Bifurcation Control in Systems Governed by Functional Differential Equations*

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Abstract

This paper provides explicit sufficient conditions under which a Hopf bifurcation in systems described by functional differential equations can be stabilized. The main assumption is that the bifurcating modes are linearly unstabilizable and all other modes are linearly stabilizable. Stabilization of a Hopf bifurcation is defined as the existence of sufficiently smooth feedback control laws such that the Hopf bifurcation for the closed loop systems is supercritical. The construction of stabilizing control laws is explicit. We also give an example to illustrate the theory.

1 Introduction

Much research has been done on control of Hopf bifurcations in ordinary differential equations (For example, [1, 2, 5, 7, 8], to name just a few). One of the major scenarios is when the bifurcating mode is linearly unstabilizable so it is impossible to design a feedback to eliminate the bifurcation. Since a subcritical Hopf bifurcation is usually associated with abrupt inception of instabilities characterized by large amplitude limit cycles and hysteresis, it is sometimes desirable in practice to design feedbacks such that the bifurcation of the closed loop system is supercritical. Thereby the hysteresis associated with the subcritical bifurcation is eliminated and the instability inception becomes much more benign. In [7, 8], algebraic necessary and sufficient conditions of stabilizability of Hopf bifurcations were derived. When the Hopf bifurcation is stabilizable, explicit construction of a stabilizing feedback was given.

On the other hand, there are many physical systems that are modeled by infinite dimensional dynamical systems, including delay differential equations, integral equations, and partial differential equations. For example, reduced order models describing combustion instabilities in gas turbine engines are in the form of delay differential equations and the dynamics exhibits limit cycling behavior (see [3]). The limit cycles are detrimental and active control techniques that reduce or eliminate the limit cycles are desirable. Moreover, linear stabilizability of all modes might not be achievable by practically reliable actuators which are usually not distributed.

Many infinite dimensional systems including some partial differential equations can be described abstractly by functional differential equations of which the state space is a Banach space. The theory of center manifold and Hopf bifurcation has been developed for functional differential equations (see [4] and the references therein). In particular, the formula dictating the criticality of the Hopf bifurcation is also derived, and it is analogous to the case of ordinary differential equations.

In this paper, we investigate the stabilizablity of Hopf bifurcations in systems described by functional differential equations (including delay differential equations) by providing explicit sufficient conditions. The main assumption is that the bifurcating modes are linearly unstabilizable, and all other modes are linearly stabilizable. With this assumption, the essential dynamics is similar to the case of ordinary differential equations (ODEs) in that in both cases the dynamics of the controlled system near the equilibrium has a stable three dimensional center manifold which is foliated by periodic orbits. Despite the similarities, we need semigroup and spectral theory in the infinite dimensional case and the results cannot be viewed as a corollary of those developed in [7, 8] for the ODE case. The analysis in this paper is constructive so that the stabilizing feedbacks are given explicitly. We also give an example to illustrate the construction of stabilizing feedbacks.

2 Preliminaries

In this section we introduce some concepts that will be use in later sections. For more theory on semigroups and functional differential equations, see [4, 6].

Let $\{X, \|\cdot\|\}$ be a Banach space, the *dual space* of a Banach space $\{X, \|\cdot\|\}$ is the space of all linear functionals denoted by X^*. For any $x^* \in X^*$ and $x \in X$, we have $x^*(x) := \langle x^*, x\rangle \in \mathbb{R}$ or $\mathbb{C}$. The norm on X^* is

*Funding for this research was provided in part by AFOSR grant F49620-95-1-0409.

the induced norm defined by

$$\|x^*\| = \sup_{\|x\| \le 1} \langle x^*, x \rangle.$$

The *adjoint operator* of a bounded linear operator L: $X \longrightarrow X$ is defined as $L^*: X^* \longrightarrow X^*$, such that for every $x \in X$, $x^* \in X^*$, we have $\langle x^*, Lx \rangle = \langle L^* x^*, x \rangle$.

Definition 2.1 A C_0 *semigroup* on a Banach space $\{X, \|\cdot\|\}$ is a family of bounded linear operators $T = \{T(t)\}_{t \ge 0}$ satisfying

(1) $T(0) = I$ (the identity map),

(2) $T(t)T(s) = T(t+s)$, for $t, s \ge 0$,

(3) for any $\varphi \in X$, $\lim_{t \to 0+} \|T(t)\varphi - \varphi\| = 0$.

