Global output regulation through singularities

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Abstract

In this paper, global output regulation problem using state-feedback for nonlinear systems with no relative degree is concidered. We introduce an invariant manifold on which tracking error tends to zero, and through which we derive a coordinate transformation of state. By means of the forwarding design method, it is shown that a simple feedback on the new coordinate attains the global output regulation through singularities.

1 Introduction

In this paper, we present a new method of the global output regulation through singularities.

Tracking control for nonlinear systems that have no relative degree has been addressed by the approximating method[1] and the switching method, but the exact tracking problem for such systems has been a very difficult problem. On the other hand, these systems may be stabilized to an equilibrium point using forwarding[4, 5, 6] or backstepping. In this work, we convert the tracking problem to a stabilization problem, in the framework of output regulation.

Output regulation of nonlinear systems has been studied actively in recent years. The reference signals to be tracked are generated by a stable autonomous system, so called exo-system. The first paper[7] to examine the nonlinear output-regulation problem considers local stability alone. Global or semi-global stability for minimum-phase systems was considered in following papers[8, 9, 10]. These studies assumed the system is minimum-phase, thus needing a relative degree.

We extend the target systems of global output regulation using state feedback to the system with no relative degree. We introduce new sort of PDE that defines an invariant manifold on which tracking error tends to zero. Using linear-growth conditions and some assumptions, it is proven that a use of feedback such that the state tends to the manifold attains the global output regulation for such a system, by means of the forwarding method.

2 Problem statements

We consider a tracking problem in a single-input single-output affine system in the following form:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_\rho &= x_{\rho+1} + \phi_0(x,\xi)u \\
\dot{x}_{\rho+1} &= \psi(x,\xi) + \phi_1(x,\xi)u \\
\dot{\xi} &= \eta(x,\xi)
\end{aligned} \tag{1}
$$

$$
y = h(x) = x_1 \tag{2}
$$

where $(x_1, \ldots, x_{\rho+1}, \xi^T)^T \in \Re^n$ denotes a state vector, $u \in \Re$ an input, and $y \in \Re$ an output. The system has an equilibrium point $x = 0$, $\xi = 0$ such that $\psi(0,0) = 0$ and $\eta(0,0) = 0$. Note that we do not assume that $\phi_0(x,\xi)$ is nonzero for all x, ξ, which means the relative degree of the system may not exist. We rewrite the system (1) shortly as

$$
\frac{d}{dt}\begin{pmatrix} x \\ \xi \end{pmatrix} = f(x,\xi) + g(x,\xi)u. \tag{3}
$$

The reference signal is generated by an exo-system

$$
\dot{w} = s(w) \tag{4}
$$

$$
y_d = \gamma(w). \tag{5}
$$

The exo-system has an equilibrium point $w = 0$ which satisfies $s(0) = 0$ and $\gamma(0) = 0$. The exo-system is assumed to be Lyapunov stable, i.e. there exists $U(w)$ such that

$$
\begin{aligned}
&U(w) > 0 \quad (w \neq 0), \qquad U(0) = 0 \\
&\frac{\partial U(w)}{\partial w}\gamma(w) \leq 0.
\end{aligned} \tag{6}
$$

The purpose of control is to find a controller with which tracking error $e = y - y_d$ converges to zero, and all state variables of (1) are bounded. The state may pass through the point such that $\phi_0(x,\xi) = 0$, therefore we address an output regulation problem through singularities.

3 Transformation of system

We make the following assumption that is one of the solvability conditions for a local version of the tracking problem.

Assumption 1 *There exist functions* $\pi(w) \in \Re^n$ *and* $c(w)$ *which satisfy*

$$f(\pi(w)) + g(\pi(w))c(w) = \frac{\partial \pi(w)}{\partial w} s(w) \tag{7}$$

$$\pi_1(w) = \gamma(w). \tag{8}$$

Let $\pi_A(w)$ denote $(\pi_1(w), \ldots, \pi_\rho(w))^T$, $\pi_B(w) = \pi_{\rho+1}(w)$, and $\pi_C(w) = (\pi_{\rho+2}(w), \ldots, \pi_n(w))^T$. Moreover, we assume the existence of another invariant manifold $x_{\rho+1} = \pi'(x_1, \ldots, x_\rho, \xi, w)$ including $(x^T, \xi^T)^T = \pi(w)$.

