Block Control of Linear Time Invariant Systems with Delay

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Abstract

This paper applies the block control method to form a decomposed control law suitable for multivariable linear time-delay systems. A Block Controllable Form is introduced, and a non-singular transformation that reduces the system to this form, is proposed. Conditions of stability of the closed-loop system are derived.

1 Introduction

The feedback stabilization of time-delay systems remains is one of most interest problems in control. This problem has been extensively studied and several controllers and stability criteria based on optimal control method [1] including H_∞ and LMI approaches [2], have been proposed. The common feature of the referred works is that their derivations are based on analysis of full order system.

In this paper, to stabilize linear time invariant systems with delayed state and input, we use the block control principle which is fruitful and relatively simple, especially when dealing with multivariable systems because the control problem is decomposed into a number of simpler sub-problems of lower dimensions. In order to achieve this, a special state representation must be used which will be referred to as the Block Controllable Form (or BC-form), consisting of a set of controlled blocks. This approach has successfully been employed for decomposition and control of linear time systems [3]. Here, the possibility of applying the same method for stabilization of linear delayed systems is investigated.

2 BC-Form for Systems with Delay

Consider a linear time-delay system described by the following state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{C}\mathbf{x}(t-\tau) + \mathbf{B}\mathbf{u}(t) + \mathbf{D}\mathbf{u}(t-\tau) \qquad (1)$$

where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$, $\mathbf{x}(t) = \boldsymbol{\varphi}(t) \quad \forall t \in [t_0 - \tau, t_0]$, $\boldsymbol{\varphi}(t)$ is a continuous vector-valued initial function.

The essential feature of the proposed method is the conversation of the system (1) to the BC-form consisting of *r* blocks:

$$\begin{aligned}
\dot{\mathbf{x}}_r(t) &= \mathbf{A}_r \mathbf{x}_r(t) + \mathbf{C}_r \mathbf{x}_r(t-\tau) + \mathbf{B}_r[\mathbf{x}_{r-1}(t) + \mathbf{P}_r \mathbf{x}_{r-1}(t-\tau)] \\
\dot{\mathbf{x}}_i(t) &= \mathbf{A}_i \bar{\mathbf{x}}_i(t) + \mathbf{C}_i \bar{\mathbf{x}}_i(t-\tau) + \mathbf{B}_i[\mathbf{x}_{i-1}(t) + \mathbf{P}_i \mathbf{x}_{i-1}(t-\tau)] \\
\dot{\mathbf{x}}_1(t) &= \mathbf{A}_1 \bar{\mathbf{x}}_1(t) + \mathbf{C}_1 \bar{\mathbf{x}}_1(t-\tau) + \mathbf{B}_1[\mathbf{u}(t) + \mathbf{P}_1 \mathbf{u}(t-\tau)] \\
& i = 2,\dots,r-1 \qquad (2)
\end{aligned}$$

where $\mathbf{x} = (\mathbf{x}_1,\dots,\mathbf{x}_r)^T$, $\bar{\mathbf{x}}_i = (\mathbf{x}_1,\dots,\mathbf{x}_i)^T$, $\mathbf{x}_i \in \mathbf{R}^{n_i}$, and

$$rank\mathbf{B}_i = n_i,\ i = 1,\dots,r,\ \sum_{i=1}^{r} n_i = n.$$

The procedure for transforming plant (1) to the form (2) comprises $r-1$ steps. On the first step we introduce the following assumptions

A11) $rank\mathbf{B} = n_1 = m$.

A12) There is a matrix $\mathbf{P}_1 \in R^{n_1 \times n_1}$ such that

$$\mathbf{D} = \mathbf{B}\mathbf{P}_1$$

Under these assumptions, a change of variables

$$\mathbf{x}' = \mathbf{M}_1 \mathbf{x},\ \mathbf{x}' = \mathbf{M}_1 = \begin{bmatrix} \mathbf{I}_{n-n_1} & -\mathbf{B}_{12}\mathbf{B}_1^{-1} \\ 0 & \mathbf{I}_{n_1} \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} \mathbf{B}_{12} \\ \mathbf{B}_1 \end{bmatrix}$$