Definition 2.2 The *infinitesimal generator* of the semigroup T is a linear map A: $X \longrightarrow TX$ (TX is the tangent bundle of X) defined as

$$A\varphi = \lim_{t \to 0+} \frac{1}{t}(T(t)\varphi - \varphi).$$

The domain of A is defined as

$$\mathcal{D}(A) = \left\{ \varphi \mid \lim_{t \to 0+} \frac{1}{t}(T(t)\varphi - \varphi) \text{ exists} \right\}.$$

Definition 2.3 Let $\{X, \|\cdot\|\}$, and L: $\mathcal{D}(L) \subset X \longrightarrow X$ be a linear operator. The *point spectrum* $\sigma(L) \subset \mathbb{C}$ is the set of those λ such that $\lambda I - L$ is not one to one, i.e., there exists $\phi \in X$ and $\phi \ne 0$ such that $L\phi = \lambda\phi$. We call λ the *eigenvalue* of L and ϕ is the corresponding *eigenvector*. The *generalized eigenspace* of eigenvalue λ, denoted by $\mathcal{M}_\lambda(L)$, is defined as the smallest closed linear subspace contains all $\mathcal{N}\left((\lambda I - L)^j\right)$ for $j = 1, 2, \cdots$. If λ is an isolated point in $\sigma(L)$ and the dimension of its eigenspace is finite, then λ is called an *eigenvalue of finite type*. We call λ a *simple eigenvalue* if the dimension of its eigenspace is one.

Proposition 2.1 *([4]) Let L: $\mathcal{D}(L) \subset X \longrightarrow X$ a closed linear operator. Suppose* $\dim \mathcal{N}(\lambda I - L) = 1$, *and define $\mathbf{P}_\lambda$ as the projection operator to the eigenspace $\mathcal{M}_\lambda(L)$. Then for any $\psi \in X$,*

$$\mathbf{P}_\lambda \psi = \langle \phi^*, \psi \rangle \phi,$$

where ϕ and ϕ^* are eigenvectors of L and L^* corresponding to the eigenvalue λ satisfying $\langle \phi^*, \phi \rangle = 1$.

Definition 2.4 Let $T = \{T(t)\}_{t \ge 0}$ be a C_0 semigroup on a Banach space $\{X, \|\cdot\|\}$, the *abstract integral equation* is defined as

$$x(t) = T(t-s)x(s) + \int_s^t T(t-\tau)R(x(\tau), \mu)\, d\tau, \quad (1)$$

where $x(t) \in X$, R: $X \times \mathbb{R} \longrightarrow X$ is a nonlinear map, and μ is the bifurcation parameter.

Note in [4] the abstract integral differential equation is denoted by

$$x(t) = T(t-s)x(s) + \int_s^t T^{\odot *}(t-\tau)R(x(\tau), \mu)\, d\tau,$$

where $T^{\odot *}$ is the semigroup on $X^{\odot *}$, and $X^{\odot *}$ is the dual space of the closure of $\mathcal{D}(A)$, denoted by $X^{\odot} := \text{cl}\{\mathcal{D}(A)\}$. $X^{\odot *}$ is usually larger than X, and X is an embedding on $X^{\odot *}$. In this paper we identify $T^{\odot *}$ with T to avoid cumbersome notations.

Define the interval $\mathcal{I}_\epsilon := [\mu_0 - \epsilon, \mu_0 + \epsilon]$. Let f: $X_1 \times X_2 \times \cdots \times X_l \longrightarrow X$, where X and X_k ($k = 1, \cdots, n$) are Banach spaces, then we denote $D^m_{k_1 k_2 \cdots k_m} f(x_1, \cdots, x_l)$ as the m-th derivative of f with respect to the argument $x_{k_1}, x_{k_1}, \cdots, x_{k_m}$. Assume system (1) satisfies

H1. The map $(x, \mu) \mapsto R(x, \mu)$ from $X \times \mathcal{I}_\epsilon$ into X is C^k smooth, where $k \ge 3$.

H2. For all $\mu \in \mathcal{I}_\epsilon$, we have $R(0, \mu) = 0$ and $D_1 R(0, \mu) = 0$.

H3. Let A be the infinitesimal generator of the C_0 semigroup T. At $\mu = \mu_0$, A has simple eigenvalues at $\pm i\omega$ with $\omega > 0$. For $\mu \in \mathcal{I}_\epsilon$, all other eigenvalues have negative real parts.

H4. Let ϕ and ϕ^* be the eigenvectors of A and A^* corresponding to the eigenvalue $i\omega$. ϕ and ϕ^* are normalized such that $\langle \phi^*, \phi \rangle = 1$. We assume $\text{Re}\langle \phi^*, D_1 D_2 R(0, \mu_0)\phi \rangle \ne 0$.

