

$$p_F(x_d) = \begin{cases} \frac{L_f^*V + \alpha(|x|)}{|L_gV_1|^2} & \text{if } \; L_f^*V + \alpha(|x|) > 0 \\ 0 & \text{if } \; L_f^*V + \alpha(|x|) \leq 0 \end{cases}$$

The time derivative of V along the trajectories of the closed loop system satisfies

$$\begin{aligned} \dot{V} &= L_f^*V - p_F|L_gV_1|^2 = \min\{L_f^*V(x_d), -\alpha(|x|)\} \\ &\leq -\alpha(|x|) \end{aligned}$$

and the origin is globally asymptotically stable.

By using Lemma 1, one can show that the control laws (3.1) and (3.4) are smooth everywhere except possibly at the origin. Moreover, the control laws (3.1) and (3.4) are continuous at $x = 0$ if the CLF satisfies the *small control property*: for each $\epsilon > 0$, we can find $\delta(\epsilon) > 0$ such that, if $0 <\| x_d \|< \delta$, there exists u which satisfies $L_f^*V + (L_gV_1)^T u < 0$ and $\| u \| < \epsilon$ (c.f. [14]).

The third control law under consideration in this paper is the domination redesign formula [13]:

$$u(x_d) = -\gamma(V(x_d))(L_gV_1)^T \tag{3.5}$$

where the scalar domination function $\gamma(\cdot)$ must satisfy $\gamma(s) > 0$, and $\lim_{T\to\infty} \int_0^T \gamma(s)\, ds = \infty$. Because $V(x_d)$ is, in general, not radially unbounded, that is, $\|x_d\|$ may converge to infinity while $V(x_d)$ stays bounded, the gain $\gamma(V(x_d))$ is not strong enough to guarantee global stability. Therefore, the stability we can establish is semiglobal, for which we assume that $\|x_d(0)(\cdot)\| \leq M$ for some arbitrary large $M > 0$.

If, for all $\|x_d\| < \delta$, and for some $c > 0$

$$\frac{L_f^*V}{|L_gV_1|^2} < c, \tag{3.6}$$

there exists a smooth function γ^* such that for all γ such that $\gamma(s) > \gamma^*(s)$

$$\dot{V} = L_f^*V - \gamma(V)|L_gV|^2 \leq -\mu(|x|)$$

that is, the closed loop system (2.3), (3.5) is asymptotically stable. The condition (3.6) is the same one employed in [7] to prove the global asymptotic stabilization by CLF based domination redesign control law for non-delay systems and in [5] for the case of CLRF's and time-delay systems.

One reason for considering the CLF based control laws, such as Sontag's formula, Freeman's formula, and the domination redesign formula, is the robustness guarantee of the resulting control laws to input uncertainties. So, let us consider the "nominal" closed loop system system

$$(H, k): \quad \begin{cases} \dot{x} = f(x_d) + g(x_d)u \\ y = k(x_d) \end{cases} \tag{3.7}$$

where $k(x_d)$ is chosen such that $u = -k(x_d)$, achieves

$$\dot{V} = L_f^*V - L_gV_1 k(x_d) \leq -\mu(|x|) \tag{3.8}$$

When the vector fields $f(x_d)$ and $g(x_d)$ are of the form (2.3), and a CLKF for (3.7) is known, $k(x_d)$ exists such that (3.8) holds. In our consideration $u = -k(x_d)$ represents a control law obtained by one of the universal formulas discussed above.

To consider the robustness of the system (3.7) we insert an uncertainty Δ into the feedback loop as depicted in Figure 1. First we consider the case when the uncertainty Δ is memoryless:

$$\Delta(s, t) = diag\{\varphi_1(s, t), \ldots, \varphi_m(s, t)\} \tag{3.9}$$

where s is a scalar variable, t is the time and, for all $t > 0$, the functions $\varphi_i(\cdot, t)$ are sector[2] nonlinearities.

Proposition 1 If that the memoryless uncertainty Δ is of the form (3.9) then the following statements are true:

1. If the control law $u = -k(x_d)$ is given by the Sontag's formula (3.1) then the perturbed closed

[2] A function $\varphi : \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}$ is said to belong to a sector (a, b) if $as^2 < s\varphi(s, t) < bs^2$ for all $s \neq 0$, and all $t \geq 0$.

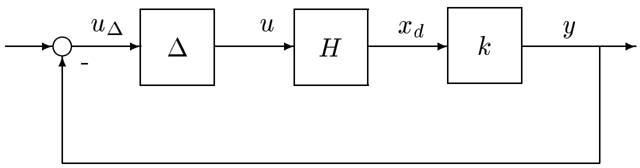

Figure 1: Perturbed feedback loop formed of the nominal feedback system (H, k) and an input uncertainty Δ.

loop system is globally stable provided that φ_i, $i = 1, \ldots, m$, belong to the sector $[\frac{1}{2}, \infty)$ and globally asymptotically stable if φ_i, $i = 1, \ldots, m$, belong to the sector $(\frac{1}{2} + \delta, \infty)$ for some $\delta > 0$.

