Global Finite-Time Stabilization: from State Feedback to Output Feedback

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Abstract

This paper addresses the problem of global finite-time design via dynamic output feedback. The results are obtained in the form of the so-called "separation principle", i.e., the design methods for the finite-time output feedback laws are based on finite-time state feedback laws and finite-time observers, which can be designed separately. Conditions are given to assure that the constructed output feedback laws render the closed-loop systems globally finite-time convergent or even finite-time stable.

Keywords: Output feedback finite-time stabilization, separation principle, fractional power control, homogeneity.

1. Introduction

Because of its disturbance-rejection properties and speedy response, finite-time control design via continuous time-invariant feedback laws, especially controllers consisting of terms with fractional powers, the so-called *fractional power control* (FPC), has been studied in recent years. Most of the existing results are on finite-time stabilization by state feedback ([2] [3] [8] [9]).

However, in some practical situations, state information is not readily available and one has to consider the problem of control design via measurement feedback, namely, the output feedback. In the context of finite-time output feedback design, a rather direct approach is to combine finite-time state feedback laws with finite-time observers. To this end, a typical approach is to use finite-time dynamic state observers, together with finite-time static state feedback controllers, to arrive at nonsmooth dynamic output controllers. Unfortunately, results on simple-structured finite-time observers are relatively scarce. Some results related to finite-time observers have been given in [12] [14] [16] using different techniques. Furthermore, the problem of output feedback stabilization in finite time was studied in [14], where the system under consideration is a double integrator and the proposed finite-time observer possesses homogeneous properties.

The focus of this paper is not on particular methods of constructing finite-time state feedback laws and finite-time observers. Rather, the objective here is to obtain design conditions in the spirit of the "separation principle" to render global finite-time-stabilizing output feedback by combining finite-time state feedback laws and finite-time observers, which can be designed separately, and also to give some theoretic guideline for nonsmooth design. Although, there exist many results on discontinuous or time-varying finite-time control in the contexts of optimality and controllability,

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this paper concentrates on continuous time-invariant finite-time design. In particular, we extend the results of [14] in at least two aspects: the systems considered hereby can be nth order systems; the finite-time controllers and observers may not be homogeneous.

It is well known that the observer design and output feedback design in nonlinear control systems are much more difficult than the case of state feedback design. In addition, the so-called separation principle, which is available for linear systems [6], may not hold in general. The separation principle is quite desirable in decomposing the difficult task of design of dynamic output feedback into designs of state feedback control and observer. A even more challenging problem is the separation principle for global finite-time design because it is naturally related to nonlinearity and even nonsmoothness. Moreover, unlike the results in linear case, it is shown in [17] that, in nonlinear control systems, global complete observability and global stabilizability by state feedback are not sufficient to guarantee global stabilizability by dynamic output feedback. In fact, it is not easy at all to obtain finite-time stabilization via output feedback based on finite-time stabilizing state feedback, especially in global sense.

The paper is organized as follows. In Section 2, the mathematical formulation of the problem is presented. Then, the global finite-time convergence problem via output feedback is studied in Sections 3 and 4, by avoiding finite-time escape. After that, in Section 5, the global finite-time stabilization via output feedback is considered for a class of nonlinear systems. Concluding remarks are collected in Section 6.

2. Problem formulation

To start with, some concepts related to finite-time designs are introduced, which can be found in the existing literature such as [3] [9] [16].

Definition 2.1. Consider a system in the form of

$$\dot{x} = f(x), \qquad f(0) = 0, \quad x \in R^n, \tag{1}$$

where f is continuous. The equilibrium $x = 0$ of the system is FINITE-TIME CONVERGENT if there is a settling time function $T : R^n \to [0, \infty)$, such that, $\forall x_0$, its solution $x(t; 0, x_0)$ of system (1) with x_0 as the initial condition is defined and

$$\lim_{t \to T(x_0)} x(t; 0, x_0) = 0, \tag{2}$$

The equilibrium is FINITE-TIME STABLE if it is Lyapunov stable and finite-time convergent as well.