Proposition 2.2 *([4]) Under assumptions H1–H4, there is a stable three dimensional center manifold passing through $(0, \mu_0)$ in $X \times \mathcal{I}_\epsilon$. Locally the center manifold is foliated by a family of periodic orbits whose frequency approaches ω when μ approaches μ_0. Define*

$$\begin{aligned} \tilde{\alpha} \;=\; & \frac{1}{2}\langle \phi^*, D^3_{111} R(0, \mu_0)(\phi, \phi, \bar{\phi}) \rangle \\ & + \langle \phi^*, D^2_{11} R(0, \mu_0)\left((-A)^{-1} D^2_{11} R(0, \mu_0)(\phi, \bar{\phi}), \phi\right) \rangle \\ & + \frac{1}{2}\langle \phi^*, D^2_{11} R(0, \mu_0)\left((2i\omega I - A)^{-1} \cdot \right. \\ & \left. D^2_{11} R(0, \mu_0)(\phi, \phi), \bar{\phi}\right) \rangle, \end{aligned}$$

where $\bar{\phi}$ denotes the complex conjugate of ϕ. Define $\alpha := \text{Re}\,\tilde{\alpha}$. If $\alpha > 0$, then the bifurcation is subcritical and the periodic orbits are unstable. If $\alpha < 0$, then the bifurcation is supercritical and the periodic orbits are stable. The case when $\alpha = 0$ is called degenerate since the stability of the periodic orbits are determined by higher order terms in the dynamics of the center manifold.

A linear autonomous retarded functional differential equation is given by

$$\begin{cases} \dot{x}(t) = \int_0^h d\zeta(\theta) x(t-\theta), & t \ge 0, \\ x(\theta) = \varphi(\theta), & -h \le \theta \le 0. \end{cases} \quad (2)$$

The state space of (2) is the Banach space $X = \mathbb{C}^n \times L^\infty([-h,0],\mathbb{C}^n)$. The abstract form of equation (2) is given by

$$\frac{dw}{dt} = Aw, \tag{3}$$

where $w := (a,\varphi) \in X$, and A is defined by $Aw := (\langle \zeta, \varphi\rangle, \dot{\varphi})$. The semigroup defined by (3) is given by $T(t) = e^{At}$, and A is the infinitesimal generator.

Proposition 2.3 *([4]) Let A be the infinitesimal generator of equation (3), then the spectrum of A consists of eigenvalues of finite type only. Furthermore, we have*

$$\sigma(A) = \{\lambda | \det \Delta(\lambda) = 0\},$$

where the characteristic matrix $\Delta(\lambda)$ is a mapping from $\mathbb{C}^n$ to $\mathcal{L}(\mathbb{C}^n)$ (the Banach space of all bounded linear operators on X):

$$\Delta(z) = zI - \int_0^h e^{-z\theta} d\zeta(\theta).$$

Proposition 2.4 *([4]) Let A be the infinitesimal generator of the semigroup defined by equation (3), and $\Delta(z)$ be the characteristic matrix. Assume the kernel ζ is real-valued. Suppose λ is an eigenvalue of A, and $p \in \mathbb{C}^{n\times 1}$, $q \in \mathbb{C}^{1\times n}$ satisfy $\Delta(\lambda)p = 0$, and $q\Delta(\lambda) = 0$. Define*

$$\begin{aligned} \phi(\theta) &= pe^{\lambda\theta}, \quad -h \le \theta \le 0, \\ \phi^*(\tau) &= \frac{q}{q\Delta'(\lambda)p}\left(I + \int_0^\tau \int_\sigma^h e^{\lambda(\sigma - s)} d\zeta(s) d\sigma\right), \\ & \quad 0 \le \tau \le h, \end{aligned}$$

where Δ' is the derivative of Δ. Then ϕ and ϕ^* are eigenvectors of A and A^* corresponding to the eigenvalue λ. Furthermore, we have $\langle \phi^*, \phi\rangle = 1$.

Note that for $\varphi \in \mathcal{C} := C([-h,0],\mathbb{R}^n)$, and $\varphi^* \in \mathcal{C}^* := C([0,h],\mathbb{R}^n)$, we have the bilinear function

$$\langle \varphi^*, \varphi\rangle := \int_0^h d\varphi^*(\tau)\varphi(-\tau),$$

where φ is a column vector and φ^* is a row vector, and $\mathcal{C}^*$ is the dual space of $\mathcal{C}$.

Now consider the nonlinear retarded functional differential equation

$$\begin{cases} \dot{x}(t) = \int_0^h d\zeta(\theta,\mu)x(t-\theta) + g(x_t,\mu), & t \ge 0, \\ x(\theta) = \varphi(\theta), & -h \le \theta \le 0. \end{cases} \tag{4}$$

We make the following assumptions for system (4):

Hd1. g is a C^k mapping from $X \times \mathcal{I}_\epsilon$ to $\mathbb{R}^n$, $k \ge 3$.

