Lyapunov Coupled Equations For Infinite Jump Linear Systems [1]

Marcelo D. Fragoso

Department of Systems and Control
National Lab. for Scient. Comput. - LNCC/CNPq
Av.Getulio Vargas 333,
25651-070, Petrópolis, RJ, Brazil
email: frag@lncc.br

Jack Baczynski

Systems and Computing Eng. Program
Federal University of Rio de Janeiro
Bloco H, Ilha do Fundão,
21945-970, Rio de Janeiro, RJ, Brazil
email: jack@lncc.br

Abstract

This paper deals with Lyapunov equations for continuous-time Markov jump linear systems (MJLS). Out of the bent which wends most of the literature on MJLS, we focus here on the case in which the Markov chain has a *countably infinite* state space . It is shown that the infinite MJLS is stochastically stabilizable (SS) if and only if the associated countably infinite coupled Lyapunov equations have a unique norm bounded strictly positive solution. It is worth mentioning here that this result do not hold for mean square stabilizability (MSS), since SS and MSS are no longer equivalent in our set up (see, e.g., [2]). To some extent, tools from operator theory in Banach space and, in particular, from semigroup theory are the very technical underpinning of the paper.

1 Introduction

Our main concern in this paper is with the so-called class of continuous time Markov jump linear systems (MJLS). Considering a finite set for the jump Markov chain state space, the following stability result was established in [19] (Theorem 1): *a MJLS is stochastically stabilizable if and only if the associated Lyapunov equation has a unique positive definite solution.* Our aim here is to extend this result for the *countably infinite* state space case. This paper is a continuation of our previous effort of extending the results available for MJLS to the countably infinite state space case (see, e.g., [2] and [16]), by framing the problems in terms of operator theory.

A cursory examination of the proof of Theorem 1 in [19] reveals that a fundamental aspect in the "sufficiency" part of the proof is the explicit use of the fact that the Markov chain takes values in a finite set. Since it was not possible to adequate the technique used in [19] to our setting, we get around this problem by making use of a germane result proved in [2] (reproduced here as Lemma 2), which shows that the concept of stochastic stabilizability (SS) is bounded up with the spectrum of a certain infinite dimensional Banach space operator.

For the "necessity" part of the proof of our main result, we follow closely the spirit of the proof in [19]. However we deal with some technical issues which appears specifically in the countably infinite scenario and unveil some significant differences from the proof in [19]. For instance, we must assure that the infinite sum of the $o(\cdot)$ functions related to the transition probability function of $\{\theta\}$ is an $o(\cdot)$ function. In addition, differentiability must focus on certain decomplexifications ${}^{R}g$ *in lieu* of the original functions g (see Section 3 of the Appendix). Moreover, we strongly benefited from Lemma 2, mentioned above, to rigorously extend a finite dimensional auxiliary result.

The more general complex setting to the problem is necessary not just for the sake of comprehensiveness, but because it is a well-known fact that in dealing with the spectrum of an operator, it is always tacitly more convenient to assume that the operator is defined on a complex space. Indeed, this is true, for instance, to perform in [2] the proof of Lemma 2 mentioned above. On the other hand, this more general framework prevent us from employing the usual procedures to specify the infinitesimal generator of the Markov process $\{x, \theta\}$ applied to a certain (nonholomorfic) quadratic functional with domain in the complex space $\mathbb{C}^n$. For this case, as in [2], we consider a natural adaptation of the decomplexification concept, described in [1, Section 18], for nonlinear complex functions with range in $\mathbb{R}$, from which we establish a certain version of a gradient concept and, from this, the linear approximation to nonholomorfic functionals. With such a tool in hand, we are allowed to conveniently specify the above infinitesimal generator.

There is nowadays an extensive theory surrounding MJLS. An initial trickle of papers ([26], [28]) soon grew to a considerable amount (see, e.g., [2], [3], [5], [7] - [14], [16], [17], [19], [20], [23], [26], [28], and references therein), with a sober eye towards applications, as befits a maturing field (see, e.g., [4],[18], [22], [23], [26] and

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references therein). Potential applications include, *inter alia*, safety-critical and high-integrity systems (e.g., aircraft, chemical plants, nuclear power station, robotic manipulator systems, large scale flexible structures for space stations such as antenna, solar arrays, etc.), typically, systems which may experience abrupt changes in their structure. In addition, the recent unfolded connection between linear dynamically varying (LDV) systems and jump linear systems, which has been exploited in [6], will certainly give a new impetus to LSMJP in the intervening years.

The outline of the content of this paper is as follows. In Section 2 we provide the bare essential of notations and assumptions. The model description is stated in Section 3. Some preliminaries are given in Section 4. The main result is exhibited in Section 5. We conclude the paper with an appendix which provides some support results.