2. If the control law $u = -k(x_d)$ is given by the Freeman's formula (3.4) then the perturbed closed loop system is globally asymptotically stable provided that φ_i, $i = 1, \ldots, m$, belong to the sector $[1, \infty)$[3].

The proof follows closely the derivation for the CLF case [7, 13] and is omitted.

If the uncertainty Δ is dynamic, Sontag's and Freeman's formulas, in general do not guarantee stability of the closed loop system. We can prove robustness to a class of dynamic input uncertainties by using the domination redesign formula. The class of dynamic uncertainties under consideration is characterized by the following two conditions:

(a) Δ is of the form

$$\begin{aligned} \dot{\zeta} &= q(t, \zeta, u_\Delta) \\ y_\Delta &= h(t, \zeta, u_\Delta) \end{aligned}$$

where u_Δ and y_Δ are m-dimensional input and output vectors of the system Δ.

(b) There exists a positive definite, radially unbounded storage function $S(\zeta)$ such that

$$\dot{S} \leq y_\Delta^T u_\Delta - \nu u_\Delta^T u_\Delta \tag{3.10}$$

with $0 < \nu < 1$.

As a special case, this class contains memoryless uncertainties of the form (3.9). By setting the "excess of passivity" $\nu = \frac{1}{2}$ we establish a connection to the above mentioned robustness of optimal systems: $(\frac{1}{2}, infty)$ gain margin and $\pm 60°$ phase margin.

[3]To achieve $\dot{V} \leq -\mu(|x|)$ this controller uses the minimum possible effort. Thus, any reduction may leave the system unstable. Better stability margins can be obtained by multiplying the control law by a positive constant larger than 1.

Theorem 1 Assume that a CLKF $V(x_d)$ for the system (2.3) is known and that the uncertainty Δ satisfies conditions (a) and (b). Then the origin of the perturbed closed loop system in Figure 1 remains stable and trajectories remain bounded provided that $\|x_d(0)(\cdot)\| \leq M$, and $\int_0^{V(x_d(0)(\cdot))} \gamma(s)\, ds + S(\zeta(0)) \leq \int_0^{\beta_1(M)} \gamma(s)\, ds$.

Proof: Let us consider the perturbed closed loop system

$$\begin{aligned} \dot{x} &= f(x_d) + g(x_d)u \\ \dot{\zeta} &= q(t, \zeta, u_\Delta) \\ u &= y_\Delta \\ u_\Delta &= -k(x_d) \end{aligned}$$

where $k(x_d) = \gamma(V) L_g V_1$, and a Lyapunov-Krasovsky functional $V_{aug}(x_d, \zeta) = \int_0^{(V(x_d))} \gamma(s)\, ds + S(\zeta)$. Assume that $\|x_d\| < M$ at some time $t_1 \geq 0$, which implies the existence of a function γ^* such that

$$L_f^* V - \gamma^*(V)|L_g V_1|^2 \leq -\mu(|x|)$$

Select $\gamma(s) = \frac{1}{\nu}\gamma^*(s)$ and compute the time derivative of V_{aug} along trajectories of the closed loop system:

$$\begin{aligned} \dot{V}_{aug} &\leq \gamma(V)(L_f^* V + L_g V_1 y_\Delta) + y_\Delta^T u_\Delta - \nu u_\Delta^T u_\Delta \\ &= \gamma(V)(L_f^* V - \nu\gamma(V)|L_g V_1|^2) \\ &\leq \gamma(V)(L_f^* V - \gamma^*(V)|L_g V_1|^2) \leq -\mu(|x|) \end{aligned} \tag{3.11}$$

which implies that V_{aug} is non increasing and, in turn, that $\|x_d\| < M$ for $t > t_1$. Because, $\|x_d\| < M$ holds by assumption at $t = 0$, (3.11) holds for all $t \geq 0$ and stability of the origin for the perturbed closed loop system follows. □

If the conditions on the dynamic uncertainty Δ are strengthen to a strong dissipativity,

$$\dot{S} \leq -a(|\zeta|) + y_\Delta^T u_\Delta - \nu u_\Delta^T u_\Delta \ ,$$

instead of Lyapunov stability one can prove asymptotic stability of the the origin for the perturbed closed loop system.

