An Integral Inequality in the Stability Problem of Time-Delay Systems[1]

Keqin Gu
Department of Mechanical & Industrial Engineering
Southern Illinois University at Edwardsville
Edwardsville, Illinois 62026, USA
Telephone: 618-650-2803
Fax: 618-650-2555
E-mail: kgu@siue.edu

Abstract

An integral inequality is derived, and applied to the stability problem of time-delay systems using discretized Lyapunov functional formulation. As the result, a simpler stability criterion is derived.

Notations

W^T	Transpose of matrix W
$W > (<)0$	matrix W is positive (negative) definite
$W \geq (\leq)0$	matrix W is positive (negative) semi-definite
$\mathcal{R}$, $\mathcal{R}^k$, $\mathcal{R}^{k\times m}$	The set of real numbers, k-vectors and k by m matrices
r	Time-delay
n	Number of states
$\mathcal{C}$	The set of continuous $\mathcal{R}^n$ valued functions on $[-r, 0]$
N	Number of divisions of the interval $[-r, 0]$ in discretization
h	Length of each division $= \frac{r}{N}$
I	Identity matrix of appropriate dimensions
$O_{k\times m}$	kn by mn dimensional zero matrix
I_r	Matrix $[O_{N\times 1}, I]$
I_l	Matrix $[I, O_{N\times 1}]$
I_d	$= I_l - I_r$

1 Introduction

Time-delay systems are frequently encountered in engineering, biology, economy, and other areas [10]. In the wake of intensive research on the robust stability and control theory, the stability and control of time-delay systems has received renewed interests. The development of efficient computational algorithm for nonsmooth convex optimization problem [19], which made it possible to efficiently solve Linear Matrix Inequalities (LMI) [1] [4], inspired intensive activities to formulate such problems in a LMI form. We will only mention some more recent activities in the time-domain approaches using Lyapunov-Krasovskii and Razumikhim methods. For a more comprehensive survey, see [20] [14] [13].

For systems with small coefficient matrix for the delayed term, a delay-independent stability criterion, based on a rather simple Lyapunov-Krasovskii functional argument, is often sufficient in practice. There are still some recent research on the design synthesis techniques , tracking and model following [16] [21]. This formulation consider the delayed term of the system as always detrimental to stability. For many practical systems, the delayed inputs are often needed to stabilize the system. Obviously, the delay independent stability would be inappropriate in such situations.

For systems with small delay, a model transformation technique is often used to transform the point-wise delay system into a distributed delay system, and Razumikhin stability criterion is used to the resulting system. There are still a relatively large number of new publications in this approach, see, for example, [3][18]. The resulting stability criterion explicitly depends on the delay (delay-dependent), and often reflects the reality better. For relatively large delay, however, this method can be rather conservative. One source of conservatism is due to the application of Razumikhin theory as numerical examples show [5]. The other conservatism is due to the fact that the model transformation may introduce additional poles which are not present in the original system, and one of these additional poles may cross the imaginary axis before any of the poles

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of the original system do as the delay increases from zero [9] [12]. However, for systems where the time-delay may be fast time-varying, Razumikhin approach remains the only method available.

Recently, there are a number of interesting new ideas on the Lyapunov-Krasovskii methods with improved results on delay-dependent stability [22] [15] and mixed delay-independent and delay-dependent result [17].

For linear systems with constant time-delays, the existence of a quadratic Lyapunov-Krasovskii functional is a necessary and sufficient condition [11]. A piecewise linear discretization proposed in [5] enables one to write the stability criterion in LMI form. Even with the most coarse discretization of $N = 1$, the result shows significant improvements over many stability results using time-domain approach. The analytical result can be approached as the discretization becomes finer. The convergence and the result for uncertain system are further improved by using a more general Lyapunov functional [6]. The result has been further improved by allowing some parameters to depend on uncertainties, with the resulting formulation no more complicated since it is possible to eliminate some variables in the resulting LMI [7].

In this article, an integral inequality will be used in the discretized Lyapunov functional and its derivative, resulting in a alternative stability criterion which requires less computation than [6] and [7].