Hd2. $g(0,\mu) = 0$ and $D_1 g(0,\mu) = 0$ for all $\mu \in \mathcal{I}_\epsilon$.

Hd3. The mapping $(\mu,\varphi) \mapsto \int_0^h d\zeta(\theta,\mu)\varphi(-\theta)$ from $X \times \mathcal{I}_\epsilon$ to $\mathbb{R}^n$ is C^k, $k \ge 3$.

Hd4. $z = \pm i\omega$ are simple roots of $\det \Delta(z,\mu_0) = 0$, and all other roots of $\det \Delta(z,\mu) = 0$ have negative real parts for $\mu \in \mathcal{I}_\epsilon$.

Hd5. $\mathrm{Re}\{qD_2(i\omega,\mu_0)p\} \ne 0$.

Proposition 2.5 *([4]) Under the assumptions Hd1–Hd5, there is a stable three dimensional center manifold passing through $(0,\mu_0)$ in $X \times \mathcal{I}_\epsilon$, where $X = \mathbb{C}^n \times L^\infty([-h,0],\mathbb{C}^n)$. The center manifold is foliated by a family of periodic orbits whose frequency approaches ω when μ goes to μ_0. Let p and q satisfy $\Delta(i\omega,\mu_0)p = 0$, and $q\Delta(i\omega,\mu_0) = 0$, and $qD_1\Delta(i\omega,\mu_0)p = 1$. Define*

$$\begin{aligned} \tilde{\alpha} \;=\; & \frac{1}{2}\, qD_{111}^3 g(0,\mu_0)(\phi,\phi,\bar{\phi}) \\ & + qD_{11}^2 g(0,\mu_0)\left(e^{0\cdot}\Delta(0,\mu_0)^{-1} D_{11}^2 g(0,\mu_0)(\phi,\bar{\phi}),\phi\right) \\ & + \frac{1}{2}\, qD_{11}^2 g(0,\mu_0)\big(e^{2i\omega\cdot}\Delta(2\omega i,\mu_0)^{-1} \\ & D_{11}^2 g(0,\mu_0)(\phi,\phi),\bar{\phi}\big), \end{aligned}$$

and let $\alpha = \mathrm{Re}\,\tilde{\alpha}$. Then $\alpha > 0$ and $\alpha < 0$ correspond to subcritical and supercritical Hopf bifurcation, respectively. If $\alpha = 0$, then the bifurcation is degenerate in that criticality is determined by higher order terms in the dynamics on the center manifold.

3 Main Results

Let $T = \{T(t)\}_{t\ge 0}$ be a C_0 semigroup on a Banach space $\{X, \|\cdot\|\}$. Let $\{Y, \|\cdot\|\}$ be a Banach space. The abstract integral equation with control is defined as

$$x(t) = T(t-s)x(s) + \int_s^t T(t-\tau)R(x(\tau),u(\tau),\mu)\, d\tau, \tag{5}$$

where $x(t) \in X$, R: $X \times Y \times \mathbb{R} \longrightarrow X$ is a nonlinear map, $\mu \in \mathbb{R}$ is the bifurcation parameter. We make the following assumptions:

Hc1. The mapping R: $X \times Y \times \mathbb{R} \longrightarrow X$ is C^k smooth.

Hc2. $R(0,0,\mu) = 0$, $D_1 R(0,0,\mu) = 0$ for any $\mu \in \mathcal{I}_\epsilon$.

Let A be the infinitesimal generator of the C_0 simigroup T and define $B = D_2 R(0,0,\mu)$, i.e., B: $Y \longrightarrow X$ is a bounded linear mapping. The linearization of the abstract integral equation (6) is given by

$$x(t) = T(t-s)x(s) + \int_s^t T(t-\tau)Bu(t)\, d\tau, \tag{6}$$

or equivalently,

$$\frac{dx}{dt}(t) = Ax(t) + Bu(\tau), \tag{7}$$

where A is the infinitesimal generator of the semigroup T. Suppose λ is an eigenvalue of A, we call the eigenspace $\mathcal{M}_\lambda(A)$ a "mode" corresponding to λ.

Definition 3.1 A mode $\mathcal{M}_\lambda(A)$ is *stabilizable* if there exists a linear feedback $K\colon X \longrightarrow Y$, $x \mapsto u = Kx$, such that the projection of (7) onto $\mathcal{M}_\lambda(A)$ is asymptotically stable.