transforms the system (1) into

$$\begin{aligned}
\dot{\mathbf{x}}'_2(t) &= \mathbf{A}'_{22}\mathbf{x}'_2(t) + \mathbf{C}'_{22}\mathbf{x}'_2(t-\tau) + \mathbf{B}'_2\mathbf{x}_1(t) + \mathbf{D}'_2\mathbf{x}_1(t-\tau) \\
\dot{\mathbf{x}}_1(t) &= \mathbf{A}_{12}\mathbf{x}'_2(t) + \mathbf{C}_{12}\mathbf{x}'_2(t-\tau) + \mathbf{A}_{11}\mathbf{x}_1(t) + \mathbf{C}_{11}\mathbf{x}_1(t-\tau) \\
&\quad + \mathbf{B}_1[\mathbf{u}(t) + \mathbf{P}_1\mathbf{u}(t-\tau)]
\end{aligned}$$

where $\mathbf{x}' = (\mathbf{x}_1, \mathbf{x}'_2)^T \quad \mathbf{x}_1 \in \mathbf{R}^{n_i}$, $\mathbf{x}'_2 \in \mathbf{R}^{n-n_i}$.

Lemma: Let

i) *The system* (1) *be controllable*.

ii) *Assumptions* **A11** *and* **A12** *hold*.

iii) *In the system*

$$\begin{aligned}
\dot{\mathbf{x}}'_k(t) &= \mathbf{A}'_k\mathbf{x}'_k(t) + \mathbf{C}'_k\mathbf{x}'_k(t-\tau) + \mathbf{B}'_k[\mathbf{x}_{k-1}(t) + \mathbf{D}'_k\mathbf{x}_{k-1}(t-\tau)] \\
\dot{\mathbf{x}}_i(t) &= \mathbf{A}_i\bar{\mathbf{x}}_i(t) + \mathbf{C}_i\bar{\mathbf{x}}_i(t-\tau) + \mathbf{B}_i[\mathbf{x}_{i-1}(t) + \mathbf{P}_i\mathbf{x}_{i-1}(t-\tau)] \\
\dot{\mathbf{x}}_1(t) &= \mathbf{A}_1\bar{\mathbf{x}}_1(t) + \mathbf{C}_1\bar{\mathbf{x}}_1(t-\tau) + \mathbf{B}_1[\mathbf{u}(t) + \mathbf{P}_1\mathbf{u}(t-\tau)],\ i = 2,\dots,k-1
\end{aligned}$$

with $rank\mathbf{B}_i = n_i$, $i = 1,\dots,k-1$, *obtained at the* $(k-1)^{th}$ *step of the transformation procedure, the following conditions hold*

Ak1) $rank\mathbf{B}'_k = n_k = m$.

Ak2) *There is a matrix* $\mathbf{P}_k \in R^{n_k \times n_k}$ *such that*

$$\mathbf{D}'_k = \mathbf{B}'_k\mathbf{P}_k$$

Then, there exists an integer $r \le n$ *such that the system* (1) *takes the form* (2).

3 Block Control Design

It is more conveniently to renumber the variables of (2) as

$$\dot{\mathbf{x}}_1(t) = \mathbf{A}_1\mathbf{x}_1(t) + \mathbf{C}_1\mathbf{x}_1(t-\tau) + \mathbf{B}_1\mathbf{v}_1(t)$$
$$\dot{\mathbf{x}}_i(t) = \mathbf{A}_i\bar{\mathbf{x}}_i(t) + \mathbf{C}_i\bar{\mathbf{x}}_i(t-\tau) + \mathbf{B}_i\mathbf{v}_i(t),\ i = 2,...,r-1 \quad (3)$$
$$\dot{\mathbf{x}}_r(t) = \mathbf{A}_r\bar{\mathbf{x}}_r(t) + \mathbf{C}_r\bar{\mathbf{x}}_r(t-\tau) + \mathbf{B}_r\mathbf{v}_r(t)$$

where $\mathbf{v}_i(t) = \mathbf{x}_{i+1}(t) + \mathbf{P}_i\mathbf{x}_{i+1}(t-\boldsymbol{\tau})$, $i = 1,...,r-1$, and

$$\mathbf{v}_r(t) = \mathbf{u}(t) + \mathbf{P}_r\mathbf{u}(t-\boldsymbol{\tau}).$$

A control strategy can be designed for (3) considering $\mathbf{v}_i$ as a fictitious control vector in the i^{th} block of (3). This procedure is outlined in the following.