2 Notations and Assumptions

As usual, $\mathbb{C}^n$ (resp. $\mathbb{R}^n$) stands for the n-dimensional Euclidean space over the field of complex (resp. real) numbers $\mathbb{C}$ (resp. $\mathbb{R}$) and $\mathbb{N} = \{1, 2, ...\}$. We define by $\mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$ the normed linear space of all n by m complex matrices and, for simplicity, write $\mathbb{M}(\mathbb{C}^n)$ whenever $n = m$. We use the superscripts $^-$, $'$ and * for complex conjugate, transpose and conjugate transpose, respectively. The notation $L \geq 0$ and $L > 0$ is adopted if a self-adjoint matrix is positive semidefinite or positive definite, respectively. We denote $\mathbb{M}(\mathbb{C}^n)^+ = \{L \in \mathbb{M}(\mathbb{C}^n); L = L^* \geq 0\}$ and write $\|\cdot\|_L$ for the norm in $\mathbb{C}^n$ induced by the inner product $\langle x, y\rangle_L = x^* L y$ whenever the matrix $L = L^* \geq 0$. Furthermore, $\|\cdot\|_Y$ indicates a norm in the space Y. Except otherwise mentioned, $\|\cdot\|$ represents either the Euclidean norm in $\mathbb{C}^n$ or the spectral induced norm in $\mathbb{M}(\mathbb{C}^n)$. In addition, to avoid notational confusion with the summation index i and j, we denote by ι the pure imaginary complex number.

Remark 1 *Every element in $\mathbb{M}(\mathbb{C}^n)$ has a Cartesian self adjoint decomposition [24, pg 376] and every self adjoint operator in $\mathbb{M}(\mathbb{C}^n)$ can be decomposed in positive and negative parts [24, pg 464]. Thus, for any $L \in \mathbb{M}(\mathbb{C}^n)$, there exist X^+, X^-, Y^+, Y^- in $\mathbb{M}(\mathbb{C}^n)^+$ such that $L = (X^+ - X^-) + \iota (Y^+ - Y^-)$. Moreover $X^+ \leqslant X^+ + X^- = (L + L^*)/2$ and thus $\|X^+\| \leqslant \|L\|$. Similarly $\|X^-\| \leqslant \|L\|$, $\|Y^+\| \leqslant \|L\|$ and $\|Y^-\| \leqslant \|L\|$.*

Set $\mathcal{H}_1^{m,n}$ (resp. $\mathcal{H}_\infty^{m,n}$) the linear space made up of all infinite sequences of complex matrices $H = (H_1, H_2, ...)$, $H_i \in \mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$ such that $\sum_{i=1}^\infty \|H_i\| < \infty$ (resp. $\sup\{\|H_i\|,\ i \in \mathcal{S}\} < \infty$). For $H \in \mathcal{H}_1^{m,n}$ (resp. $H \in \mathcal{H}_\infty^{m,n}$) we define a norm in $\mathcal{H}_1^{m,n}$ (resp. $\mathcal{H}_\infty^{m,n}$) by $\|H\|_1 = \sum_{i=1}^\infty \|H_i\|$ (resp. $\|H\|_\infty = \sup\{\|H_i\|,\ i \in \mathcal{S}\}$). We shall write $\mathcal{H}_1^n$ and $\mathcal{H}_\infty^n$ whenever $n = m$, the positive semidefinite sets $\mathcal{H}_1^{n^+} = \{H \in \mathcal{H}_1^n,\ H_i \in \mathbb{M}(\mathbb{C}^n)^+,\ i \in \mathcal{S}\}$ and $\mathcal{H}_\infty^{n^+} = \{H \in \mathcal{H}_\infty^n,\ H_i \in \mathbb{M}(\mathbb{C}^n)^+,\ i \in \mathcal{S}\}$. For $H = (H_1, H_2, ...)$ and $L = (L_1, L_2, ...)$ in $\mathcal{H}_1^{n^+}$ we shall use the notation $H \leqslant L$ to indicate that $H_i \leqslant L_i$ for each i in $\mathbb{N}$. We have that

$$H \leqslant L \Rightarrow \|H\|_1 \leqslant \|L\|_1 \tag{1}$$

We shall write $\mathring{\mathcal{H}}_\infty^{n^+} = \{H \in \mathcal{H}_\infty^{n^+},\ H_i > \alpha_H I$ for some $\alpha_H > 0,\ i \in \mathcal{S}\}$ and shall say that each element of $\mathring{\mathcal{H}}_\infty^{n^+}$ is strictly positive.

Remark 2 *Consider $Q = (Q_1, Q_{2,...}) \in \mathcal{H}_1^n$. From Remark 1, $Q_i = (X_i^+ - X_i^-) + \iota (Y_i^+ - Y_i^-)$, where X_i^+, X_i^-, Y_i^+ and Y_i^- belong to $\mathbb{M}(\mathbb{C}^n)^+$. Now, define $X^+ = (X_1^+, X_2^+, ...)$, $X^- = (X_1^-, X_2^-, ...)$, $Y^+ = (Y_1^+, Y_2^+, ...)$ and $Y^- = (Y_1^-, Y_2^-, ...)$. Since $Q \in \mathcal{H}_1^n$, it follows, again from Remark 1, that X^+, X^-, Y^+ and Y^- also belong to $\mathcal{H}_1^n$. Therefore, Q can always be decomposed as $Q = (X^+ - X^-) + \iota (Y^+ - Y^-)$ with X^+, X^-, Y^+ and Y^- in $\mathcal{H}_1^{n^+}$.*

For any complex Banach space X, we denote by $Blt(X)$ the Banach space of all bounded linear transformations of X into X equipped with the uniform induced norm represented by $\|\cdot\|$, and for $L \in Blt(X)$ we denote $\sigma(L)$ the spectrum of L.