Assumption 2 *There exist functions* $\pi'(\bar{x}, \xi, w) \in \Re$ *and* $c'(\bar{x}, \xi, w) \in \Re$ *such that*

$$\begin{aligned} &\psi(\bar{x}, \pi'(\bar{x}, \xi, w), \xi) + \phi_1(\bar{x}, \pi'(\bar{x}, \xi, w), \xi)c'(\bar{x}, \xi, w) \\ &\quad = \left(\sum_{i=1}^{\rho-1} \frac{\partial \pi'}{\partial x_i} x_{i+1} \right) + \frac{\partial \pi'}{\partial x_\rho} \{\pi'(\bar{x}, \xi, w) \\ &\qquad + \phi_0(\bar{x}, \pi'(\bar{x}, \xi, w), \xi)c'(\bar{x}, \xi, w)\} \\ &\qquad + \frac{\partial \pi'}{\partial \xi} \eta(\bar{x}, \pi'(\bar{x}, \xi, w), \xi) + \frac{\partial \pi'}{\partial w} s(w) \end{aligned} \tag{9}$$

$$\begin{aligned} &\pi'(\bar{x}, \xi, w) + \phi_0(\bar{x}, \pi'(\bar{x}, \xi, w), \xi)c'(\bar{x}, \xi, w) \\ &\quad - \frac{\partial \pi_\rho}{\partial w} s(w) + \alpha_0(x_1 - \pi_1(w)) + \\ &\qquad \cdots + \alpha_{\rho-1}(x_\rho - \pi_\rho(w)) = 0 \end{aligned} \tag{10}$$

$$c'(\pi_A(w), \pi_C(w), w) = c(w) \tag{11}$$

$$\pi'(\pi_A(w), \pi_C(w), w) = \pi_B(w) \tag{12}$$

$$\phi_1(\bar{x}, z, \xi) - \frac{\partial \pi'}{\partial x_\rho} \phi_0(\bar{x}, z, \xi) \neq 0 \tag{13}$$

where $\bar{x}$ *denotes* $(x_1, \ldots, x_\rho)^T$*, and* α*'s are constants such that polynomial*

$$s^\rho + \alpha_{\rho-1}s^{\rho-1} + \cdots + \alpha_1 s + \alpha_0 \tag{14}$$

is Hurwitz.

Equations (9),(10) mean that $x_{\rho+1} - \pi'(\bar{x}, \xi, w) = 0$ is an invariant manifold under the input $u = c'(\bar{x}, \xi, w)$, and on the manifold the error dynamics become

$$\frac{d^\rho e}{dt^\rho} + \alpha_{\rho-1} \frac{d^{\rho-1} e}{dt^{\rho-1}} + \cdots + \alpha_1 \dot{e} + \alpha_0 e = 0. \tag{15}$$

Under these assumptions, the system (1) can be transformed to

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ &\vdots \\ \dot{\tilde{x}}_{\rho-1} &= \tilde{x}_\rho \\ \dot{\tilde{x}}_\rho &= -\alpha_0 \tilde{x}_1 - \cdots - \alpha_{\rho-1}\tilde{x}_\rho + z \\ &\quad + \tilde{\phi}_0(\tilde{x}, z, \tilde{\xi}, w)(\tilde{u} + \tilde{c}(\tilde{x}, \tilde{\xi}, w)) \\ &\quad - \tilde{\phi}_0(\tilde{x}, 0, \tilde{\xi}, w)\tilde{c}(\tilde{x}, \tilde{\xi}, w) \\ \dot{z} &= \tilde{\psi}(\tilde{x}, z, \tilde{\xi}, w) - \tilde{\psi}(\tilde{x}, 0, \tilde{\xi}, w) \\ &\quad + \tilde{\phi}_1(\tilde{x}, z, \tilde{\xi}, w)(\tilde{u} + \tilde{c}(\tilde{x}, \tilde{\xi}, w)) \\ &\quad - \tilde{\phi}_1(\tilde{x}, 0, \tilde{\xi}, w)\tilde{c}(\tilde{x}, \tilde{\xi}, w) \\ &\quad - \frac{\partial \pi'}{\partial x_\rho}\{z + \tilde{\phi}_0(\tilde{x}, z, \tilde{\xi}, w)(\tilde{u} + \tilde{c}(\tilde{x}, \tilde{\xi}, w)) \\ &\qquad - \tilde{\phi}_0(\tilde{x}, 0, \tilde{\xi}, w)\tilde{c}(\tilde{x}, \tilde{\xi}, w)\} \\ &\quad - \frac{\partial \pi'}{\partial \xi}\{\tilde{\eta}(\tilde{x}, z, \tilde{\xi}, w) - \tilde{\eta}(\tilde{x}, 0, \tilde{\xi}, w)\} \\ \dot{\tilde{\xi}} &= \tilde{\eta}(\tilde{x}, z, \tilde{\xi}, w) - \tilde{\eta}(0, 0, 0, w) \end{aligned} \tag{16}$$