2 An integral inequality

The following integral inequality plays a central role for the result of the paper.

Lemma 1 *For any constant matrix $M \in \mathcal{R}^{m\times m}$, $M = M^T > 0$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \to \mathcal{R}^m$ such that the integrations in the following are well defined, then*

$$\gamma \int_0^\gamma \omega^T(\beta) M \omega(\beta) d\beta \geq \left(\int_0^\gamma \omega(\beta) d\beta\right)^T M \left(\int_0^\gamma \omega(\beta) d\beta\right) \quad (1)$$

Proof: It is easy to see, using Shur's complement, that

$$\begin{pmatrix} \omega^T(\beta)M\omega(\beta) & \omega^T(\beta) \\ \omega(\beta) & M^{-1} \end{pmatrix} \geq 0$$

for any $0 \leq \beta \leq \gamma$. Integration of the above inequality from 0 to γ yields

$$\begin{pmatrix} \int_0^\gamma \omega^T(\beta)M\omega(\beta)d\beta & \int_0^\gamma \omega^T(\beta)d\beta \\ \int_0^\gamma \omega(\beta)d\beta & \gamma M^{-1} \end{pmatrix} \geq 0$$

Use Shur's complement. ∎

For the scalar case ($m = 1$), the results is well known in calculus. A simple extension of the above lemma is the following.

Lemma 2 *For any constant matrix $M \in \mathcal{R}^{m\times m}$, $M = M^T > 0$ and vector function $\omega, \omega_1, \omega_2 : [0, \alpha] \to \mathcal{R}^m$, such that the integrations in the following are well defined, then*

$$\begin{aligned} & \int_0^1 [(1-\alpha)\omega_1^T(\alpha)M\omega_1(\alpha) + \alpha\omega_2^T(\alpha)M\omega_2(\alpha)]d\alpha \\ \geq & \int_0^1 \left(\int_0^\alpha \omega_1(\beta)d\beta + \int_\alpha^1 \omega_2(\beta)d\beta\right)^T M \\ & \left(\int_0^\alpha \omega_1(\beta)d\beta + \int_\alpha^1 \omega_2(\beta)d\beta\right) d\alpha \end{aligned} \quad (2)$$

$$\begin{aligned} & \int_0^1 (1-\alpha)\omega^T(\alpha)M\omega(\alpha)d\alpha \\ \geq & \int_0^1 \left(\int_0^\alpha \omega(\beta)d\beta\right)^T M \left(\int_0^\alpha \omega(\beta)d\beta\right) d\alpha \end{aligned} \quad (3)$$

$$\begin{aligned} & \int_0^1 \alpha\omega^T(\alpha)M\omega(\alpha)d\alpha \\ \geq & \int_0^1 \left(\int_\alpha^1 \omega(\beta)d\beta\right)^T M \left(\int_\alpha^1 \omega(\beta)d\beta\right) d\alpha \end{aligned} \quad (4)$$

Proof: Use Lemma 1 for the case $\gamma = 1$, and

$$\omega(\beta) = \begin{cases} \omega_1(\beta), & \text{if } 0 \leq \beta \leq \alpha \\ \omega_2(\beta), & \text{if } \alpha < \beta \leq 1 \end{cases}$$

to obtain

$$\begin{aligned} & \int_0^\alpha \omega_1^T(\beta)M\omega_1(\beta)d\beta + \int_\alpha^1 \omega_2^T(\beta)M\omega_2(\beta)d\beta \\ \geq & \left(\int_0^\alpha \omega_1(\beta)d\beta + \int_\alpha^1 \omega_2(\beta)d\beta\right)^T \\ & M\left(\int_0^\alpha \omega_1(\beta)d\beta + \int_\alpha^1 \omega_2(\beta)d\beta\right) d\alpha \end{aligned}$$

Integrate the above from $\alpha = 0$ to $\alpha = 1$, and exchange the order of integration of the left hand side to arrive at inequality (2). Inequality (3) can be arrived by setting $\omega_1 = \omega$ and $\omega_2 = 0$ in (2). Similarly, setting $\omega_1 = 0$ and $\omega_2 = \omega$ yields (4). ∎