Lemma 3.1 *(1) If* $\operatorname{Re}\lambda < 0$*, then* $\mathcal{M}_\lambda(A)$ *is stabilizable.*

(2) If $\operatorname{Re}\lambda \geq 0$*,* λ *is a finite type eigenvalue, then the mode is stabilizable if and only if* $B^*\phi_j^* \neq 0$ $(j = 1, \cdots, m)$*, where* ϕ_j^* $(j = 1, \cdots, m) \in \mathcal{M}_\lambda(A^*)$ *are all the eigenvectors of* A^* *corresponding to the eigenvalue* λ*, and* $B^*\colon X^* \longrightarrow Y^*$ *is the dual of* B.

Proof: The first statement is trivial since $K = 0$ is the stabilizing controller. For the second statement, choose a set of basis in $\mathcal{M}_\lambda(A)$, denoted by V, such that the projection of A onto $\mathcal{M}_\lambda(A)$ is given by the Jordan form $J = \operatorname{Diag}[J_1, \cdots, J_m]$. More specifically, letting $\mathbf{P}_{\mathcal{M}_\lambda}$ be the projection operator from X to $\mathcal{M}_\lambda(A)$, then

$$\mathbf{P}_{\mathcal{M}_\lambda} x(t) = \langle V^*, x(t)\rangle V,$$

where V^* is the basis of $\mathcal{M}_\lambda(A^*)$ for eigenvalue λ satisfying $\langle V^*, V\rangle = I_N$, where $N = \dim \mathcal{M}_\lambda(A)$. Define $y(t) = \langle V^*, x(t)\rangle \in \mathbb{C}^N$, then equation (7) becomes

$$\begin{aligned}\frac{dy}{dt}(t) &= \langle V^*, Ax(t)\rangle + \langle V^*, Bu(t)\rangle, \\ &= Jy(t) + \langle B^*V^*, u(t)\rangle,\end{aligned}$$

where $y(t) = [y_{11}(t) \ \cdots \ y_{1d_1}(t) \ y_{21}(t) \ \cdots \ y_{md_m}(t)]$, then we have

$$\frac{dy_{k1}}{dt}(t) = \lambda y_{k1}(t) + \langle B^*\phi_k^*, u(t)\rangle, \ \ k = 1, \cdots, m.$$

The result follows easily. ∎

We make following assumptions for system (5):

Hc1. The map $(x, u, \mu) \mapsto R(x, u, \mu)$ from $X \times Y \times \mathcal{I}_\epsilon$ into X is C^k smooth, where $k \geq 3$.

Hc2. For all $\mu \in \mathcal{I}_\epsilon$, we have $R(0, 0, \mu) = 0$ and $D_1 R(0, 0, \mu) = 0$.

Hc3. Let A be the infinitesimal generator of the C_0 semigroup T. At $\mu = \mu_0$, A has simple eigenvalues at $\pm i\omega$ with $\omega > 0$. The modes corresponding to $\pm i\omega$ are linearly unstabilizable. All other modes are stabilizable.

Hc4. Let ϕ and ϕ^* be the eigenvectors of A and A^* corresponding to the eigenvalue $i\omega$, we have $\operatorname{Re}\langle \phi^*, D_1 D_2 R(0, \mu_0)\phi\rangle \neq 0$.

Theorem 3.1 *Suppose assumptions Hc1–Hc4 are satisfied, define*

$$\begin{aligned}\tilde{\alpha}_0 &= \frac{1}{2}\langle \phi^*, D_{111}^3 R(0, 0, \mu_0)(\phi, \phi, \bar{\phi})\rangle \\ &\quad + \langle \phi^*, D_{11}^2 R(0, 0, \mu_0)\big((-A)^{-1} \cdot \\ &\quad D_{11}^2 R(0, 0, \mu_0)(\phi, \bar{\phi}), \phi\big)\rangle \\ &\quad + \frac{1}{2}\langle \phi^*, D_{11}^2 R(0, 0, \mu_0)\big((2\omega i I - A)^{-1} \cdot \\ &\quad D_{11}^2 R(0, 0, \mu_0)(\phi, \phi), \bar{\phi}\big)\rangle, \\ \Theta_1 &= \operatorname{Re}\{\langle \phi^*, D_{12}^2 R(0, 0, \mu_0)(\phi, Id) + \\ &\quad D_{11}^2 R(0, 0, \mu_0)(\phi, (-A)^{-1} D_2 R(0, 0, \mu_0))\rangle\}, \\ \Theta_2 &= \langle \phi^*, D_{12}^2 R(0, 0, \mu_0)(\bar{\phi}, Id) + \\ &\quad D_{11}^2 R(0, 0, \mu_0)(\bar{\phi}, (2i\omega I - A)^{-1} D_2 R(0, 0, \mu_0))\rangle,\end{aligned}$$

and define $\alpha_0 = \operatorname{Re}\tilde{\alpha}_0$*. The Hopf bifurcation for the closed loop system can be made supercritical if one of the following conditions holds:*