where

$$\tilde{x} = \bar{x} - \pi_A(w) \tag{17}$$

$$\tilde{\xi} = \xi - \pi_C(w) \tag{18}$$

$$\tilde{\pi}(\tilde{x}, \tilde{\xi}, w) = \pi'(\tilde{x} + \pi_A(w), \tilde{\xi} + \pi_C(w), w) \tag{19}$$

$$z = x_{\rho+1} - \tilde{\pi}(\tilde{x}, \tilde{\xi}, w) \tag{20}$$

$$\begin{aligned} &\tilde{\phi}_0(\tilde{x}, z, \tilde{\xi}, w) = \\ &\quad \phi_0(\tilde{x} + \pi_A(w), z + \tilde{\pi}(\tilde{x}, \tilde{\xi}, w), \tilde{\xi} + \pi_C(w)) \end{aligned} \tag{21}$$

$$\begin{aligned} &\tilde{\phi}_1(\tilde{x}, z, \tilde{\xi}, w) = \\ &\quad \phi_1(\tilde{x} + \pi_A(w), z + \tilde{\pi}(\tilde{x}, \tilde{\xi}, w), \tilde{\xi} + \pi_C(w)) \end{aligned} \tag{22}$$

$$\begin{aligned} &\tilde{\psi}(\tilde{x}, z, \tilde{\xi}, w) = \\ &\quad \psi(\tilde{x} + \pi_A(w), z + \tilde{\pi}(\tilde{x}, \tilde{\xi}, w), \tilde{\xi} + \pi_C(w)) \end{aligned} \tag{23}$$

$$\begin{aligned} &\tilde{\eta}(\tilde{x}, z, \tilde{\xi}, w) = \\ &\quad \eta(\tilde{x} + \pi_A(w), z + \tilde{\pi}(\tilde{x}, \tilde{\xi}, w), \tilde{\xi} + \pi_C(w)) \end{aligned} \tag{24}$$

$$\tilde{c}(\tilde{x}, \tilde{\xi}, w) = c'(\tilde{x} + \pi_A(w), \tilde{\xi} + \pi_C(w), w) \tag{25}$$

$$\tilde{u} = u - \tilde{c}(\tilde{x}, \tilde{\xi}, w). \tag{26}$$

By expanding $\tilde{\phi}_0$, $\tilde{\phi}_1$, $\tilde{\psi}$, and $\tilde{\eta}$ as

$$\tilde{\phi}_0(\tilde{x}, z, \tilde{\xi}, w) = \tilde{\phi}_0(\tilde{x}, 0, \tilde{\xi}, w) + P_0(\tilde{x}, z, \tilde{\xi}, w)z \tag{27}$$

$$\tilde{\phi}_1(\tilde{x}, z, \tilde{\xi}, w) = \tilde{\phi}_1(\tilde{x}, 0, \tilde{\xi}, w) + P_1(\tilde{x}, z, \tilde{\xi}, w)z \tag{28}$$

$$\tilde{\psi}(\tilde{x}, z, \tilde{\xi}, w) = \tilde{\psi}(\tilde{x}, 0, \tilde{\xi}, w) + P_2(\tilde{x}, z, \tilde{\xi}, w)z \tag{29}$$

$$\begin{aligned} \tilde{\eta}(\tilde{x}, z, \tilde{\xi}, w) &= \tilde{\eta}(0, 0, 0, w) + E_1(\tilde{\xi}, w)\tilde{\xi} \\ &\quad + E_2(\tilde{x}, \tilde{\xi}, w)\tilde{x} + E_3(\tilde{x}, z, \tilde{\xi}, w)z, \end{aligned} \tag{30}$$

the system (16) becomes

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ &\vdots \\ \dot{\tilde{x}}_{\rho-1} &= \tilde{x}_\rho \\ \dot{\tilde{x}}_\rho &= \{-\alpha_0 \tilde{x}_1 - \cdots - \alpha_{\rho-1}\tilde{x}_\rho\} + z \\ &\quad + \tilde{\phi}_0(\tilde{x}, z, \tilde{\xi}, w)\tilde{u} + \tilde{c}(\tilde{x}, \tilde{\xi}, w)P_0(\tilde{x}, z, \tilde{\xi}, w)z \end{aligned} \tag{31}$$