3 Lyapunov functional

Consider the stability problem of time-delay system

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-r), \quad (5)$$

with initial condition

$$x(t) = \phi(t), \quad t \in [-r, 0],$$

where $x(t) \in \mathcal{R}^n$ is the state; $A(t) \in \mathcal{R}^{n\times n}$ and $B(t) \in \mathcal{R}^{n\times n}$ are uncertain matrices, which are unknown and possibly time-varying, but are known to be bounded by some compact set Ω, i.e.,

$$(A(t), B(t)) \in \Omega \subset R^{n\times 2n}, \qquad \text{for all } t \in (0, \infty). \quad (6)$$

Define $\mathcal{C}$ as the set of $\mathcal{R}^n$ valued continuous function in the interval $[-r, 0]$, and let $x_t \in \mathcal{C}$ be a segment of system trajectory defined as

$$x_t(\theta) = x(t + \theta), \quad -r \le \theta \le 0$$

Choose a Lyapunov functional $V(x_t)$ of a quadratic form:

$$\begin{aligned} V &: \mathcal{C} \mapsto \mathcal{R}, \\ V(\phi) &= \frac{1}{2}\phi^T(0)P\phi(0) \\ &+\phi^T(0)\int_{-r}^{0} Q(\xi)\phi(\xi)d\xi \\ &+\frac{1}{2}\int_{-r}^{0} d\xi \int_{-r}^{0} \phi^T(\xi)R(\xi,\eta)\phi(\eta)d\eta \\ &+\frac{1}{2}\int_{-r}^{0} \phi^T(\xi)S(\xi)\phi(\xi)d\xi \qquad (7) \end{aligned}$$

where $P = P^T \in \mathcal{R}^{n\times n}, Q(\xi) \in \mathcal{R}^{n\times n}, R(\xi,\eta) = R^T(\eta,\xi) \in \mathcal{R}^{n\times n}, S(\xi) = S^T(\xi) \in \mathcal{R}^{n\times n}$. For systems without uncertainty, the existence of such a quadratic Lyapunov functional is necessary and sufficient[11]. Choose Q, R and S as continuous piecewise linear, i.e.,

$$\begin{aligned} Q(\theta_{i-1} + \alpha h) &= Q^i(\alpha) \qquad (8) \\ &= (1-\alpha)Q_{i-1} + \alpha Q_i \\ S(\theta_{i-1} + \alpha h) &= S^i(\alpha) \qquad (9) \\ &= (1-\alpha)S_{i-1} + \alpha S_i \end{aligned}$$

$$R(\theta_{i-1} + \alpha h, \theta_{j-1} + \beta h) = R^{ij}(\alpha, \beta) \qquad (10)$$

$$= \begin{cases} (1-\alpha)R_{i-1,j-1} + \beta R_{ij} + (\alpha-\beta)R_{i,j-1}, & \alpha \ge \beta \\ (1-\beta)R_{i-1,j-1} + \alpha R_{ij} + (\beta-\alpha)R_{i-1,j}, & \alpha < \beta \end{cases}$$

for $0 \le \alpha \le 1$, $0 \le \beta \le 1$, where

$$\begin{aligned} \theta_i &= -r + ih, \\ h &= r/N, \end{aligned}$$

i.e., N is the number of divisions of the interval $[-r, 0]$, and h is the length of each division. As a result, similar to [6], the Lyapunov functional can be written as

$$\begin{aligned} V(\phi) &= \frac{1}{2}\int_0^1 (\phi^T(0), h[\tilde{\psi}^T(1)I_r - \tilde{\psi}^T(\alpha)I_d]) \\ &\begin{pmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R} \end{pmatrix} \begin{pmatrix} \phi(0) \\ h[I_r^T\tilde{\psi}(1) - I_d^T\tilde{\psi}(\alpha)] \end{pmatrix} d\alpha \\ &+\frac{1}{2}\int_{-r}^{0} \phi^T(\xi)S(\xi)\phi(\xi)d\xi \qquad (11) \end{aligned}$$