Finally, we denote by $1_A\{.\}$ the Dirac measure for some measurable set A, write $\{\eta\}$ for any process $\{\eta(t),\ 0 \leqslant t \leqslant T\}$, whenever it is clear whether T is finite or not, and adopt $E[\cdot]$ for the usual expectation. A function $f : [0, \infty) \to \mathbb{E}$, $\mathbb{E}$ standing for $\mathbb{R}$ or $\mathbb{C}$, is said $o(\delta)$ if $\lim_{\delta \downarrow 0} \frac{|f(\delta)|}{\delta} = 0$.

3 Problem Statement

Let us fix an underlying complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ carrying a right continuous filtration $\{\mathcal{F}_t,\ t \geq 0\}$ and consider the following class of stochastic differential equations

$$\dot{x}(t) = A_{\theta(t)} x(t) + B_{\theta(t)} u(t), \quad t \geq 0 \tag{2}$$

$$x(0) = x_0,\ \theta(0) = \theta_0 \tag{3}$$

where $x(t) \in \mathbb{C}^n$ denotes the state vector and $u(t) \in \mathbb{C}^m$ the control input adapted to $\mathcal{F}_t$. The system parameters are functions of a homogeneous Markov process $\{\theta(t), t \geq 0\}$ with right continuous trajectories and an infinite countable state space which, for convenience, we assign to the set $\mathcal{S} = \{1, 2, ...\}$. We assume that $\{\theta\}$ has stationary standard transition probability matrix function (see [21, pg 138]) $\{P_\tau(i,j)\}_{i,j \in \mathcal{S}}$ in that, for $s \geq 0$,

$$P_s(i,j) =: P\{\theta(t+s) = j | \theta(t) = i\} = \begin{cases} \lambda_{ij} s + o(s) & i \neq j \\ 1 + \lambda_{ii} s + o(s) & i = j \end{cases} \tag{4}$$

with infinitesimal matrix $\Lambda = [\lambda_{ij}]_{i,j \in \mathcal{S}}$, where $\lambda_{ij} \geq 0$ for $i \neq j$. The Markov process $\{\theta\}$ is conservative in that $\sum_{j=1, j\neq i}^{\infty} \lambda_{ij} = -\lambda_{ii}, \quad i \in \mathcal{S}$. Moreover, we assume the coefficients $-\lambda_{ii}$ to be bounded above, uniformly on i, say by a constant c. We assume that $\{A_{(.)}, B_{(.)}\}$ are such that for any $j \in \mathcal{S}$ and for $\theta(t) = j$, $A_{\theta(t)} = A_j$ and $B_{\theta(t)} = B_j$, with A_j, B_j being constant matrices in $\mathbb{M}(\mathbb{C}^n)$ and $\mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$, respectively. In addition, the parameters are supposed norm bounded in that $A = (A_1, A_2, ...) \in \mathcal{H}_{\infty}^{n}$ and $B = (B_1, B_2, ...) \in \mathcal{H}_{\infty}^{m,n}$. We consider x_0 a second order random variable (r.v.) which may depend on the r.v. θ_0, and we shall denote $\vartheta_0 = \vartheta_0(x_0, \theta_0)$ the joint initial distribution of x_0 and θ_0.We shall denote system (2)-(4) by (A, B, Λ).

We consider the time homogeneous Borel measurable control functions $u : \{\mathbb{C}^n, \mathcal{S}\} \to \mathbb{C}^m$ such that $u(x, i) = -G_i x$, $G = (G_1, G_2, ...) \in \mathcal{H}_{\infty}^{n,m}$.

We shall show that (A, B, Λ) is stochastically stabilizable (see Definition 1) if and only if the associated Lyapunov equation (given by (11)) has a unique solution in $\mathring{\mathcal{H}}_{\infty}^{n+}$ (i.e., a unique norm bounded strictly positive definite solution).

4 Preliminaries

Definition 1 (Stochastic stabilizability) *We say that the system (A, B, Λ) is stochastically stabilizable (SS) if there exists $G \in \mathcal{H}_{\infty}^{n,m}$ such that for any joint initial distribution ϑ_0, we have that $\int_0^{\infty} E[\|x(t)\|^2]dt < \infty$, where $x(t)$ is given by (2) with $u(t) = -G_{\theta(t)}x(t)$, i.e., $\dot{x}(t) = F_{\theta(t)}x(t), \quad t > 0$, with $F_{\theta(t)} = A_{\theta(t)} - B_{\theta(t)}G_{\theta(t)}$. In this case we say that $\dot{x}(t) = F_{\theta(t)}x(t)$ is stochastically stable and G stabilizes (A, B, Λ).*

The following lemma and corollary are specializations of the ones given in Section 4.2 of [2].