Definition 2.2. If there is a feedback law $u = \mu(x)$ such that the equilibrium $x = 0$ of a nonlinear system

$$\begin{cases} \dot{x} = f(x) + G(x)u & x \in R^n, \\ y = h(x), & u \in R^m, \quad y \in R^l \end{cases} \tag{3}$$

where f and G are smooth with $f(0) = 0$, is finite-time convergent (or stable), then $u = \mu(x)$ is called a finite-time convergent (or stable) controller. If there is an operator $\mathcal{F}$ with

$$\hat{x}(t) = \mathcal{F}(y, u, t) \tag{4}$$

such that $\hat{x}(t) = x(t)$ after a finite time T, then (4) is called a finite-time convergent observer. Moreover, suppose that for any $\varepsilon > 0$, there is a $\delta > 0$ such that when $||e(0)|| \le \delta$, $||e(t)|| \le \varepsilon$, where $e(t)$ denotes $e(t) = \hat{x}(t) - x(t)$. Then (4) is a finite-time (stable) observer.

The goals of the paper are to study, under what conditions,

1. the dynamic feedback

$$u = \mu(\hat{x}), \qquad \hat{x}(t) = \mathcal{F}(y, u, t) \tag{5}$$

will become finite-time convergent output feedback, i.e., to make system (3) convergent to the equilibrium $x = 0$ from any initial condition x within finite time, where $u = \mu(x)$ is finite-time convergent state feedback and (4) is a finite-time convergent observer; and

2. the dynamic feedback (5) will be finite-time stable output feedback, i.e., (3) is finite-time stable where $u = \mu(x)$ is finite-time stabilizing state feedback and (4) is a finite-time observer.

In what follows, the concept of homogeneity is recalled (referring to [5] and [1] for details).

Definition 2.3. A function $V(x)$ is called homogeneous of degree q with respect to dilation $(r_1, ..., r_n)$, where $r_i > 0$, $i = 1, ..., n$, if

$$V(\epsilon^{r_1} x_1, ..., \epsilon^{r_n} x_n) = \epsilon^q V(x), \ \epsilon > 0$$

A vector field $f(x) = (f_1(x), ..., f_n(x))^T$ is called homogeneous of degree k with respect to $(r_1, ..., r_n)$ where $r_i > 0$, $i = 1, ..., n$, if

$$f_i(\epsilon^{r_1} x_1, ..., \epsilon^{r_n} x_n) = \epsilon^{k + r_i} f_i(x), \ \epsilon > 0$$

In addition, system (1) is called homogeneous if its vector field is so.

3. Preliminary results

In this and next sections, we focus on the problem of finite-time convergence of a closed loop system via output feedback, i.e., the problem 1 stated above. Let us first consider what may happen in a finite-time design with output feedback. In fact, once the state of a proposed finite-time observer reaches the real state of the system, say at $t = T_e$, then the output feedback problem becomes a state feedback one from this point on. Therefore, to study finite-time convergent behavior of the closed-loop system in question, we focus on its dynamic behavior within the time interval $[0, T_e]$. In fact, its behavior does not influence the convergence if the closed-loop systems under feedback law (5) cannot escape during $t \in [0, T_e]$. If so, (5) will be finite-time convergent output feedback, where $u = \mu(x)$ is a finite-time convergent state feedback law and $\hat{x}$ is the state of a finite-time convergent observer. In order words, the condition that the system does not escape in finite time is sufficient to guarantee the finite-time convergence.

Thus, we will study the problem of avoiding finite-time escape. That is, the task is to prove that the closed-loop system

$$\dot{x} = f(x) + G(x)\mu(x+e), \qquad \hat{x} = x + e \tag{6}$$

with $||e(t)|| \le N_*$ and $N_* > 0$ that may depend on $e(0)$, cannot blow up in finite time, i.e., the system does not escape in finite time with a bounded signal $e(t)$ contained in control law μ.