and its derivative

$$\dot{V}(\phi) = \frac{d}{dt}V(x_t)|_{x_t=\phi}$$

along the system trajectory can be written as

$$\dot{V}(\phi) = -\frac{1}{2}\int_0^1 (\phi^T(0), \phi^T(-r), h\tilde{\phi}^T(\alpha))$$

$$\begin{pmatrix} \Delta_{11} & -\Delta_{12} & -(1-\alpha)D_1^0 - \alpha D_1^1 \\ & \Delta_{22} & -(1-\alpha)D_2^0 - \alpha D_2^1 \\ \text{symmetric} & & \frac{1}{h}S_d \end{pmatrix}$$

$$\begin{pmatrix} \phi(0) \\ \phi(-r) \\ h\tilde{\phi}(\alpha) \end{pmatrix} d\alpha - \frac{h^2}{2}\int_0^1 \tilde{\phi}^T(\alpha)d\alpha R_d \int_0^1 \tilde{\phi}(\alpha)d\alpha.$$

where

$$\begin{aligned} \tilde{R} &= \begin{pmatrix} R_{00} & R_{01} & \dots & R_{0N} \\ R_{10} & R_{11} & \dots & R_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N0} & R_{N1} & \dots & R_{NN} \end{pmatrix} \\ \tilde{W} &= (W_0, W_1, ..., W_N), \\ \tilde{Q} &= (Q_0, Q_1, ..., Q_N), \\ \Delta_{11} &= -PA - A^TP - Q_N - Q_N^T - S_N, \\ \Delta_{12} &= PB - Q_0, \\ \Delta_{22} &= S_0, \\ S_d &= \text{diag}(S_{d1}, S_{d2}, ..., S_{dN}), \\ S_{di} &= \frac{1}{h}(S_i - S_{i-1}), \\ R_d &= \begin{pmatrix} R_{d11} & R_{d12} & \dots & R_{d1N} \\ R_{d21} & R_{d22} & \dots & R_{d2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{dN1} & R_{dN2} & \dots & R_{dNN} \end{pmatrix}, \\ R_{dij} &= \frac{1}{h}(R_{ij} - R_{i-1,j-1}), \\ D_j^k &= (D_{j1}^k, D_{j2}^k, ..., D_{jN}^k), \\ D_{1i}^k &= A^TQ_{i-1+k} - \frac{1}{h}(Q_i - Q_{i-1}) + R_{i-1+k,N}^T, \\ D_{2i}^k &= B^TQ_{i-1+k} - R_{i-1+k,0}^T, \end{aligned}$$

and

$$\begin{aligned} \tilde{\phi}(\alpha) &= (\phi^{1T}(\alpha), \phi^{2T}(\alpha), ..., \phi^{NT}(\alpha))^T, \\ \tilde{\psi}(\alpha) &= (\psi^{1T}(\alpha), \psi^{2T}(\alpha), ..., \psi^{NT}(\alpha))^T, \\ \phi^i(\alpha) &= \phi(\theta_{i-1} + \alpha h). \\ \psi^i(\alpha) &= \int_0^\alpha \phi^i(\tau)d\tau, \end{aligned}$$

4 Stability conditions

In this section, the integral inequality will be used to find the alternative conditions for Lyapunov functional to be "positive" and its derivative to be "negative", thus arriving at the stability condition of the system.

Proposition 3 ***(Alternative Lyapunov positive condition)*** *There exists an $\varepsilon > 0$ such that the Lyapunov functional (7) satisfies*

$$V(\phi) \geq \varepsilon \phi^T(0)\phi(0)$$

if

$$\tilde{S} > 0 \tag{12}$$

and

$$\begin{pmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R} + \frac{1}{h}\tilde{S} \end{pmatrix} > 0$$

where

$$\tilde{S} = \mathrm{diag}(S_0, S_1, ..., S_N)$$

Proof: In view of (11), it is sufficient to prove

$$\begin{aligned} V_S(\phi) &= \frac{1}{2}\int_{-r}^{0} \phi^T(\xi)S(\xi)\phi(\xi)d\xi \\ &\geq \frac{h^2}{2}\int_0^1 [\tilde{\psi}^T(1)I_r - \tilde{\psi}^T(\alpha)I_d] \\ &\quad \tilde{S}[I_r^T\tilde{\psi}(1) - I_d^T\tilde{\psi}(\alpha)]d\alpha \end{aligned} \tag{13}$$