Lemma 1 *Let the bounded linear operator $\mathcal{A} : Y \to Y$ be the infinitesimal generator of the uniformly continuous semigroup $T_{\mathcal{A}}(t) : Y \to Y$, Y a Banach space. Then the following assertions are equivalent.*

1. $\sup\{Re\lambda : \lambda \in \sigma(\mathcal{A})\} < 0$
2. *There are constants $M \geq 1$ and $\omega > 0$ such that $\|T_{\mathcal{A}}(t)\| \leqslant M \exp(-\omega t)$.*
3. *$\int_0^{\infty} \|T_{\mathcal{A}}(t)y\| \, dt < \infty$, for every $y \in Y$.*

Corollary 1 *If Assertion 2 of Lemma 1 holds, then $\beta \|y\|$, with $\beta = \frac{M}{\omega}$, is an upper bound for Assertion 3.*

The following propositions and lemma in this section are proved in Sections 4.2, 6.1 and 6.3 of [2].

For $H = (H_1, H_2, ...) \in \mathcal{H}_1^n$, let us define the operator $\mathcal{D}$, with $\mathcal{D}(H) = (\mathcal{D}_1(H), \mathcal{D}_2(H), ...)$, such that

$$\mathcal{D}_i(H) = F_i H_i + H_i F_i^* + \sum_{j=1}^{\infty} \lambda_{ji} H_j, \tag{5}$$

Proposition 1 $\mathcal{D} \in Blt(\mathcal{H}_1^n)$.

Proposition 2 *For $\mathcal{D}$ given by (5), consider the Banach space Riccati differential equation*

$$\begin{cases} \dot{Q}(t) = \mathcal{D}(Q(t)) \quad , \quad t \geq 0 \\ Q(0) = Q^{0+} \in \mathcal{H}_1^{n^+} \end{cases} \tag{6}$$

or equivalently, the countably infinite set of interconnected Riccati differential equations $\dot{Q}_i(t) = \mathcal{D}_i(Q(t))$, $t \geq 0$, with initial value $Q_i(0) = Q_i^{0+}, \quad i \in \mathcal{S}$. Then there is a unique $\mathcal{H}_1^{n^+}$-valued function, say $Q(t)$, continuous for $t \geq 0$ and continuously differentiable for $t > 0$, satisfying (6). Moreover, $Q(t)$ is expressed by

$$Q(t) = T(t)Q^{0+} \in \mathcal{H}_1^{n^+}, \; t \geq 0 \tag{7}$$

where $T(t) : \mathcal{H}_1^n \to \mathcal{H}_1^n$, $t \geq 0$ is the C_0-semigroup (actually a uniformly continuous semigroup) of bounded linear transformations generated by $\mathcal{D}$, its infinitesimal operator.

Lemma 2 *Consider the operator $\mathcal{D}$ given by (5) and some $G = (G_1, \; G_2, ...) \in \mathcal{H}_{\infty}^{n,m}$. Then the system (A, B, Λ) is SS with stabilizing G if and only if*

$$\sup\{Re\lambda : \lambda \in \sigma(\mathcal{D})\} < 0, \; F_i = A_i - B_i G_i \tag{8}$$

For arbitrary initial condition (x_0, θ_0), let us now consider the homogeneous dynamic system

$$\begin{cases} \dot{x}(t) = F_{\theta(t)}x(t), \quad t > 0 \\ x(0) = x_0, \; \theta(0) = \theta_0 \end{cases} \tag{9}$$

where $F = (F_1, F_2, ...) \in \mathcal{H}_{\infty}^n$, and define $Q(t) = (Q_1(t), \; Q_2(t), ...) \in \mathcal{H}_1^{n^+}, \; t \geq 0$, with

$$Q_i(t) = E[x(t)x(t)^* 1_{\{\theta(t)=i\}}] \in \mathbb{M}(\mathbb{C}^n)^+, \; i \in \mathcal{S} \tag{10}$$

Proposition 3 *The unique $\mathcal{H}_1^{n^+}$-valued solution to (6) initialized by $Q_i^{0+} = E[x_0 x_0^* 1_{\{\theta_0=i\}}], \quad i \in \mathcal{S}$, can be expressed by (10) where $x(t)$ is given by (9).*

5 The Countably Infinite Coupled Lyapunov Equations

Lemma 3 (sufficiency) *Consider $G = (G_1, ...) \in \mathcal{H}_{\infty}^{n,m}$. The system (A, B, Λ) is SS with stabilizing G if, for some $V \in \mathring{\mathcal{H}}_{\infty}^{n+}$, there is $S \in \mathring{\mathcal{H}}_{\infty}^{n+}$ satisfying the countably infinite set of coupled Lyapunov equations given by*

$$F_i^* S_i + S_i F_i + \sum_{j=1}^{\infty} \lambda_{ij} S_j = -V_i, \;\; i \in \mathcal{S} \tag{11}$$

with $F_i = A_i - B_i G_i$, $i \in \mathcal{S}$.