Step 1. Let the fictitious control $\mathbf{v}_1(t)$ in (3) be chosen as

$$\mathbf{v}_1(t) = \mathbf{v}_{1c}(t) + \mathbf{B}_1^{-1}[\mathbf{K}_1\mathbf{y}_1(t) + \mathbf{y}_2(t)] \quad (4)$$

where $\mathbf{y}_1 = \mathbf{x}_1$, $\mathbf{y}_2 \in R^{n_2}$, $\mathbf{K}_1$ is a Hurwitz matrix, and $\mathbf{v}_{1c}$ is calculated from $\dot{\mathbf{y}}_1(t) = 0$ of the form

$$\mathbf{v}_{1c}(t) = -\mathbf{B}_1^{-1}[\mathbf{A}_1\mathbf{x}_1(t) + \mathbf{C}_1\mathbf{x}_1(t-\tau)] \quad (5)$$

The transformed first block of (3) in new coordinates $\mathbf{y}_1$ and $\mathbf{y}_2$ with input (4) and (5) has the desired form

$$\dot{\mathbf{y}}_1(t) = \mathbf{K}_1\mathbf{y}_1(t) + \mathbf{y}_2(t) \quad (6)$$

The algorithm (4)-(5) defines the transformation for $\mathbf{y}_2(t)$

$$\mathbf{y}_2(t) = (\mathbf{A}_1 - \mathbf{K}_1)\mathbf{x}_1(t) + \mathbf{C}_1\mathbf{x}_1(t-\boldsymbol{\tau}) + \mathbf{B}_1\mathbf{v}_1(t) \quad (7)$$
$$\mathbf{v}_1(t) = \mathbf{x}_2(t) + \mathbf{P}_1(t)\mathbf{x}_2(t-\tau)$$

Step 2. Taking the derivative of (7), we have

$$\dot{\mathbf{y}}_2(t) = \sum_{j=1}^{2}[\mathbf{A}_{2,j}\mathbf{x}_j(t) + \mathbf{C}_{2,j}^1\mathbf{x}_j(t-\tau) + \mathbf{C}_{2,j}^2\mathbf{x}_j(t-2\tau)] + \bar{\mathbf{B}}_2\mathbf{v}_2^1(t)$$

where $\bar{\mathbf{B}}_2 = \mathbf{B}_1\mathbf{B}_2$ and $\mathbf{v}_2^1(t) = \mathbf{v}_2(t) + \mathbf{B}_2^{-1}\mathbf{P}_1\mathbf{B}_2\mathbf{v}_2(t-\tau)$.

As on the first step the fictitious control input $\mathbf{v}_2^1(t)$ is chosen similar to (4) and (5), namely

$$\mathbf{v}_2^1(t) = \mathbf{v}_{2c}^1(t) + \bar{\mathbf{B}}_2^{-1}[\mathbf{K}_2\mathbf{y}_2(t) + \mathbf{y}_3(t)] \quad (8)$$

where $\mathbf{y}_3 \in R^{n_3}$, $\mathbf{K}_2$ is a Hurwitz matrix, and $\mathbf{v}_{2c}^1(t)$ again is calculated from $\dot{\mathbf{y}}_2(t) = 0$ as

$$\mathbf{v}_{2c}^1(t) = -\bar{\mathbf{B}}_2^{-1}\sum_{j=1}^{2}[\mathbf{A}_{2,j}\mathbf{x}_j(t) + \mathbf{C}_{2,j}^1\mathbf{x}_j(t-\tau) + \mathbf{C}_{2,j}^2\mathbf{x}_j(t-2\tau)]$$

Thus, equation for $\mathbf{y}_2$ with (8) takes the same form of (6):

$$\dot{\mathbf{y}}_2(t) = \mathbf{K}_2\mathbf{y}_2(t) + \mathbf{y}_3(t)$$

Then algorithm (8) gives the following transformation

$$\mathbf{y}_3 = \sum_{j=1}^{2}[\mathbf{A}_{2,j}\mathbf{x}_j(t) + \mathbf{C}_{2,j}^1\mathbf{x}_j(t-\boldsymbol{\tau}) + \mathbf{C}_{2,j}^2\mathbf{x}_j(t-2\boldsymbol{\tau})]$$
$$-\mathbf{K}_2\mathbf{y}_2(t) + \bar{\mathbf{B}}_2\mathbf{v}_2^1(t),$$

$\mathbf{v}_2^1(t) = \mathbf{v}_2(t) + \mathbf{B}_2^{-1}\mathbf{P}_1\mathbf{B}_2\mathbf{v}_2(t-\boldsymbol{\tau})$, $\mathbf{v}_2(t) = \mathbf{x}_3(t) + \mathbf{P}_2\mathbf{x}_3(t-\tau)$