With the piecewise expression of S,

$$V_S(\phi) = \frac{h}{2}\sum_{i=1}^{N}\int_0^1 \phi^{iT}(\alpha)\left((1-\alpha)S_{i-1} + \alpha S_i\right)\phi^i(\alpha)d\alpha$$

or

$$\begin{aligned} V_S(\phi) &= \frac{h}{2}\int_0^1 \Big((1-\alpha)\phi^{1T}(\alpha)S_0\phi^1(\alpha) \\ &\quad + \alpha\phi^{NT}(\alpha)S_N\phi^N(\alpha) \\ &\quad + \sum_{i=1}^{N-1}\Big((1-\alpha)\phi^{i+1T}(\alpha)S_i\phi^{i+1}(\alpha) \\ &\quad + \alpha\phi^{iT}(\alpha)S_i\phi^i(\alpha)\Big)\Big)d\alpha \end{aligned}$$

Use (3) in the first term, (4) for the second term, and (2) for the remaining terms within the summation, one obtains

$$\begin{aligned} V_S(\phi) &\geq \frac{h}{2}\int_0^1 \left(\left(\int_0^\alpha \phi^1(\beta)d\beta\right)^T S_0 \left(\int_0^\alpha \phi^1(\beta)d\beta\right)\right. \\ &\quad + \left(\int_\alpha^1 \phi^N(\beta)d\beta\right)^T S_N \left(\int_\alpha^1 \phi^N(\beta)d\beta\right) \\ &\quad + \sum_{i=1}^{N-1}\left(\left(\int_\alpha^1 \phi^i(\beta)d\beta + \int_0^\alpha \phi^{i+1}(\beta)d\beta\right)^T\right. \\ &\quad \left.\left. S_i\left(\int_\alpha^1 \phi^i(\beta)d\beta + \int_0^\alpha \phi^{i+1}(\beta)d\beta\right)\right)\right)d\alpha \\ &= \frac{h}{2}\int_0^1 \Big(\psi^{1T}(\alpha)S_0\psi^1(\alpha) \\ &\quad + \left(\psi^N(1) - \psi^N(\alpha)\right)^T S_N \left(\psi^N(1) - \psi^N(\alpha)\right) \\ &\quad + \sum_{i=1}^{N-1}\left(\left(\psi^i(1) - \psi^i(\alpha) + \psi^{i+1}(\alpha)\right)^T\right. \\ &\quad \left. S_i\left(\psi^i(1) - \psi^i(\alpha) + \psi^{i+1}(\alpha)\right)\right)\Big)d\alpha \end{aligned}$$

which is (13). ■

Compare the above result with Corollary 5 of [7], the matrix $\tilde{R}$ is no longer required to be positive definite, which tends to make the result less conservative. On the other hand, $\tilde{S}$ here differs from $\hat{S}$ in Corollary 5 of [7] in the first and last entries, which tends to make the result more conservative. The computation here is less demanding, since one no longer needs to check the positive definiteness of $\tilde{R}$.

The Lyapunov derivative condition can also have alternative condition using the integral inequality. For the sake of convenience, the following notations will be introduced

$$\Delta = \begin{pmatrix} \Delta_{11} & -\Delta_{12} \\ -\Delta_{12}^T & \Delta_{22} \end{pmatrix}$$

$$D^s = \frac{1}{2}\begin{pmatrix} D_1^0 + D_1^1 \\ D_2^0 + D_2^1 \end{pmatrix}$$

$$D^a = \frac{1}{2}\begin{pmatrix} D_1^0 - D_1^1 \\ D_2^0 - D_2^1 \end{pmatrix}$$