Before further analysis, preliminary results are given first. Consider a nonlinear system

$$\dot{x} = f_0(t, x), \quad x(0) = x_0 \tag{7}$$

The following result is useful in the analysis of finite-time escape problem.

Lemma 3.1. Suppose $f_0(t, x)$ is continuous in a domain $E = \{(t, x) : T_0 < t < T_1, ||x|| < \infty\}$, where $x(t)$ is a solution of (7), and there exists a scalar function $\rho(x) \ge 0$, which is continuous and satisfies:

$$\lim_{|x| \to \infty} \rho(x) = \infty, \tag{8}$$

Moreover, if there is a continuous function $\psi(\rho) \ge 0$ in $\rho \ge 0$ and is positive when $\rho > 0$, satisfying:

$$|\frac{d\rho}{dt}|_{(7)}| \le \psi(\rho) \tag{9}$$

and

$$\int_{\rho_0}^{\infty} \frac{d\rho}{\psi(\rho)} = \infty, \tag{10}$$

for a $\rho_0 > 0$, then the solution $x(t)$ can be extended over $T_0 < t < T_1$.

Remark 3.1. When $\rho(x)$ is taken as a norm in space x and the condition (9) becomes $||f_0(t, x)|| \le \psi(\rho)$, the result becomes Wintner Theorem [11]. In fact, Lemma 3.1 can be viewed as an extended form of Wintner Theorem.

The following lemmas are useful in the analysis of given specific systems.

Lemma 3.2 (Young's inequality). for any $a >, b > 0$ and $c > 0$, we have

$$ab \le a^{1+c} + b^{1+\frac{1}{c}}$$

Lemma 3.3. When a, b, and $c < 1$ are all positive numbers, the following inequality holds:

$$(a+b)^c \le a^c + b^c$$

This result is quite straightforward.

The following example illustrates the proposed analysis procedure about the absence of finite-time escape for finite-time convergent systems.

Example 3.1. Consider a system in polar coordinates $(\bar{\rho}, \bar{\theta})$

$$\begin{cases} \dot{\bar{\rho}} = -\sin(\frac{\bar{\theta}+\pi}{2})\bar{\rho}^{\frac{1}{2}} \\ \dot{\bar{\theta}} = -\bar{\theta} \end{cases}$$

with $\bar{\rho} \ge 0$ and $\bar{\theta} \in [0, 2\pi)$.

The system is finite-time convergent. Note that during $\bar{\theta} \in [\pi, 2\pi)$, the system is divergent, but it does not blow up in any finite time. Since $\dot{\bar{\theta}} = -\bar{\theta}$, there is a moment $T_*(\bar{\rho}(0), \bar{\theta}(0))$ so that, after $t > T_*$, $\bar{\theta} \in [0, \pi/2)$, where the system is finite-time convergent. However, the system is not finite-time stable because it is not Lyapunov stable. In fact, during $\bar{\theta} \in [3\pi/2, 2\pi)$, the system behaves as a finite-time repeller [4].

Take $\rho = |\bar{\rho}| + |\bar{\theta}|$ and $\psi(\rho) = M + \rho$ where $M > \sup_t ||e(t)|| + 1$. By Lemma 3.1, with bounded signals $e_i(t), i = 1, 2$, the system

$$\begin{cases} \dot{\bar{\rho}} = -\sin(\frac{\bar{\theta}+e_2+\pi}{2})(\bar{\rho}+e_1)^{\frac{1}{2}} \\ \dot{\bar{\theta}} = -(\bar{\theta}+e_2) \end{cases}$$

with $\bar{\rho} + e_1 \ge 0$, does not blow up in finite time according to Lemmas 3.2 and 3.3.