$$
\begin{aligned}
\dot{z} &= \theta(\tilde{x}, z, \tilde{\xi}, w)z \\
&\quad + \left\{ \tilde{\phi}_1(\tilde{x}, z, \tilde{\xi}, w) - \frac{\partial \tilde{\pi}}{\partial \tilde{x}_\rho} \tilde{\phi}_0(\tilde{x}, z, \tilde{\xi}, w) \right\} \tilde{u} \\
\dot{\tilde{\xi}} &= E_1(\tilde{\xi}, w)\tilde{\xi} + E_2(\tilde{x}, \tilde{\xi}, w)\tilde{x} + E_3(\tilde{x}, z, \tilde{\xi}, w)z,
\end{aligned}
$$

where

$$
\begin{aligned}
\theta(\tilde{x}, z, \tilde{\xi}, w) &= P_2(\tilde{x}, z, \tilde{\xi}, w) - \frac{\partial \tilde{\pi}}{\partial \tilde{x}_\rho} - \frac{\partial \tilde{\pi}}{\partial \tilde{\xi}} E_3(\tilde{x}, z, \tilde{\xi}, w) \\
&\quad + \tilde{c}(\tilde{x}, \tilde{\xi}, w) \left\{ P_1(\tilde{x}, z, \tilde{\xi}, w) - \frac{\partial \tilde{\pi}}{\partial \tilde{x}_\rho} P_0(\tilde{x}, z, \tilde{\xi}, w) \right\}
\end{aligned} \tag{32}
$$

If one can stabilize (31) globally, then global output regulation problem for system (1),(2) is solvable.

By applying a feedback

$$
\tilde{u} = \frac{-\theta(\tilde{x}, z, \tilde{\xi}, w)z + v}{\tilde{\phi}_1(\tilde{x}, z, \tilde{\xi}, w) - \dfrac{\partial \tilde{\pi}}{\partial \tilde{x}_\rho} \tilde{\phi}_0(\tilde{x}, z, \tilde{\xi}, w)} \tag{33}
$$

the system (31) is converted to

$$
\dot{\tilde{\xi}} = E_1(\tilde{\xi}, w)\tilde{\xi} + E_2(\tilde{x}, \tilde{\xi}, w)\tilde{x} + E_3(\tilde{x}, z, \tilde{\xi}, w)z \tag{34}
$$

$$
\begin{aligned}
\dot{\tilde{x}} &= \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -\alpha_0 & \cdots & \cdots & -\alpha_{\rho-1} \end{bmatrix} \tilde{x} \\
&\quad + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \delta(\tilde{x}, z, \tilde{\xi}, w)z \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \zeta(\tilde{x}, z, \tilde{\xi}, w) \end{pmatrix} v
\end{aligned} \tag{35}
$$

$$
\dot{z} = v \tag{36}
$$

where

$$
\zeta(\tilde{x}, z, \tilde{\xi}, w) = \frac{\tilde{\phi}_0(\tilde{x}, z, \tilde{\xi}, w)}{\tilde{\phi}_1(\tilde{x}, z, \tilde{\xi}, w) - \dfrac{\partial \tilde{\pi}}{\partial \tilde{x}_\rho} \tilde{\phi}_0(\tilde{x}, z, \tilde{\xi}, w)} \tag{37}
$$

$$
\begin{aligned}
\delta(\tilde{x}, z, \tilde{\xi}, w) &= 1 + \tilde{c}(\tilde{x}, \tilde{\xi}, w) P_0(\tilde{x}, z, \tilde{\xi}, w) \\
&\quad - \zeta(\tilde{x}, z, \tilde{\xi}, w)\theta(\tilde{x}, z, \tilde{\xi}, w).
\end{aligned} \tag{38}
$$

The system (34),(35),(36) is not a feedforward system because $\tilde{\xi}$ is included in equation (35). However, the terms including $\tilde{\xi}$ vanish in (35) when $z = 0$ and $v = 0$. Therefore, a method similar to forwarding design can be applied to this system.

4 Global output regulation with state feedback

In this section, global stabilization of the system (34), (35),(36) will be discussed, after we prove the following lemma:

Lemma 3 *Consider the system*

$$
\begin{aligned}
\dot{x} &= f(x, w) + g(x, z, w)z \\
\dot{z} &= q(z) \\
\dot{w} &= s(w)
\end{aligned} \tag{39}
$$

where $\dot{w} = s(w)$ is stable in the definition of Lyapunov, $\dot{x} = f(x, w)$ is globally asymptotically stable with a radially unbounded Lyapunov function $W(x, w)$ which satisfies

$$
W(x, w) > 0 \quad (x \neq 0), \quad W(0, w) = 0, \tag{40}
$$

$$
\frac{\partial W}{\partial x} f(x, w) + \frac{\partial W}{\partial w} s(w) < 0 \quad (x \neq 0), \tag{41}
$$

$$
\begin{aligned}
&\left\| \frac{\partial W}{\partial x} \right\| \|x\| \leq K(w) \|W(x, w)\|, \\
&\quad {}^{\exists}K(w) > 0, \; \textit{for } \|x\| \geq {}^{\exists}M > 0,
\end{aligned} \tag{42}
$$

and $\dot{z} = q(z)$ is globally asymptotically stable and locally exponentially stable with a radially unbounded Lyapunov function $U(z)$. Moreover, $g(x, z, w)$ satisfies linear-growth condition with respect to x. Then, $(x^T, z^T)^T$ converges to the origin.

