A Set Theoretic Approach to the Simultaneous Localization and Map Building Problem

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Abstract

Self localization of mobile robots is one of the most important problems in long range autonomous navigation. When moving in an unknown environment, the navigator must exploit measurements from exteroceptive sensors to build a map, identify landmarks and, at the same time, localize itself with respect to them. This problem is known as Simultaneous Localization And Mapping (SLAM).

In this paper a set theoretic approach to the SLAM problem is presented. Estimates of the position of the robot and the selected landmarks are derived in terms of uncertainty regions, under the hypothesis that the errors affecting all sensor measurements are unknown but bounded. Set approximation techniques are adopted in order to provide efficient recursive algorithms, suitable for on-line implementation.

1 Introduction

Recent years have witnessed a growing interest towards mobile robotics and their use in hostile and unknown environments [1, 2]. In many of these applications, the robot cannot rely on the intervention of human operators, thus introducing a strong need for autonomy and reliability of the performed tasks. Self localization is then one of the most important problems to tackle for long range autonomous navigation, as it is required to ensure the mission fulfillment and the environment representation consistency. Though the position of a mobile vehicle can be continuously updated through the use of odometric or inertial sensors, wheel slippage, sensor drift an noise cause error accumulation, thus leading to erroneous estimates. Therefore, additional means for periodically localizing the robot must be considered (see e.g. [1, 2]). It is a common technique to measure some elements of the environment using exteroceptive sensors, and then exploit geometric properties, together with a priori knowledge, to obtain information on the robot position. However, when moving in an unknown environment, little (if any) a priori knowledge is available. This calls for a second task the robot has to perform during its exploration, i.e. map building from the measurements performed on the environment. The whole task, often called Simultaneous Localization And Mapping (SLAM) problem, is well-known and studied in literature, where several different kind of sensors, as well as environment representations, have been employed to achieve satisfactory solutions [3]-[8].

In this paper, differently from what is done in most of the literature, no statistical assumption is made on the errors that affect all the sensors; the only hypothesis is that they are bounded in norm by some quantity. This leads quite naturally to a set theoretic approach to the problem. Estimates of the position of the robot and the selected landmarks are derived in terms of *feasible uncertainty sets*, defined as those regions where the robot and the landmarks are guaranteed to lie, according to the available information. Since exteroceptive measurements are generally complicated nonlinear functions of robot and landmark positions, set approximation techniques are needed.

The basic reference theory for the technical development of the paper relies in the recently developed set membership estimation theory (see, e.g., [9]-[11]). Though most of this theory is developed for linear estimation problems, we will exploit the specific structure of the nonlinear SLAM problem to get efficient solutions, based on recursive approximations of the uncertainty regions through simply shaped sets.

The paper is structured as follows. Section 2 introduces the framework of the general SLAM problem and the set theoretic approach. Section 3 provides the main contribution on recursive approximation of the uncertainty sets. Section 4 reports results of numerical simulation experiments performed in order to assess the validity of the proposed procedures. Some concluding remarks are given in Section 5.

2 Set membership SLAM

When considering the SLAM problem, one has to deal with a moving vehicle, whose kinematic model is (partially or roughly) known, moving through an unknown environment in which it is possible to identify a number of features (or landmarks). The robot is equipped with sensors that can take measurements of relative position of the vehicle with respect to visible landmarks. The absolute location of each environment feature, as well as the robot starting position, are not known.

2.1 Vehicle and landmarks models

We assume that the environment and the robot can be adequately described by considering their projection on a plane (*flat* landscape assumption). Let $p(k) = [x(k)\ y(k)]'$ be the coordinates of the vehicle at time kT_s, where T_s denotes the sampling time. Under the assumption of slow dynamics, and since the odometers provide direct measurements of the vehicle displacement, the robot dynamics can be described by the linear discrete-time model

$$p(k+1) = p(k) + m(k) + G(k)w(k), \qquad (1)$$

where $m(k) \in \mathbb{R}^2$ are the x-axis and y-axis displacement measurements provided by encoders, and $w(k) \in \mathbb{R}^2$ are the errors affecting such measurements (possibly shaped by a suitable matrix $G(k)$).

Due to the presence of uncertainties in the model, the vehicle must collect measurements from the environment to localize itself. Moreover, since the robot has no a priori description of the environment, an efficient way to map it has to be pursued. In this work, the robot will select static landmarks, such that the time evolution of their position $L_i(k) = [x_{L_i}(k)\ \ y_{L_i}(k)]'$ can be described by $L_i(k+1) = L_i(k)$, their initial condition $L_i(0)$ being unknown.