4. Global finite-time convergence

In this section, we study the conditions for which the system with output feedback can avoid finite-time escape. As discussed in the previous section, the absence of finite-time escape in (6) implies that, with any given finite-time convergent observers, (5) becomes finite-time convergent output controller. In other words, we will discuss a "separation principle" for finite-time convergence, by avoiding finite-time escape phenomena. Once the closed-loop system (6) does not escape in finite time, the "separation principle" is obtained.

In fact, it is not easy to construct finite-time output feedback in global sense. In this paper, although the proposed methods may be used in other cases, we mainly study a class of controllable and observable systems of form:

$$\begin{cases} \dot{x}_1 = x_2^{j_1} \\ \cdots \\ \dot{x}_{n-1} = x_n^{j_n} \\ \dot{x}_n = \sum_{i=1}^n c_i x_i + u \\ y = x_1 \end{cases} \tag{11}$$

where $j_i > 0, i = 1, ..., n-1$ are odd numbers, and u and y are the control input and the output variable, respectively. When $j_i = 1, i = 1, ..., n-1$, the system becomes a typical controllable and observable linear system. In fact, a class of nonlinear systems proposed in [5] can be homogeneously approximated by system (11) with respect to some homogeneous dilations.

System (11) can be globally finite-time stabilizable by state feedback. With the help of the results in [7] and [2], the existence of its finite-time stabilizing state feedback $u = -\sum_{i=1}^n c_i x_i + \mu(x)$ is directly guaranteed if

$$\mu(\epsilon^{r_1} x_1, ..., \epsilon^{r_n} x_n) = \epsilon^{k+r_n} \mu(x), \quad \epsilon > 0, \tag{12}$$

of homogeneity degree k with respect to $(r_1, ..., r_n)$ satisfying

$$r_1 = 1, \quad r_i = \frac{r_{i-1} + k}{j_{i-1}} > -k > 0, \quad i = 2, ..., n \tag{13}$$

A following lemma can be proved with the help of Lemmas 3.1, 3.2, and 3.3.

Lemma 4.1. Consider a nonlinear system

$$\begin{cases} \dot{x}_1 = x_2^{j_1} \\ \cdots \\ \dot{x}_{n-1} = x_n^{j_{n-1}} \\ \dot{x}_n = a_0 + \sum_{i=1}^n a_i |x_i + e_i(t)|^{\alpha_i} \end{cases} \tag{14}$$

where $0 < \alpha_i \leq \frac{1}{j_i \cdots j_{n-1}}, i = 1, ..., n-1; 0 < \alpha_n < 1$, $j_i, i = 1, ..., n-1$ are positive integers; $e_i(t), i = 1, ..., n$ is continuous and bounded, say, $|e_i(t)| \leq M, i = 1, ..., n$ when $t \in [t_0, \infty)$ (t_0 is a finite nonnegative number). Then the existence interval of any solution $x(t) = (x_1(t), \cdots, x_2(t))$ of (14) can be extended over $t_0 \leq t < \infty$.

Then we have

Proposition 4.1. System (11) under the feedback $u(x+e)$ with $u(x) = -\sum_{i=1}^n c_i x_i - \mu(x)$, where μ satisfies (12), cannot blow up in finite time if $e(t)$ is bounded.

Proof: Since μ in the form of (12) is a homogeneous function of degree $k + r_n > 0$ with respect to dilation r, we have $|\mu(x)| \leq a \sum_{i=1}^n |x_i|^{\frac{k+r_n}{r_i}}$ for some $a > 0$. Thus,

$$|\sum_{i=1}^n c_i x_i + u(x+e)| \leq \sum_{i=1}^n |c_i e_i| + a \sum_{i=1}^n |x_i + e_i|^{\frac{k+r_n}{r_i}}$$

where $e(t)$ is a bounded signal. Therefore, $\sum_{i=1}^n |c_i e_i| \leq a_0$ for some suitable positive constant a_0. Note that $r_i, i = 1, ..., n$ are defined as in (13). Clearly,

$$\frac{r_i}{r_{i-1}} = \frac{r_{i-1} + k}{r_{i-1}} \frac{1}{j_{i-1}} < \frac{1}{j_{i-1}}, \quad i = 2, ..., n$$

Therefore,

$$0 < \frac{k + r_n}{r_i} < \frac{r_n}{r_i} \leq \frac{1}{j_i \cdots j_{n-1}}, \quad i = 1, ..., n-1.$$

Then according to Lemma 4.1, the conclusion follows. Q.E.D.