and

$$\phi_{0r} = \begin{pmatrix} \phi(0) \\ \phi(-r) \end{pmatrix}$$

Then

$$\begin{aligned} \dot{V}(\phi) &= -\frac{1}{2}\phi_{0r}^T\Delta\phi_{0r} \\ &\quad - h\phi_{0r}^T\int_0^1 (D^s + (1-2\alpha)D^a)\tilde{\phi}(\alpha)d\alpha \\ &\quad - \frac{h^2}{2}\int_0^1 \tilde{\phi}^T(\alpha)\frac{1}{h}S_d\tilde{\phi}(\alpha)d\alpha \\ &\quad - \frac{h^2}{2}\int_0^1 \tilde{\phi}^T(\alpha)d\alpha R_d\int_0^1 \tilde{\phi}(\alpha)d\alpha. \end{aligned} \tag{14}$$

Proposition 4 *There exists an $\varepsilon > 0$ such that the derivative of Lyapunov functional satisfies*

$$\dot{V}(\phi) \leq -\varepsilon\phi^T(0)\phi(0) \tag{15}$$

if there exist $U = U^T \in \mathcal{R}^{n\times n}$ *and* $W = W^T \in \mathcal{R}^{nN\times nN}$ *such that*

$$\begin{pmatrix} \Delta - U & D^s \\ D^{sT} & R_d + W \end{pmatrix} > 0 \qquad (16)$$

$$\begin{pmatrix} U & D^a \\ D^{aT} & \frac{1}{h}S_d - W \end{pmatrix} > 0 \qquad (17)$$

$$W > 0 \qquad (18)$$

Proof: Equation (14) can be rewritten as

$$\dot{V}(\phi) = -\frac{1}{2}\begin{pmatrix} \phi_{0r}^T & h\int_0^1 \tilde{\phi}^T(\alpha) \end{pmatrix}\begin{pmatrix} \Delta - U & D^s \\ D^{sT} & R_d + W \end{pmatrix}\begin{pmatrix} \phi_{0r} \\ h\int_0^1 \tilde{\phi}(\alpha)d\alpha \end{pmatrix} - \frac{1}{2}\int_0^1 \begin{pmatrix} \phi_{0r}^T & h\tilde{\phi}^T(\alpha) \end{pmatrix} \begin{pmatrix} U & (1-2\alpha)D^a \\ (1-2\alpha)D^{aT} & \frac{1}{h}S_d - W \end{pmatrix}\begin{pmatrix} \phi_{0r} \\ h\tilde{\phi}(\alpha) \end{pmatrix} d\alpha - \frac{h^2}{2}\left(\int_0^1 \tilde{\phi}^T(\alpha)W\tilde{\phi}(\alpha)d\alpha - \int_0^1 \tilde{\phi}^T(\alpha)W\int_0^1 \tilde{\phi}(\alpha)d\alpha\right)$$

The last term is non-positive if (18) is satisfied according to Lemma 1. Therefore, (15) is satisfied if (18), (16) and

$$\begin{pmatrix} U & (1-2\alpha)D^a \\ (1-2\alpha)D^{aT} & \frac{1}{h}S_d - W \end{pmatrix} > 0 \qquad (19)$$

are satisfied for all $0 \leq \alpha \leq 1$. But since α appears linearly, (19) is satisfied for $0 \leq \alpha \leq 1$ if and only if it is satisfied for $\alpha = 0$ and $\alpha = 1$, which is (17) and

$$\begin{pmatrix} U & -D^a \\ -D^{aT} & \frac{1}{h}S_d - W \end{pmatrix} > 0$$

but the above is equivalent to (17). ■

In practice, W can be chosen as block-diagonal, resulting in a significant savings of computational as compared to the corresponding conditions in [6] and [7].

For large N, the computation can be rather demanding. However, it turns out the convergence to analytical solution can be accelerated by refining the bounding using a combination of the integral inequality and elimination of variables. This will allow estimate of stability bounds very close to the analytical solution with very small N, and therefore, requiring computation which is comparable to most other methods. The details will be discussed in [8].

5 Conclusions

An integral inequality is used to derive the stability results of time-delay systems based on discretized Lyapunov functional method. The resulting stability criterion can be verified with significantly less computation.

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