Proof: Let us first define the Lyapunov function $\phi_S : \mathcal{H}_1^{n^+} \to \mathbb{R}$, $S \in \mathcal{H}_\infty^{n^+}$, as $\phi_S(H) = \sum_{i=1}^\infty tr(S_i H_i)$, noting that $0 < \phi_S(H) \leqslant n \|S\|_\infty \|H\|_1 < \infty$ for nonzero S and H. Now, for the continuously differentiable function $Q(t)$ given as in Proposition 2 and $S \in \mathring{\mathcal{H}}_\infty^{n^+}$ satisfying (11), let us first obtain an expression for the derivative of $\phi_S(Q(t)$, as follows.

$$\frac{d}{dt}\phi_S(Q(t)) = \lim_{M\to\infty} \sum_{i=1}^M \frac{d}{dt} tr(S_i Q_i(t)) = \lim_{M\to\infty} \sum_{i=1}^M tr(S_i \dot{Q}_i(t)) = \lim_{M\to\infty} \sum_{i=1}^M tr(S_i \{F_i Q_i(t) + Q_i(t) F_i^* + \sum_{j=1}^\infty \lambda_{ji} Q_j(t)\}) = \lim_{M\to\infty} \sum_{i=1}^M tr(\{S_i F_i + F_i^* S_i + \sum_{j=1}^\infty \lambda_{ij} S_j\} Q_i(t)) = -\sum_{i=1}^\infty tr(V_i Q_i(t)) = -\phi_V(Q(t)) \tag{12}$$

where we rely, in this order, on the uniform convergence of $\sum_{i=1}^M tr(S_i Q_i(t))$, on the linearity of the differential operator $\frac{d}{dt}$, on (6), on (5), on the fact that $tr(AB) = tr(BA)$ for square matrices A and B, on the continuity and linearity of the trace operator and on (11).

Let us assume for the moment that there is a positive constant ρ, which may depend on S and V, such that

$$\phi_V(Q(t)) > \rho \phi_S(Q(t)), t \geq 0 \tag{13}$$

Hence, defining $\psi(t) = \phi_S(Q(t))$, we have, from (12), that $\psi(t)$ satisfies $\frac{d}{dt}\psi(t) < -\rho\psi(t)$, $t \geq 0$, initialized by $\psi(0) = \psi^0 = \phi_S(Q^{0+})$, $Q^{0+} \in \mathcal{H}_1^{n^+}$. To apply a comparison theorem, let us define the scalar differential equation given by $\frac{d}{dt}\zeta(t) = -\rho\zeta(t)$, $t \geq 0$, with initial value $\zeta(0) = \psi^0$, which is satisfied by $\zeta(t) = \psi^0 e^{-\rho t}$, $t \geq 0$. Now, $\psi(t)$ and $\zeta(t)$ are continuous and continuously differentiable real valued functions (recall that $Q(t)$ is continuous and continuously differentiable) and $\psi(0) = \zeta(0)$. Hence we may apply Lemma 5 in the Appendix to obtain that $0 \leqslant \psi(t) < \zeta(t) = \psi^0 e^{-\rho t}$, $t \geq 0$, where $\psi^0 \geq 0$. Since $S \in \mathring{\mathcal{H}}_\infty^{n^+}$, there is $\alpha > 0$ such that $S_i > \alpha I$ for every $i \in \mathcal{S}$. Since, in addition, $Q_i(t) \geq 0$, it follows that $tr(S_i Q_i(t)) \geq tr(\alpha I Q_i(t))$. Hence, $\psi^0 e^{-\rho t} > \psi(t) = \sum_{i=1}^\infty tr(S_i Q_i(t)) \geq \sum_{i=1}^\infty tr(\alpha I Q_i(t)) \geq \alpha \sum_{i=1}^\infty \|Q_i(t)\| = \alpha \|Q(t)\|_1$ where we used the fact that $tr(A) \geq \|A\|$ for every positive semidefinite matrix A. Integrating the expression above and recalling (7), we have, for every $Q^{0+} \in \mathcal{H}_1^{n^+}$, that, $\int_0^\infty \|T(t)Q^{0+}\|_1 dt < \frac{\phi_S(Q^{0+})}{\alpha} \int_0^\infty e^{-\rho t} dt = \frac{\phi_S(Q^{0+})}{\alpha\rho} < \infty$. Now, from Remark 2, there exists X^+, X^-, Y^+ and Y^- in $\mathcal{H}_1^{n^+}$ such that $Q^0 = (X^+ - X^-) + \iota(Y^+ - Y^-)$, for every $Q^0 \in \mathcal{H}_1^n$. Hence $\int_0^\infty \|T(t)Q^0\|_1 dt < \infty$. Now, appealing to implication ($3 \Rightarrow 1$) of Lemma 1 it follows that $\sup\{Re\lambda : \lambda \in \sigma(\mathcal{D})\} < 0$. But from Lemma 2, this means that system (A, B, Λ) is stochastically stabilizable, with stabilizing $G = (G_1, G_2, ...) \in \mathcal{H}_\infty^{n,m}$.