Therefore, if there is a finite-time convergent observer for system (11), then we can construct its finite-time convergent output feedback, based on (12). Unfortunately, up to now, we have not constructed a simple-structured finite-time convergent observer for all systems of the form (11).

Remark 4.1. In fact, a class of finite-time fractional power controllers are constructed in [8] in a backstepping-like way: $\mu(x) = u_n(x)$, where

$$\begin{cases} u_0 = 0 \\ u_{i+1}(x) = -l_{i+1} sig[sig(x_{i+1})^{j_i \beta_i} \\ -sig(u_i(x))^{\beta_i}]^{\frac{r_{i+1}+k}{r_{i+1} j_i \beta_i}}, \quad i = 0, 1, ..., n-1 \end{cases} \tag{15}$$

with (13) and $j_0 = 1, \beta_0 = 1, (j_i \beta_i + 1) r_{i+1} \geq (j_{i-1} \beta_{i-1} + 1) r_i, i = 1, ..., n-1$, and $l_i > 0$ are suitable constants for $i = 1, 2, ...n$. Here, for convenience, define $sig(z)^\alpha = |z|^\alpha sgn(z)$, where $sgn(\cdot)$ is the sign function.

Other cases, different from that of Proposition 4.1, can still be studied with the proposed method as illustrated in the following example.

Example 4.1. Consider a system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = c_1 x_1^2 + 2 c_2 x_1 x_2 + u \\ y = x_1 \end{cases} \tag{16}$$

with $c_i, i = 1, 2$ constants. Then the control law, $u(x) = -c_1 x_1^2 - 2 c_2 x_1 x_2 - sat(x_1^{\frac{1}{5}}) - sat(x_2^{\frac{1}{3}})$, $sat(\cdot)$ denotes the saturation function, is globally finite-time stabilizing [10]. In fact, when $c_1 = c_2 = 0$, the control is bounded.

Take $z_1 = x_1, z_2 = -c_2x_1^2 + x_2$. Then we have

$$\begin{cases} \dot{z}_1 = z_2 + c_2z_1^2 \\ \dot{z}_2 = c_1z_1^2 + u \\ y = z_1 \end{cases}$$

With the help of [14], its finite-time observer of the system can be taken as

$$\begin{cases} \dot{\hat{z}}_1 = \hat{z}_2 + c_2y^2 - (\hat{z}_1 - y)^{\frac{3}{5}} \\ \dot{\hat{z}}_2 = c_1y^2 + u - (\hat{z}_1 - y)^{\frac{1}{5}} \end{cases}$$

Define $\hat{x}_1 = \hat{z}_1$, $\hat{x}_2 = \hat{z}_2 + c_2\hat{z}_1^2$. Note that $e_1 = \hat{x}_1 - x_1$ and $e_2 = \hat{x}_2 - x_2$ are bounded and will become 0 in finite time. Assume $|e_1| + |e_2| < M$ for all $t > 0$ with $M > 1$ depending on the initial condition $e(0)$.

Take $T_0 = 0, T_1 = \infty$, and $\rho = ||x||$ and $\psi(\rho) = 2+(|c_1|+2|c_2|)M^2+2(1+|c_1|+|c_2|)M\rho$ in Lemma 3.1, we have that the solution of the closed loop system with output feedback $u(x+e)$ exists in $[0,\infty)$, and therefore, cannot blow up in finite time. Thus, the finite-time convergence of the closed-loop system via output feedback can be obtained.