This procedure can be performed iteratively obtaining on the k^{th} step, $k = 3,...,r-1$ the recursive transformation

$$\mathbf{y}_{k+1} = \sum\nolimits_{j=1}^{k}[\mathbf{A}_{k,j}\mathbf{x}_j(t) + \mathbf{C}_{k,j}^1\mathbf{x}_j(t-\boldsymbol{\tau}) + \cdots + \mathbf{C}_{k,j}^k\mathbf{x}_j(t-k\boldsymbol{\tau})]$$
$$-\mathbf{K}_k\mathbf{y}_k(t) + \bar{\mathbf{B}}_k\mathbf{v}_k^1(t),$$

$$\mathbf{v}_k^1(t) = \mathbf{v}_k^2(t) + \bar{\mathbf{B}}_k^{-1}\cdots\bar{\mathbf{B}}_2^{-1}\mathbf{P}_1\bar{\mathbf{B}}_2\cdots\bar{\mathbf{B}}_k\mathbf{v}_k^2(t-\tau),$$
$$\mathbf{v}_k^2(t) = \mathbf{v}_k^3(t) + \bar{\mathbf{B}}_{k-1}^{-1}\cdots\bar{\mathbf{B}}_3^{-1}\mathbf{P}_2\bar{\mathbf{B}}_3\cdots\bar{\mathbf{B}}_{k-1}\mathbf{v}_k^3(t-\tau),$$
$$\cdots$$
$$\mathbf{v}_k^{k-1}(t) = \mathbf{v}_k(t) + \bar{\mathbf{B}}_k^{-1}\mathbf{P}_{k-1}\bar{\mathbf{B}}_k\mathbf{v}_k(t-\tau),$$
$$\mathbf{v}_k(t) = \mathbf{x}_{k+1}(t) + \mathbf{P}_k\mathbf{x}_{k+1}(t-\tau).$$

where $\mathbf{P}_k$ is a Hurwitz matrix, and $\bar{\mathbf{B}}_k = \mathbf{B}_1\cdots\mathbf{B}_k$.

On the last step, the system (2) can be presented in the new variables of the form

$$\dot{\mathbf{y}}_i(t) = \mathbf{K}_i\mathbf{y}_i(t) + \mathbf{y}_{i+1}(t),\ i = 1,...,r-1 \quad (9)$$

$$\dot{\mathbf{y}}_r = \sum_{j=1}^{r}[\mathbf{A}_{r,j}\mathbf{x}_j(t) + \mathbf{C}_{r,j}^1\mathbf{x}_j(t-\tau) + \cdots + \mathbf{C}_{r,j}^r\mathbf{x}_j(t-r\tau)] + \bar{\mathbf{B}}_r\mathbf{v}_r^1$$

where $\mathbf{v}_r^1(t) = \mathbf{v}_r^2(t) + \bar{\mathbf{B}}_r^{-1}\cdots\bar{\mathbf{B}}_2^{-1}\mathbf{P}_1\bar{\mathbf{B}}_2\cdots\bar{\mathbf{B}}_r\mathbf{v}_r^2(t-\tau)$,

$$\cdots$$
$$\mathbf{v}_r(t) = \mathbf{u}(t) + \mathbf{P}_r\mathbf{u}(t-\tau).$$

A choice of the control $\mathbf{v}_r^1(t)$ similar to (4)-(5) form

$$\mathbf{v}_r^1(t) = \mathbf{v}_{rc}^1(t) + \bar{\mathbf{B}}_r^{-1}\mathbf{K}_r\mathbf{y}_r(t) \quad (10)$$

$$\mathbf{v}_{rc}^1 = -\bar{\mathbf{B}}_r^{-1}\sum_{j=1}^{r}[\mathbf{A}_{r,j}\mathbf{x}_j(t) + \mathbf{C}_{r,j}^1\mathbf{x}_j(t-\tau) + \cdots + \mathbf{C}_{r,j}^r\mathbf{x}_j(t-r\tau)]$$

provides the closed-loop dynamics (9) and (10) as

$$\dot{\mathbf{y}}_i(t) = \mathbf{K}_i\mathbf{y}_1(t) + \mathbf{y}_{i+1}(t),\quad i = 1,...,r-1$$
$$\dot{\mathbf{y}}_r(t) = \mathbf{K}_r\mathbf{y}_r(t)$$

The stability conditions of the closed loop system are presented in the following theorem.

Theorem: *If all eigenvalues of matrices* $\mathbf{P}_i$, $i = 1,...,r$ *are located inside unit circle, then the system* (2) *with control strategy* (10) *is asymptotically stable.*

4 Conclusions

The block control method has been formulated for control of a class of linear time-delay systems, which can be transformed into BC-form. The proposed transformation and control design procedures have step-by step character and simplify the solution of the problem. This method enables to solve one of the important classical problem: design of pole placement state feedback for linear systems with delayed state and control input.

References

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