(1) $\alpha_0 < 0$*,*

(2) $\alpha_0 \geq 0$*, either* $\Theta_1 \neq 0$*, or* $\Theta_2 \neq 0$*.*

Without loss of generality, suppose the stabilizable modes are stable. Then the control law is given by

$$\begin{aligned}u(x)(t) &= K_1\langle \phi^*, x(t)\rangle\langle \bar{\phi}^*, x(t)\rangle + K_2\langle \phi^*, x(t)\rangle^2 \\ &\quad + \bar{K}_2\langle \bar{\phi}^*, x(t)\rangle^2,\end{aligned}$$

where K_1*,* K_2 *are selected such that*

$$\Theta_1 K_1 + \operatorname{Re}\{\Theta_2 K_2\} + \alpha_0 < 0.$$

The idea in the proof of Theorem 3.1 is as follows. First, substitute the control law into the system equations, and notice the important fact that the critical eigenvalues and eigenvectors remain unchanged for the closed loop system. Then the closed loop system $\tilde{\alpha}$ can be obtained by utilizing Proposition 2.1 and a lengthy but straightforward calculations of different derivatives. We omit the details of the proof here due to space constraint.

Consider the retarded functional differential equation with control

$$\begin{aligned}\dot{x}(t) &= \int_0^h d\zeta(\theta, \mu)x(t - \theta) + g(x_t, u(t), \mu), \ \ t \geq 0, \\ x(\theta) &= \varphi(\theta), \ \ -h \leq \theta \leq 0, \qquad (8)\end{aligned}$$

where $u \in Y$, Y is a Banach space. We make the following assumptions:

Hdc1. g is a C^k mapping from $X \times Y \times \mathcal{I}_\epsilon$ to $\mathbb{R}^n$, $k \geq 3$.

Hdc2. $g(0, 0, \mu) = 0$ and $D_1 g(0, 0, \mu) = 0$ for all $\mu \in \mathcal{I}_\epsilon$.

Hdc3. The mapping $(\mu, \varphi) \mapsto \int_0^h d\zeta(\theta, \mu)\varphi(-\theta)$ from $X \times \mathcal{I}_\epsilon$ to $\mathbb{R}^n$ is C^k, $k \geq 3$.

Hdc4. $z = \pm i\omega$ are simple roots of $\det \Delta(z, \mu_0) = 0$. $\pm i\omega$ are linearly unstabilizable, i.e., $qD_2 g(0, 0, \mu_0) = 0$, where q satisfies $q\Delta(i\omega, \mu_0) = 0$. All other modes are stabilizable.

Hdc5. $\mathrm{Re}\{qD_2(i\omega, \mu_0)p\} \neq 0$ with $q \neq 0$, $p \neq 0$, $q\Delta(i\omega, \mu_0) = 0$, and $\Delta(i\omega, \mu_0)p = 0$.

Theorem 3.2 *Suppose assumptions Hdc1–Hdc5 are true. Let p and q satisfy $\Delta(i\omega, \mu_0)p = 0$, $q\Delta(i\omega, \mu_0) = 0$, and $qD_1\Delta(i\omega, \mu_0)p = 1$. Define*

$$\begin{aligned}
\tilde{\alpha}_0 &= \frac{1}{2}\, qD^3_{111} g(0, \mu_0)(\phi, \phi, \bar{\phi}) \\
&\quad + qD^2_{11} g(0, \mu_0)\big(e^{0\cdot}\Delta(0, \mu_0)^{-1} \\
&\quad D^2_{11} g(0, \mu_0)(\phi, \bar{\phi}), \phi\big) \\
&\quad + \frac{1}{2}\, qD^2_{11} g(0, \mu_0)\big(e^{2i\omega\cdot}\Delta(2\omega i, \mu_0)^{-1} \cdot \\
&\quad D^2_{11} g(0, \mu_0)(\phi, \phi), \bar{\phi}\big), \\
\Theta_1 &= \mathrm{Re}\,\big\{q\big[D^2_{12} g(0, 0, \mu_0)(\phi, Id) + \\
&\quad D^2_{11} g(0, 0, \mu_0)(\phi, e^{0\cdot}\Delta(0, \mu_0)^{-1} D_2 g(0, 0, \mu_0))\big]\big\}, \\
\Theta_2 &= q\big[D^2_{12} g(0, 0, \mu_0)(\bar{\phi}, Id) + D^2_{11} g(0, 0, \mu_0) \\
&\quad \big(\bar{\phi}, e^{2i\omega\cdot}\Delta(2\omega i, \mu_0)^{-1} D_2 g(0, 0, \mu_0)\big)\big],
\end{aligned}$$

and define $\alpha_0 = \mathrm{Re}\,\tilde{\alpha}_0$. The Hopf bifurcation for the closed loop system can be made supercritical if one of the following conditions holds.