Proof: First of all, boundedness of x will be shown. Because $g(x, z, w)$ satisfies linear-growth condition with respect to x, there exist $G_0(z, w) \geq 0$ and $G_1(z, w) \geq 0$ such that

$$
\|g(x, z, w)\| \leq G_0(z, w) + G_1(z, w)\|x\|. \tag{43}
$$

By evaluating the value of W

$$
\begin{aligned}
\dot{W} &= \frac{\partial W}{\partial x} \{ f(x, w) + g(x, z, w)z \} + \frac{\partial W}{\partial w} s(w) \\
&\leq \frac{\partial W}{\partial x} g(x, z, w)z \\
&\leq \left\| \frac{\partial W}{\partial x} \right\| \{ G_0(z, w)\|z\| + G_1(z, w)\|x\|\|z\| \} \\
&\leq W(x, w)K(w)(G_0(z, w) + G_1(z, w))\|z\|, \\
&\qquad (\|x\| \geq \max(1, M))
\end{aligned} \tag{44}
$$

is obtained. Because of the exponential stability of $\dot{z} = q(z)$ and the stability of $\dot{w} = s(w)$, there exist functions $\gamma_0(x(0), w(0)) > 0$, $\gamma_1(x(0), w(0)) > 0$, $K_{max}(w(0)) > 0$ and a positive constant α such that

$$
\|K(w)\| \leq K_{max}(w(0)) \tag{45}
$$

$$
\|G_0(z, w)\|\|z\| \leq \gamma_0(z(0), w(0)) \exp(-\alpha t) \tag{46}
$$

$$
\|G_1(z, w)\|\|z\| \leq \gamma_1(z(0), w(0)) \exp(-\alpha t). \tag{47}
$$

Therefore,

$$
\begin{aligned}
\dot{W} &\leq \gamma_2(z(0), w(0)) \exp(-\alpha t) W \\
&= K_{max}(w(0)) \{ \gamma_0(z(0), w(0)) + \gamma_1(z(0), w(0)) \} \\
&\qquad \cdot \exp(-\alpha t) W, \qquad (\|x\| \geq \max(1, M))
\end{aligned} \tag{48}
$$

is obtained, which derives

$$
\begin{aligned}
&W(x,w)\\
&\le \exp\left(\int_0^t \gamma_2(z(0),w(0))\exp(-\alpha\tau)d\tau\right)\\
&\qquad \cdot W(x(0),w(0))\\
&= \exp\left\{\frac{\gamma_2(z(0),w(0))}{\alpha}(1-\exp(-\alpha t))\right\}\\
&\qquad \cdot W(x(0),w(0))\\
&\le \exp\left(\frac{\gamma_2(z(0),w(0))}{\alpha}\right)W(x(0),w(0)).
\end{aligned} \tag{49}
$$

This equation shows that $W(x(t),w(t))$ is bounded. The value of $x(t)$ is also bounded because the function W is radially unbounded.

Secondly, consider the following integral:

$$
\begin{aligned}
&\Psi(x(0),z(0),w(0)) =\\
&\int_0^\infty \frac{\partial W(x(\tau),w(\tau))}{\partial x(\tau)} g(x(\tau),z(\tau),w(\tau))z(\tau)\,d\tau.
\end{aligned} \tag{50}
$$

The existence of $\Psi(\cdots)$ can be proved by the boundedness of $x(t)$, $w(t)$ and exponential stability of $\dot{z}=q(z)$. Indeed, there exists $\gamma_3(x(0),z(0),w(0))$ such that

$$
\begin{aligned}
&\left\|\frac{\partial W}{\partial x}\right\| \|g(x,z,w)\|\|z\|\\
&\quad \le \gamma_3(x(0),z(0),w(0))\exp(-\alpha t),
\end{aligned} \tag{51}
$$

and thus the integral can be evaluated as

$$
\begin{aligned}
&\Psi(x(0),z(0),w(0))\\
&\le \gamma_3(x(0),z(0),w(0))\int_0^\infty \exp(-\alpha\tau)\,d\tau\\
&= \frac{\gamma_3(x(0),z(0),w(0))}{\alpha}.
\end{aligned} \tag{52}
$$