Remark 4.2. For linear systems, another class of control laws are provided in [15]. For instance, consider $\dot{x}_1 = x_2$, $\dot{x}_2 = u$, $y = x_1$. Take $u(x) = -l_1\bar{r}(x)^{-2\tau}x_1 - l_2\bar{r}(x)^{-\tau}x_2$, with $\bar{r}$ as a radius ($\bar{r}(0) = 0$) and $l_i > 0, i = 1,2$ suitable constants. If $\tau > 0$, the control law is finite-time stabilizing [15]. In fact, when τ is small enough, we have $|\bar{r}(x)^{-(3-i)\tau}x_i| \leq M_i(1+|x_i|)$, $i = 1,2$ for suitable $M_i, i = 1,2$. Then there is a constant $M \geq 1$, depending on the boundary of e and $M_i, i = 1,2$, such that $|u(x+e)| \leq M(1+||x||)$. This implies that system (16) with feedback $u(x+e)$ and a finite-time convergent observer is finite-time convergent, by Lemma 3.1.

5. Global finite-time stability

In this section, we will give some conditions to guarantee the "separation principle" of finite-time stable design rather than only finite-time convergent design as discussed in Sections 3 and 4. In this case, the condition to avoid finite-time escape, is not sufficient because we have to care about Lyapunov stability.

Lemma 5.1. Suppose that the given observer (4) is finite-time stable and the state feedback $u = \mu(x)$ can render (3) finite-time stable. Consider the system

$$\dot{x} = F(x,e) = f(x) + G(x)\mu(x+e), \qquad (17)$$

with $x \in R^n, e \in R^n, F(0,0) = 0$ and initial condition $x(0) = x_0$. If there is a C^1 proper and positive definite function $V : R^n \to R^+$ with $V(0) = 0$, such that, by letting $\rho(x) = \sum_{i=1}^n |x_i|^{d_i}$ with $d_i > 0, i = 1,..,n$ and

$$\dot{V}|_{(17)} \leq -K_1(\rho(x)) + K_2(\rho(e)) \qquad (18)$$

for $K_i, i = 1,2$ are continuous functions that are strictly increasing when with $K_i(0) = 0, i = 1,2$, then the output feedback law of form (5) is finite-time stabilizing.

For general nonsmooth finite-time stable systems, many complex phenomena may happen, and we cannot ensure that they admit suitable C^1 Lyapunov functions [4]. To facilitate Lyapunov type of analysis, we still consider the system (11). The next proposition can be viewed as a further result of Proposition 4.1.

Define

$$||x||_r = (\sum_{i=1}^n |x_i|^{\frac{c}{r_i}})^{\frac{1}{c}}$$

where $c > \max\{r_1, ..., r_n\}$. Obviously, $||x||_r = \rho^{\frac{1}{c}}$ with $d_i = \frac{c}{r_i}$, where ρ is defined in Lemma 5.1.

Proposition 5.1. System (11) under feedback $u = -\sum_{i=1}^n c_i\hat{x}_i + \mu(\hat{x})$ with $\mu(x)$ is in the form of (12) is a finite-time stabilizing output feedback if $\hat{x}$ is the state of its finite-time stable observer.

To prove this, we need some other lemmas.

Lemma 5.2. Consider continuous functions $V_1(x), V_2(e)$, and $\hat{V}(x,e)$ ($x \in R^n$ and $e \in R^n$), which are with the same homogeneity degree with dilation $r \in R^n$, r, and $(r,r) \in R^{2n}$, respectively. Suppose that V_1 and V_2 are positive definite with respect to vectors x and e, respectively, and $\hat{V}(x,0) = 0$. Then for any given $b_1 > 0$, there is a positive constant b_2 such that