Let us now show that (13) holds. Since S is norm bounded and $S_i \geq 0$, $i \in \mathcal{S}$, it follows that $S_i < \beta I$ for some positive and finite β. By its turn, $V \in \mathring{\mathcal{H}}_\infty^{n^+}$, so that there is $\eta > 0$ such that $V_i > \eta I$. Consequently, $V_i > \frac{\eta}{\beta} S_i$, $i \in \mathcal{S}$. Moreover, since $Q_i(t) \geq 0$ and defining $\rho = \frac{\eta}{\beta}$, we have that $tr(V_i Q_i(t)) > \rho tr(S_i Q_i(t))$, which means that $\sum_{i=1}^\infty tr(V_i Q_i(t)) > \rho \sum_{i=1}^\infty tr(S_i Q_i(t))$, so that (13) follows. ■

Lemma 4 (necessity) *Consider* $G = (G_1, ...) \in \mathcal{H}_\infty^{n,m}$. *The system* (A, B, Λ) *is SS with stabilizing* G *only if, for any* $V \in \mathring{\mathcal{H}}_\infty^{n^+}$, *there is* $S \in \mathring{\mathcal{H}}_\infty^{n^+}$ *satisfying the countably infinite set of coupled Lyapunov equations given by (11) with* $F_i = A_i - B_i G_i$, $i \in \mathcal{S}$. *Moreover* S *is the unique solution to (11).*

Proof: Let us chose the stabilizing control policy $u(t) = -G_{\theta(t)} x(t)$, $t \geq 0$, so that system (A, B, Λ) of Section 3 reads as $\dot{x}(t) = F_{\theta(t)} x(t)$, $t \geq 0$, with $F_{\theta(t)} = A_{\theta(t)} - B_{\theta(t)} G_{\theta(t)}$. We shall be interested in the specialized initial condition $x(0) = x$ and $\theta(0) = i$, where x and i are deterministic and arbitrary in $\mathbb{C}^n$ and $\mathcal{S}$ respectively.

For arbitrary $V \in \mathring{\mathcal{H}}_\infty^{n^+}$, let us consider the expression

$$E_{x(t),\theta(t)} \int_t^T x(\tau)^* V_{\theta(\tau)} x(\tau) d\tau = E_{x(t),\theta(t)}[\int_t^T x(t)^* M(\tau, t, \theta(t), \zeta)^* V_{\theta(\tau)} M(\tau, t, \theta(t), \zeta) x(t) d\tau] = x(t)^* S_{\theta(t)}(T-t) x(t) \tag{14}$$

where $S_{\theta(t)}(T-t)$ is well defined in $\mathbb{M}(\mathbb{C}^n)^+$ with $M(\cdot) \in \mathbb{M}(\mathbb{C}^n)$ determined in Lemma 6. Note that the value of the expression on the left hand side of (14) does not change if we apply a same shift on t and T, preserving the same values for the conditioning r.vs. so that, in fact, it suffices to $S_{\theta(t)}(\cdot)$ to be a function of $T - t$.

Since $V_i \in \mathbb{M}(\mathbb{C}^n)^+$, it is clear, from (14) that $S_i(T - t) \in \mathbb{M}(\mathbb{C}^n)^+$ and that

$$0 \leqslant S_i(T_1 - t) \leqslant S_i(T_2 - t) \tag{15}$$

for every $T_1, T_2 \in (t, \infty)$, $T_1 < T_2$, and $i \in \mathcal{S}$. Let us assume for the moment that

$$S_i(T-t) \leqslant dI, i \in \mathcal{S}, T \in (t, \infty) \tag{16}$$

for some constant d which does not depend on i and T. Then, from (15) and (16) and bearing in mind a standard monotonicity result concerning positive semidefinite matrices, there exists $S_i \in \mathbb{M}(\mathbb{C}^n)^+$ such that

$$lim_{T\to\infty} S_i(T-t) = S_i. \tag{17}$$

Now fix i arbitrarely. From (16) and recalling (1), it follows that $\|S_i(T-t)\| \leqslant d$ for every $T \in (t, \infty)$ and so $\|S_i\| \leqslant d$. Consequently $S \in \mathcal{H}_\infty^{n^+}$. Let us now show that, in fact, $S \in \mathring{\mathcal{H}}_\infty^{n^+}$. Bearing in mind that $V_i \geq \alpha I$, $i \in \mathcal{S}$, for some $\alpha > 0$, that the trajectories of $\{x\}$ are continuous from the right and, from the monotonicity property, that $S_i \geq S_i(T-t)$, we may write, for some fixed T,that