We will show that the function $\Psi(\cdot)$ belongs to $\mathcal{C}_0$-class. Let $(x_0(t),z_0(t),w_0(t))$ denote the trajectory of system (39) for an initial state $X_0=(x_0(0)^T,z_0(0)^T,w_0(0)^T)^T$. For given $\epsilon>0$, there exists T such that

$$
\begin{aligned}
&\int_T^\infty \left|\frac{\partial W}{\partial x}g(x,z,w)z\right| dt < \frac{\epsilon}{4},\\
&\qquad \|(x(0)^T,z(0)^T,z(0)^T)^T - X_0\| < 1
\end{aligned} \tag{53}
$$

because the trajectories satisfy (51). Moreover, from the continuity of the trajectory with respect to the initial state, we can show the existence of $0<d\le 1$ such that

$$
\begin{aligned}
&\int_0^T \left|\frac{\partial W}{\partial \tilde{x}}g(x,z,w)z - \frac{\partial W(x_0,w_0)}{\partial x_0}g(x_0,z_0,w_0)z_0\right| dt\\
&\quad < \frac{\epsilon}{2}, \qquad \|(x(0)^T,z(0)^T,z(0)^T)^T - X_0\| < d.
\end{aligned} \tag{54}
$$

Hence, for given ϵ there exists d such that

$$
\begin{aligned}
&\int_0^\infty \left|\frac{\partial W}{\partial \tilde{x}}g(x,z,w)z - \frac{\partial W(x_0,w_0)}{\partial x_0}g(x_0,z_0,w_0)z_0\right| dt\\
&\quad < \epsilon, \qquad \|(x(0)^T,z(0)^T,z(0)^T)^T - X_0\| < d,
\end{aligned} \tag{55}
$$

which means the continuity of the function Ψ.

Adding the function $\Psi(\cdots)$ to the Lyapunov functions W and U makes a candidate of the Lyapunov function for the system (39)

$$
V(x,z,w) = W(x,w) + U(z) + \Psi(x,z,w). \tag{56}
$$

The identical equation

$$
\begin{aligned}
&W(x(s),x(s)) = W(x(t),w(t))\\
&\qquad + \int_t^s \dot{W}(x(\tau),z(\tau),w(\tau))\,d\tau
\end{aligned} \tag{57}
$$

implies

$$
\begin{aligned}
&W(x,w) + \Psi(x,z,w) = \lim_{s\to\infty}\Bigg\{W(x(s),w(s))\\
&\quad - \int_t^s \left(\frac{\partial W}{\partial x}f(x(\tau),w(\tau)) + \frac{\partial W}{\partial w}s(w(\tau))\right) d\tau\Bigg\} \ge 0.
\end{aligned} \tag{58}
$$

This equation shows that $V(x,z,w)\ge 0$. Because (58) holds, $V(x,z,w)=0$ means $U(z)=0$ and $W(x,w)+\Psi(x,z,w)=0$. Hence, $V(x,z,w)=0$ implies $z=0$ and $x=0$ by the definition of $\Psi(\cdot)$. Therefore, $V(x,z,w)$ is positive definite with respect to x and z.

Let us prove $V(x,z,w)$ is radially unbound with respect to x and z. It is obvious from (58) that V tends to infinity when $\|z\|$ tends to infinity. So, we only have to show that $W+\Psi\to\infty$ ($\|x\|\to\infty$). The derivative of the right-hand side of (58) can be evaluated as follows:

$$
\begin{aligned}
&\dot{W} - \frac{\partial W}{\partial x}f(x,w) - \frac{\partial W}{\partial w}s(w) = \frac{\partial W}{\partial x}g(x,z,w)z\\
&\ge -\left\|\frac{\partial W}{\partial x}\right\|(G_0(x,z,w)\|z\| + G_1(x,z,w)\|x\|\|z\|)\\
&\ge -\left\|\frac{\partial W}{\partial x}\right\|(\gamma_0(z(0),w(0)) + \gamma_1(z(0),w(0))\|x\|)\\
&\qquad \cdot \exp(-\alpha t)\\
&= -\left\|\frac{\partial W}{\partial x}\right\|[\{\gamma_0(z(0),w(0)) + \gamma_1(z(0),w(0))\}\|x\|\\
&\qquad + (1-\|x\|)\gamma_0(z(0),w(0))]\exp(-\alpha t)\\
&\ge -\Bigg[(\gamma_0(z(0),w(0)) + \gamma_1(z(0),w(0)))\left\|\frac{\partial W}{\partial x}\right\|\|x\|\\
&\qquad + \gamma_0(z(0),w(0))\left(\max_{\|x\|\le 1}\left\|\frac{\partial W}{\partial x}\right\|\right)\Bigg]\exp(-\alpha t).
\end{aligned} \tag{59}
$$