$$\hat{V}(x,e) \leq b_1V_1(x) + b_2V_2(e)$$

Lemma 5.3. Suppose that a given observer (4) is finite-time stable and a state feedback controller $\mu(x)$ can render (3) finite-time stable, namely

$$\dot{x} = F(x,0), \quad x \in R^n, \qquad (19)$$

Suppose that the system (17) satisfies: for $i = 1, ..., n$,

$$F_i(\epsilon^{r_1}x_1, ..., \epsilon^{r_n}x_n, \epsilon^{r_1}e_1, ..., \epsilon^{r_n}e_n) = \epsilon^{k+r_i}F_i(x,e) \qquad (20)$$

then the output feedback law of form (5) is finite-time stabilizing.

Proof of Proposition 5.1. Obviously, system (11) with $c_i = 0, i = 1, ..., n$ under feedback $\mu(x+e)$ with $\mu(x)$ is in the form of (12) satisfies the condition (20), following Lemma 5.3. In this case, since the closed-loop system with $\mu(x)$ is homogeneous, there is a C^1 Lyapunov function $V_0(x)$ of degree σ_0 with respect to $(r_1, ..., r_n)$ given in (13), for the system [1]. Then in this case, $\dot{V}_0$ is homogeneous of degree $\sigma_0 + k$ ($-r_n < k < 0$) and negative definite.

Consider the case when $c_i, i = 1, .., n$ may not be 0. Then the control will be $u(x) = -\sum_{i=1}^n c_ix_i - \mu(x)$.

However, Lemma 5.3 cannot be employed directly because the condition (20) is not satisfied.

Still use the function V_0 for system

$$\begin{cases} \dot{x}_1 = x_2^{j_1}, \\ \cdots \\ \dot{x}_{n-1} = x_n^{j_{n-1}} \\ \dot{x}_n = -\sum_{i=1}^n c_i e_i + \mu(x+e) \end{cases} \tag{21}$$

Its derivative for system (21) becomes

$$\dot{V}(x) \leq -b_5 ||x||_r^{\sigma_0+k} + b_6 ||e||_r^{\sigma_0+k} + b_7 (\sum_{i=1}^n |c_i||e_i|)^{\sigma_0+k}$$

for some positive constants $b_i, i = 5, 6, 7$, because the following inequality holds:

$$\frac{\partial V_0}{\partial x_n}(-\sum_{i=1}^n c_i e_i) \leq b_8 ||x||_r^{\sigma_0+k} + b_7 |\sum_{i=1}^n c_i e_i|^{\frac{\sigma_0+k}{r_n+k}}$$

for $b_7 > 0, b_8 > 0$ with b_8 small enough, based on Young's inequality. This leads to the conclusion.

Q.E.D.

Example 5.1. Consider

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_1 + x_2 + u \\ y = x_1 \end{cases} \tag{22}$$

Note that $u = -x_1 - x_2 - x_1^{\frac{1}{5}} - x_2^{\frac{1}{3}}$ is finite-time stable [2].

Moreover, from [14], we can propose a class of finite-time observers in the form:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 - e_1^{\frac{5}{7}} - e_1 \\ \dot{\hat{x}}_2 = \hat{x}_1 + \hat{x}_2 + u - e_1^{\frac{3}{7}} - 2e_1 - e_1^{\frac{5}{7}} \end{cases} \tag{23}$$

where $e = \hat{x} - x$. Then, by Proposition 5.1,

$$u = -\hat{x}_1 - \hat{x}_2 - \hat{x}_1^{\frac{1}{5}} - \hat{x}_2^{\frac{1}{3}},$$

is finite-time stable output feedback, together with observer (23).

6. Conclusions

In this paper the problems of global finite-time design via dynamic output feedback are addressed. Two different problems are studied, namely, finite-time convergent output feedback and finite-time stable output feedback. The results are given in the spirit of the so-called "separation principle". With the results, the finite-time design via output feedback laws is based on finite-time state feedback laws and finite-time observers, which can be designed separately.

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