(1) $\alpha_0 < 0$,

(2) $\alpha_0 \geq 0$, *either* $\Theta_1 \neq 0$, *or* $\Theta_2 \neq 0$.

Without loss of generality, suppose the stabilizable modes are stable. Then the control law is given by

$$\begin{aligned}
u(x)(t) &= K_1\langle\phi^*, x(t)\rangle\langle\bar{\phi^*}, x(t)\rangle + K_2\langle\phi^*, x(t)\rangle^2 \\
&\quad + \bar{K_2}\langle\bar{\phi^*}, x(t)\rangle^2,
\end{aligned}$$

where K_1, K_2 are selected such that

$$\Theta_1 K_1 + \mathrm{Re}\{\Theta_2 K_2\} + \alpha_0 < 0.$$

Theorem 3.2 is a direct corollary of Theorem 3.1 and Proposition 2.4.

4 An Example

Consider the following retarded functional differential equation

$$\dot{x}(t) = -\mu x(t-1) + ax(t-1)x(t-\tau) + x(t)u(t), \quad (9)$$

where $x, u \in \mathbb{R}$, x is the "state", u is the control input. $a, \tau \in \mathbb{R}$ are parameters. It is easy to see that there are two time delays of the system, $x(t) = 0$ is an equilibrium, and the system is linearly unstabilizable around $x = 0$.

The linearized equation is given by

$$\dot{x}(t) = -\mu x(t-1), \quad (10)$$

and the characteristic matrix (1 by 1) is given by

$$\Delta(z, \mu) = s + \mu e^{-z}.$$

Straightforward analysis on the zeros of the characteristic function reveals that the linearized system (10) is stable when $0 < \mu < \frac{\pi}{2}$. When μ increases passed $\frac{\pi}{2}$, $2\pi + \frac{\pi}{2}$, $\cdots$, $2n\pi + \frac{\pi}{2}$, etc, different pairs of eigenvalues cross the imaginary axis from the left half complex plane to the right half complex plane. When μ decreases passed 0, a real eigenvalue becomes unstable. When μ decreases passed $-2\pi + \frac{\pi}{2}$, $\cdots$, $-2n\pi + \frac{\pi}{2}$, etc, different pairs of eigenvalues cross the imaginary axis from the left half complex plane to the right half complex plane. In the following we concentrate on the first Hopf bifurcation when μ is near $\mu_0 = \frac{\pi}{2}$.

At $\mu = \frac{\pi}{2}$, the characteristic function has a pair of pure imaginary zeros $\pm\frac{\pi}{2}$. The right and left eigenvectors are given by

$$\begin{aligned}
\phi(\theta) &= e^{i\frac{\pi}{2}\theta}, \quad -1 \leq \theta \leq 0, \\
\phi^*(\sigma) &= \frac{1 - \frac{\pi}{2}i}{1 + \frac{\pi^2}{4}} e^{i\frac{\pi}{2}\sigma}, \quad 0 \leq \sigma \leq 1.
\end{aligned}$$

It is easy to check that $\langle\phi^*, \phi\rangle = 1$. For the uncontrolled system, the nonlinearity is given by

$$g(x)(t) = ax(t-1)x(t-\tau).$$

It is easy to calculate that

$$\begin{aligned}
D^2_{11} g\,(0)\,(\varphi_1, \varphi_2) &= a[\varphi_1(-1)\varphi_2(-\tau) \\
&\quad +\varphi_1(-\tau)\varphi_2(-1)], \\
D^3_{111} g\,(0)\,(\varphi_1, \varphi_2, \varphi_2) &= 0.
\end{aligned}$$

According to the formula in Theorem 3.2, straightforward calculations shows

$$\begin{aligned}
\alpha_0 &= \frac{8a^2}{5\pi\,(\pi^2+4)}\Bigg[(6+\pi)\sin\pi\tau + 2(1+\pi)\cos\pi\tau \\
&\quad - \left(2 + \frac{9\pi}{2}\right)\sin\frac{\pi\tau}{2} - (1+\pi)\cos\frac{\pi\tau}{2} - \frac{5\pi}{2}\Bigg].
\end{aligned}$$

A numerical plot of α as a function of time delay τ is given in Figure 1. It can be seen for some values of τ, $\alpha > 0$ and the Hopf bifurcation is subcritical; for some other values of τ, $\alpha < 0$ and the Hopf bifurcation is supercritical.