The above inequality becomes

$$\dot{W} - \frac{\partial W}{\partial x} f(x,w) - \frac{\partial W}{\partial w} s(w) \geq \begin{cases} -[\gamma_4(z(0),w(0))W + \gamma_5(z(0),w(0))]\exp(-\alpha t) \\ \qquad \text{when } \|x\| \geq M \\ -[\gamma_6(z(0),w(0)) + \gamma_5(z(0),w(0))]\exp(-\alpha t) \\ \qquad \text{when } \|x\| < M \end{cases} \tag{60}$$

where

$$\gamma_4(x(0),z(0)) = (\gamma_0(z(0),w(0)) + \gamma_1(z(0),w(0)))K_{max}(w(0)) \tag{61}$$

$$\gamma_5(x(0),z(0)) = \gamma_0(z(0),w(0)) \left(\max_{\|x\|<1} \left\| \frac{\partial W}{\partial x} \right\| \right) \tag{62}$$

$$\gamma_6(x(0),z(0)) = (\gamma_0(z(0),w(0)) + \gamma_1(z(0),w(0))) \left(\sup_{\|x\|\leq M} \left\| \frac{\partial W}{\partial x} \right\| \|x\| \right). \tag{63}$$

We can integrate (60) as

$$\begin{aligned} W \geq & \exp\left(-\frac{\gamma_4}{\alpha}(1-\exp(\alpha t))\right) W(x(0),w(0)) \\ & + \int_0^t \exp\left(-\frac{\gamma_4}{\alpha}(\exp(\alpha\tau) - \exp(\alpha t))\right) \\ & \cdot \left(-\gamma_5 \exp(-\alpha\tau) + \frac{\partial W}{\partial x} f(x(\tau), w(\tau) \right. \\ & \left. + \frac{\partial W}{\partial w} s(w(\tau))\right) d\tau \\ \geq & \exp\left(-\frac{\gamma_4}{\alpha}\right) W(x(0),w(0)) - \frac{\gamma_5}{\alpha} \\ & + \int_0^t \frac{\partial W}{\partial x} f(x(\tau),w(\tau)) + \frac{\partial W}{\partial w} s(w(\tau)) d\tau \\ & \text{when } \|x(\tau)\| \geq M \ (0 \leq \tau \leq t), \end{aligned} \tag{64}$$

$$\begin{aligned} W \geq & W(x(0),w(0)) \\ & + \int_0^t \left(-(\gamma_5 + \gamma_6)\exp(-\alpha\tau) \right. \\ & \left. + \frac{\partial W}{\partial x} f(x(\tau), w(\tau) + \frac{\partial W}{\partial w} s(w(\tau)) \right) d\tau \\ \geq & W(x(0),w(0)) - \frac{\gamma_5}{\alpha} - \frac{\gamma_6}{\alpha} \\ & + \int_0^t \frac{\partial W}{\partial x} f(x(\tau),w(\tau)) + \frac{\partial W}{\partial w} s(w(\tau)) d\tau \\ & \text{when } \|x(\tau)\| < M \ (0 \leq \tau \leq t). \end{aligned} \tag{65}$$

Combining (64) and (65) generates an inequality

$$\begin{aligned} W - & \int_0^t \frac{\partial W}{\partial x} f(x(\tau),w(\tau)) + \frac{\partial W}{\partial w} s(w(\tau)) d\tau \\ & \geq \exp\left(-\frac{\gamma_4}{\alpha}\right) W(x(0),w(0)) - \frac{\gamma_5}{\alpha} - \frac{\gamma_6}{\alpha}. \end{aligned} \tag{66}$$

Therefore,

$$\begin{aligned} & W(x(0),w(0)) + \Psi(x(0),z(0),w(0)) \\ & \quad \geq \exp\left(-\frac{\gamma_4}{\alpha}\right) W(x(0),w(0)) - \frac{\gamma_5}{\alpha} - \frac{\gamma_6}{\alpha} \end{aligned} \tag{67}$$

is established, which means that $W+\Psi$ tends to infinity when x tends to infinity. Hence, $V(x,z,w)$ is radially unbounded with respect to x and z.

Finally, by construction

$$\dot{V} = \frac{\partial U}{\partial z} q(z) + \frac{\partial W}{\partial x} f(x,w) + \frac{\partial W}{\partial w} s(w) \leq 0 \tag{68}$$

holds, so the system is globally asymptotically stable with respect to x and z. □

The simple version of this lemma without w is a well-known theorem.