$$\begin{aligned}&x^* S_i x \geq x^* S_i(T-t)x\\&= E_{x(t)=x,\theta(t)=i} \int_t^T x(\tau)^* V_{\theta(\tau)} x(\tau) d\tau\\&\geq E_{x(t)=x,\theta(t)=i} \int_t^{t+\delta} x(\tau)^* V_{\theta(\tau)} x(\tau) d\tau\\&\geq E_{x(t)=x,\theta(t)=i} \int_t^{t+\delta} x(\tau)^* \alpha I x(\tau) d\tau = x^* \alpha\delta I x\\&+o(\delta) \geq x^* \alpha\delta I x - \gamma\delta = x^*(\alpha-\gamma)\delta I x\end{aligned}$$

where we picked $0 < \gamma < \alpha$ and $\delta = \delta(\gamma)$ sufficiently small to obtain the last inequality. Since x is arbitrary, there is $\eta = (\alpha-\gamma)\delta > 0$ such that $S_i \geq \eta I$, $i \in \mathcal{S}$.

Let us now define $g^T(t,x,i) = x^* S_i(T-t)x$ and use (14), so that

$$\begin{aligned}&\lim_{h\downarrow 0} \tfrac{1}{h} E_{x(t),\theta(t)}[g^T(t+h, x(t+h), \theta(t+h))\\&-g^T(t,x(t),\theta(t))] = \lim_{h\downarrow 0} \tfrac{1}{h} E_{x(t),\theta(t)}[E_{x(t+h),\theta(t+h)}[\\&\int_{t+h}^T x^*(\tau) V_{\theta(\tau)} x(\tau) d\tau] - E_{x(t),\theta(t)}[\int_t^T x^*(\tau)\\&V_{\theta(\tau)} x(\tau) d\tau]] = \lim_{h\downarrow 0} \tfrac{1}{h} E_{x(t),\theta(t)}[-\int_t^{t+h} x^*(\tau)\\&V_{\theta(\tau)} x(\tau) d\tau] = -\lim_{h\downarrow 0} \tfrac{1}{h} E_{x(t),\theta(t)}[x^*(t) V_{\theta(t)} x(t) h\\&+o(h)] = -x^*(t) V_{\theta(t)} x(t)\end{aligned}$$

From (16) and recalling (1), it follows that $S(T-t) \in \mathcal{H}_\infty^{n^+}$. Moreover, from (14), $t \mapsto S_i(T-t)$, $i \in \mathcal{S}$, is differentiable. Since $\{x, \theta\}$, with x satisfying $\dot{x}(t) = F_{\theta(t)} x(t)$, $t \geq 0$, $F = (F_1, F_2, ...) \in \mathcal{H}_\infty^n$, is a Markov process, its infinitesimal generator reads as (see [2] for details)

$$\begin{aligned}&\lim_{h\downarrow 0} \tfrac{1}{h} E_{x(t),\theta(t)}[g^T(t+h, x(t+h), \theta(t+h))\\&-g^T(t,x(t),\theta(t))] = x(t)^*\{\dot{S}_{\theta(t)}(T-t)\\&+F_{\theta(t)}^* S_{\theta(t)}(T-t) + S_{\theta(t)}(T-t) F_{\theta(t)}\\&+\textstyle\sum_{j=1}^\infty \lambda_{\theta(t)j} S_j(T-t)\} x(t)\end{aligned} \tag{18}$$

where $F_i = A_i - B_i G_i$, $i \in \mathcal{S}$. The above expression relies on a decomplexification technique, from which we establish a certain version of the gradient concept and from this, the linear approximation to nonholomorphic functionals. Now, both expressions above gives us that

$$\begin{aligned}&-x^*(t) V_{\theta(t)} x(t) =\\&x(t)^*\{\dot{S}_{\theta(t)}(T-t) + F_{\theta(t)}^* S_{\theta(t)}(T-t)\\&+S_{\theta(t)}(T-t) F_{\theta(t)} + \textstyle\sum_{j=1}^\infty \lambda_{\theta(t)j} S_j(T-t)\} x(t)\end{aligned} \tag{19}$$

But (19) is valid for every $x(t)$, and so

$$\begin{aligned}&-V_{\theta(t)} = \dot{S}_{\theta(t)}^T(t) + F_{\theta(t)}^* S_{\theta(t)}(T-t)\\&+S_{\theta(t)}(T-t) F_{\theta(t)} + \textstyle\sum_{j=1}^\infty \lambda_{\theta(t)j} S_j(T-t)\end{aligned}$$

Now, since $S_i(T-t)$ converges to S_i for every t, $S_i(T-t)$ is not time varying as $T \to \infty$. Therefore, taking limits as $T \to \infty$ on the above expression and denoting i instead of $\theta(t)$, we can say that, for arbitrary $V \in \mathring{\mathcal{H}}_\infty^{n^+}$, there is a unique $S \in \mathring{\mathcal{H}}_\infty^{n^+}$ given by (14) and (17), for which (11) holds.