Now consider the controlled system, the nonlinear part is given by

$$g_c(x, u)(t) = ax(t-1)x(t-\tau) + x(t)u(t).$$

It can be easily calculate that

$$\begin{aligned}
D^2_{12} g_c(0, 0)(\varphi_1, \varphi_2) &= \varphi_1(0)\varphi_2(0), \\
D^2_{11} g_c(0, 0)(\varphi_1, \varphi_2) &= a(\varphi_1(-1)\varphi_2(-\tau) \\
&\quad +\varphi_1(-\tau)\varphi_2(-1)).
\end{aligned}$$

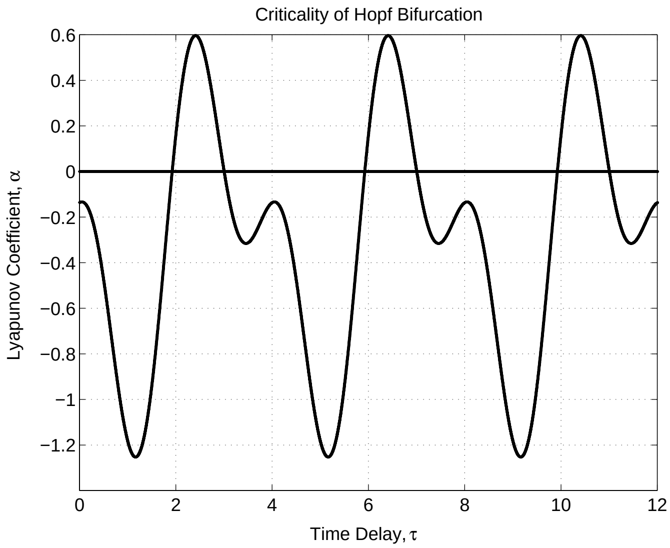

Figure 1: Lyapunov coefficient α as a function of time delay τ. In the plot, $a = 1$.

According to the difinition of Θ_1 in Theorem 3.2, we get

$$\begin{aligned}\Theta_1 &= \mathrm{Re}\left\{q \cdot \left[D_{12}^2 g_c(0,0,\mu_0)(\phi, Id) + D_{11}^2 g_c(0,0,\mu_0)\right.\right.\\ &\quad \left.\left.(\phi, e^{0\cdot}\Delta(0,\mu_0)^{-1} D_2 g_c(0,0,\mu_0))\right]\right\}\\ &= \frac{1}{1+\frac{\pi^2}{4}}.\end{aligned}$$

So the stabilizing feedback is given by

$$\begin{aligned}u(t) &= \frac{K}{1+\frac{\pi^2}{4}}\left(x(t) - \frac{\pi}{2}\int_0^1 e^{i\frac{\pi}{2}s}x(t-1+s)ds\right)\cdot\\ &\quad \left(x(t) - \frac{\pi}{2}\int_0^1 e^{-i\frac{\pi}{2}s}x(t-1+s)ds\right),\end{aligned}$$

where $K \in \mathbb{R}$ is selected such that $K > -\alpha_0\left(1+\frac{\pi^2}{4}\right)$.

5 Conclusions and Future Work

In this paper we have derived sufficient conditions under which a Hopf bifurcation in a functional differential equation can be changed from subcritical to supercritical. The conditions are explicit and stabilizing control laws can be explicitly constructed. We have also provided an example to illustrate the procedure of constructing stabilizing control laws.

Since time delay has a significant effect in many systems, such as ecological systems, biological systems, social-economic systems, and some engineering systems, the theory developed in this paper provides a tool to regulate these systems near a Hopf bifurcation point. Also, since many systems described by hyperbolic and parabolic partial differential equations (PDEs) can be expressed as abstract functional differential equations (see [6]), the theory developed in this paper makes it possible to regulate simple Hopf bifurcations in some PDEs.

The future work is to consider those scenarios when the sufficient conditions derived in this paper fail. In these cases, feedback with nonzero linear feedback gain on the unstabilizable modes is necessary, as in the ODE case (see [7, 8]). By separating the state into a stabilizable part and an unstabilizable part as in the ODE case, it is possible to derive necessary and sufficient conditions for the stabilizability of Hopf bifurcations in functional differential equations.

Acknowledgment The author would like to thank Prof. Richard Murray for many helpful discussions during the work and many corrections to the manuscript. His encouragement and suggestions are highly appreciated by the author.

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