We apply this lemma to the global stabilization problem for (34),(35),(36), with several assumptions.

The system dynamics restricted on $\tilde{x} = 0$, $z = 0$ is written as

$$\dot{\tilde{\xi}} = E_1(\tilde{\xi}, w)\tilde{\xi}. \tag{69}$$

The above dynamics correspond to the zero-error dynamics of I/O-linearizable systems ($\phi_0(\cdots) = 0$).

Assumption 4 *The system (69) is globally asymptotically stable with a Lyapunov function $W(\tilde{\xi}, w)$, i.e. there exists a Lyapunov function $W(\tilde{\xi}, w)$ such that*

$$W(0,w) = 0 \tag{70}$$

$$W(\tilde{\xi},w) > 0, \quad \tilde{\xi} \neq 0 \tag{71}$$

$$\frac{\partial W}{\partial \tilde{\xi}} E_1(\tilde{\xi},w)\tilde{\xi} + \frac{\partial W}{\partial w} s(w) < 0, \quad \tilde{\xi} \neq 0. \tag{72}$$

Moreover, there exists a positive function $K_0(w)$ and a positive constant M such that

$$\left\| \frac{\partial W}{\partial \tilde{\xi}} \right\| \|\tilde{\xi}\| < K_0(w)W, \quad \|\tilde{\xi}\| \geq M. \tag{73}$$

Let us consider the stability of the dynamics restricted on $z \equiv 0$

$$\begin{aligned} \dot{\tilde{\xi}} &= E_1(\tilde{\xi},w)\tilde{\xi} + E_2(\tilde{x},\tilde{\xi},w)\tilde{x} \\ \dot{\tilde{x}} &= \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -\alpha_0 & \cdots & \cdots & -\alpha_{\rho-1} \end{bmatrix} \tilde{x} = A\tilde{x}. \end{aligned} \tag{74}$$

To guarantee the stability of the system (74), we assume the following linear-growth condition:

Assumption 5 *The cross term $E_2(\tilde{x}, \tilde{\xi}, w)\tilde{x}$ satisfies linear-growth condition with respect to $\tilde{\xi}$, i.e. there exist positive functions $b_0(\tilde{x}, w)$, $b_1(\tilde{x}, w)$ such that*

$$\|E_2(\tilde{x}, \tilde{\xi}, w)\| \leq b_1(\tilde{x}, w)\|\tilde{\xi}\| + b_0(\tilde{x}, w). \qquad (75)$$

Under these assumptions the following theorem is established.

Theorem 6 *Under assumptions 1, 2, 4, and 5, the system (74) is globally asymptotically stable.*

Proof: The proof is straightforward from lemma 3. The Lyapunov function of (74) is

$$\begin{aligned} V_0(\tilde{x}, \tilde{\xi}, w) &= W(\tilde{\xi}, w) \\ &\quad + \int_0^\infty \frac{\partial(E_2\tilde{x})}{\partial\tilde{\xi}} E_1(\tilde{\xi}, w)\tilde{\xi} d\tau + \frac{1}{2}\tilde{x}^T P\tilde{x} \end{aligned} \qquad (76)$$

where P is a matrix satisfying

$$PA + A^T P = -I. \qquad (77)$$

□

To stabilize the dynamics of z, we adopt a feedback $v = -kz$ where k is a positive constant. Further assumptions are required for global stability.

Assumption 7 *There exist M_1 and $K_1(w)$ such that the Lyapunov function V_0 satisfies*

$$\left\| \frac{\partial V_0}{\partial(\tilde{x}, \tilde{\xi})} \right\| \cdot \left\| \begin{pmatrix} \tilde{x} \\ \tilde{\xi} \end{pmatrix} \right\| \leq K_1(w) V_0(\tilde{x}, \tilde{\xi}, w), \quad \|(\tilde{x}, \tilde{\xi})\| \geq M_1. \qquad (78)$$

Assumption 8 *The functions ζ, δ and E_3 satisfy linear-growth condition with respect to $\tilde{x}$ and $\tilde{\xi}$.*

Now, the main theorem is obtained.

Theorem 9 *Under the assumptions 1, 2, 4, 5, 7, and 8, the global output regulation problem is solved by a feedback $v = -kz$ where k is a positive constant.*

Proof: Under the feedback, z tends to zero exponentially. Therefore, global stability is obvious from lemma 3. □

5 Conclusion

We presented a method for global output regulation for nonlinear systems that have no relative degree. Future work will extend this method to error feedback case.

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