Example 2 Consider a nonlinear system with delayed state given by

$$\dot{x} = x^3(t - \tau) + (x - \frac{1}{2}x(t - \tau))u \tag{3.12}$$

and a CLKF candidate of the form $V(x_d) = V_1(x) + V_2(x_d)$:

$$V(x_d) = \frac{1}{2}x^2 + \int_{-\tau}^{0} x^4(t + \theta)\, d\theta$$

Note that V satisfies

$$\beta_1(|x|) = \frac{1}{2}x^2 \leq V \leq \|x_d\|^2 + \tau\|x_d\|^4 = \beta_2(\|x_d\|)$$

Because $L_gV_1 = x(x - \frac{1}{2}x(t-\tau)$, $L_gV_1 = 0$ implies $x = 0$ or $x = \frac{1}{2}x(t-\tau)$. For $\alpha(s) = 7s^4$ we have

(i) if $x = 0$,

$$\begin{aligned} L_f^* V &= xx^3(t-\tau) + x^4 - x^4(t-\tau) \\ &= -x^4(t-\tau) \leq 0 = -\alpha(|x|) \end{aligned}$$

(ii) if $x = \frac{1}{2}x(t-\tau)$, that is, if $x(t-\tau) = 2x$,

$$L_f^* V = 8x^4 + x^4 - 16x^4 = -\alpha(|x|)$$

Hence, V is a CLKF for the system (3.12). Therefore, one can apply one of the universal formulas to stabilize the system. For example, Freeman's formula provides the control law

$$u_F = -\frac{x^3(t-\tau)(x - x(t-\tau)) + 8x^4}{x(x - \frac{1}{2}x(t-\tau))}$$

if $xx^3(t-\tau) + 8x^4 - x^4(t-\tau) > 0$ and $u_F = 0$ if $xx^3(t-\tau) + 8x^4 - x^4(t-\tau) \leq 0$. Accordingly, this control law achieves global asymptotic stability and is robust to static (memoryless) input uncertainties.

4. Conclusion

The concept of control Lyapunov functions has already been extended to time delay-systems based on the Razumikhin stability theory providing control Lyapunov Razumikhin functions. The objective of this paper is to examine the possibility to extend the concept of CLF by applying the stability theory based on Lyapunov-Krasovsky functionals. The CLKF can be used for robust stabilization of time delay nonlinear systems and, in contrast to CLRF's, applicable control laws and their robustness properties closely resemble the case of CLF's and non-delay systems.

References

[1] B.D.O. Anderson, J.B. Moore, *Optimal Control: Linear Quadratic Methods.* Prentice-Hall, Englewood Cliffs, NJ, 1990.

[2] R.A. Freeman, P.V. Kokotović, *Robust Control of Nonlinear Systems*, Birkhauser, Boston, 1996.

[3] S.T. Glad, "Robustness of nonlinear state feedback – a survey," *Automatica*, vol. 23, pp. 425-435, 1987.

[4] J.K. Hale, S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New-York, 1993.

[5] M. Jankovic, "Control Lyapunov-Razumikhin functions for time delay systems," Proceedings of CDC, Phoenix, AZ, 1999.

[6] M. Jankovic,"Robust CLRF based nonlinear control design for time delay systems," Proceedings of the American Control Conference, Chicago, IL, June 2000.

[7] M. Jankovic, R. Sepulchre, P.V. Kokotović, "CLF Based Designs with Robustness to Dynamic Input Uncertainties," *Systems and Control Letters*, vol. 37, pp 45-54, 1999.

[8] R.E. Kalman, "When is a linear control system optimal?," *Trans. ASME Ser. D: J. Basic Eng.*, vol. 86, pp. 1-10, 1964.

[9] V.B. Kolmanovsky, V.R. Nosov, *Stability of Functional Differential Equations*, Academic Press, San Diego, 1986.

[10] V.B. Kolmanovsky, "Stability of some nonlinear functional differential equations," NoDEA, vol. 2, pp. 185-198, 1995.

[11] V.B. Kolmanovsky, S-I. Niculescu, J-P. Richard, "Some Remarks on the Stability of Linear Systems with Delayed State," *Proceedings of CDC*, pp. 299-304, Tampe, Florida, 1998.

[12] P.J. Moylan, B.D.O. Anderson, "Nonlinear regulator theory and an inverse optimal control problem," *IEEE Trans. Automatic Control*, vol. 18, pp. 460-465, 1973.

[13] R. Sepulchre, M. Jankovic, P.V. Kokotović, *Constructive Nonlinear Control*, Springer-Verlag, London, 1997.

[14] E.D. Sontag, "A universal construction of Artstein's theorem on nonlinear stabilization," *Systems & Control Letters*, vol. 13, pp. 117-123, 1989.

[15] A. R. Teel, "Connection Between Razumikhin-Type Theorems and the ISS Nonlinear Small Gain Theorem," *IEEE Trans. Automatic Control*, vol. 43, pp. 960-964, 1998.

[16] J.N. Tsitsiklis, M. Athans, "Guaranteed robustness properties of multivariable nonlinear stochastic optimal regulators," *IEEE Trans. on Automatic Control*, vol. 29, pp. 690-696, 1984.