Finally, let us show that (16) indeed holds. Recalling that $G \in \mathcal{H}_\infty^{n,m}$ is stabilizing we have, from Lemma 2, that $\sup\{Re\lambda : \lambda \in \sigma(\mathcal{D})\} < 0$ with $\mathcal{D}$ given by (5). Now, $\mathcal{D}$ generates an uniformly continuous semigroup, say $T(t)$, so we may invoke the equivalence among Assertions 1, 2 and 3 of Lemma 1, as well as Corollary 1, obtaining, for some finite constant β, that $\int_0^\infty \|T(t)Q^0\|_1 dt \leqslant \beta \|Q^0\|_1$ for every $Q^0 \in \mathcal{H}_1^n$ and, particularly, for $Q^{0+} = (Q_1^{0+}, Q_2^{0+}, ...) \in \mathcal{H}_1^{n+}$, $Q_j^{0+} = E[x(0)x(0)^* 1_{\{\theta(0)=j\}}]$. Now, from Propositions 2 and 3 we may write that

$$\int_0^\infty \|Q(t)\|_1 dt \leqslant \beta \|Q^{0+}\|_1 \tag{20}$$

where $Q(t)$ is expressed by (10). Now, with n standing for the dimension of $x(t)$, we have that $\|Q^{0+}\|_1 \leqslant E[\|x(0)\|^2] = \|x\|^2$ (which does not depend on i) and $E[\|x(t)\|^2] \leqslant n\|Q(t)\|_1$(see [2]). So, defining $d_1 = n\beta$, which does not depend on x nor on i, (20) becomes $\int_0^\infty E_{x(0)=x,\theta(0)=i}[\|x(t)\|^2]dt \leqslant d_1 \|x\|^2$ where we used the fact that $E_{x(0)=x,\theta(0)=i}[\|x(t)\|^2] = E[\|x(t)\|^2]$ for deterministic initial condition $(x(0),\theta(0))$. Now, from (14), using Fubini and defining $d = \|V\|_\infty d_1$, it follows that

$$\begin{aligned}&x^* S_i(T-t)x = E_{x(0)=x,\theta(0)=i} \int_t^T x(\tau)^* V_{\theta(\tau)} x(\tau) d\tau\\&= E_{x(0)=x,\theta(0)=i} \int_0^{T-t} x(\tau)^* V_{\theta(\tau)} x(\tau) d\tau\\&\leqslant \|V\|_\infty E_{x(0)=x,\theta(0)=i} \int_0^{T-t} \|x(\tau)\|^2 d\tau\\&\leqslant \|V\|_\infty E_{x(0)=x,\theta(0)=i} \int_0^\infty \|x(\tau)\|^2 d\tau\\&\leqslant d\|x\|^2 = x^* d I x\end{aligned}$$

and since x is arbitrary, (16) follows. ■

6 Appendix

The proof of the results in this section can be found in [2].

Lemma 5 *Let h and g be real valued functions defined on $[0,\infty)$ with $h(0) = g(0)$ and suppose that h is a continuous and continuously differentiable solution to the differential equation $\dot{h}(t) = a(h(t))$, $t \geq 0$, for some scalar function a. Let us also assume that g is continuous and continuously differentiable and satisfies the inequality $\dot{g}(t) < a(g(t))$, $t \geq 0$. Then, $g(t) < h(t)$, $t \geq 0$.*

Lemma 6 *For arbitrary t (not necessarily a jump point), $\{\theta\}$ as given in Section 3 and $F = (F_1, F_2, ...) \in \mathcal{H}_\infty^n$, consider the homogeneous system $\dot{x}(\tau) = F_{\theta(\tau)} x(\tau)$, $\tau \geq t$, with initial condition $(x(t),\theta(t))$. Then, for every $\tau \geq t$ and defining $\tau_0 = t$ and the r.v. $\zeta = (\theta(\tau_{n-1}), ..., \theta(\tau_1), \tau_{n-1}, ..., \tau_1)$, the trajectory of the state process $\{x\}$ with jump times $\tau_1 < \tau_2 ... < \tau_N$ is represented by*

$$x(\tau) = M(\tau, t, \theta(t), \zeta) x(t) \ a.s., \tau_{n-1} \leqslant \tau < \tau_n \tag{21}$$

for $n = 1, 2, ...N$, *where*

$$M(\tau, t, \theta(t), \zeta) \begin{cases} = \exp(F_{\theta(t)}(\tau - t)), n = 1 \\ = \exp(F_{\theta(\tau_{n-1})}(\tau - \tau_{n-1})) \\ \quad . \exp(F_{\theta(\tau_{n-2})}(\tau_{n-1} - \tau_{n-2}))... \\ \quad ... \exp(F_{\theta(t)}(\tau_2 - \tau_1)) \\ \quad . \exp(F_{\theta(t)}(\tau_1 - t)), n = 2, ...N \end{cases} \tag{22}$$

and N *is either finite with* $\tau_N = \infty$ *or infinite with* $\lim_{N\to\infty} \tau_N = \infty$ *a.s.*.

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