In the SLAM problem, both landmarks and robot positions must be estimated, thus leading to state estimation of a dynamic system whose dimension can be very large, since it depends on the number of features present in the environment. Indeed, when n landmarks are considered, the state vector is given by $X(k) = [p'(k)\ \ L_1'(k)\ \ \ldots\ \ L_n'(k)]'$. The state update equation becomes

$$X(k+1) = X(k) + E_2 m(k) + E_2 G(k) w(k), \qquad (2)$$

where $E_2 = [I_2\ 0\ \ldots\ 0]' \in \mathbb{R}^{2(n+1)\times 2}$.

2.2 Exteroceptive measurements model

The measurements provided by exteroceptive sensors concern the relative distance and orientation between the robot and each visible landmark:

$$\begin{aligned} \Delta_i(k) &= d_i(X(k)) + v_{d_i}(k)\ , \\ \Theta_i(k) &= \theta_i(X(k)) + v_{\theta_i}(k)\ , \end{aligned} \qquad (3)$$

where $\Delta_i(k)$ and $\Theta_i(k)$ are the actual readings provided by the sensors at time k; $v_{d_i}(k)$ and $v_{\theta_i}(k)$ are measurement noise affecting, respectively, the distance and the heading measurements; and

$$\begin{aligned} d_i(X(k)) &= d(p(k), L_i(k)) \\ &\triangleq \sqrt{(x(k) - x_{L_i}(k))^2 + (y(k) - y_{L_i}(k))^2}\ , \\ \theta_i(X(k)) &= \theta(p(k), L_i(k)) \\ &\triangleq \arctan_2\{y_{L_i}(k) - y(k), x_{L_i}(k) - x(k)\}\ , \end{aligned} \qquad (4)$$

with $\arctan_2(b, a)$ denoting the four quadrant inverse tangent. Hence, in the second equations of (3)-(4) the orientation angle between the straight line connecting the robot with the selected landmark and a fixed direction (here chosen as the positive x-axis of the reference system), is measured. This is possible if the robot is equipped with a sensor able to measure the robot absolute heading, such as a compass. Each measurement gives two (noisy) nonlinear relations among four different components of the state (the robot coordinates and the selected landmark coordinates).

2.3 Set theoretic approach

Within the framework outlined above, we now introduce the general SLAM problem.

SLAM Problem: Given the robot and features model (2) and the measurement equations (3), construct an estimator of the (relative) position of the robot $p(k)$ and all the landmarks $L_i(k)$ at each time $k = 1, 2, \ldots$.

Depending on the assumptions on the unknown disturbances w, v_{d_i}, and v_{θ_i}, the problem can be tackled in several different ways. When statistical assumptions on the disturbances are considered, the estimate of the whole system state can be computed via the Extended Kalman Filter (EKF) [3, 8]. However, several problems arise when applying this approach. First, the statistical assumptions are not always satisfied (e.g., the presence of bias is seldom accounted for) and it is generally very difficult to estimate the true parameters of noise distributions, especially in natural environments. In addition, the EKF approach tends to be cumbersome as the number of features to be mapped grows. In fact, the computations grow as n^3, while the memory needed for storing all information grows as n^2 [12]. As a matter of fact, several ad hoc solutions have been proposed in order to solve the problem in a more efficient way [8, 13]. Clearly, another obvious problem is that convergence of EKF algorithms is not guaranteed.

In this paper, a different approach is presented, based on set membership hypotheses on the uncertainties. In particular, it is assumed that the disturbances are unknown-but-bounded, i.e.

$$\begin{aligned} |w_i(k)| &\le \epsilon_i^w(k) \quad i = 1, 2 && (5) \\ |v_{d_i}(k)| &\le \epsilon_i^{v_d}(k) \quad i = 1, \ldots, n && (6) \\ |v_{\theta_i}(k)| &\le \epsilon_i^{v_\theta}(k) \quad i = 1, \ldots, n && (7) \end{aligned}$$

where $\epsilon_i^w(k)$, $\epsilon_i^{v_d}(k)$ and $\epsilon_i^{v_\theta}(k)$ are known positive scalars. The disturbance bounds are time-varying, to explicitly account for changes in the environment conditions or in the vehicle dynamics.

The aforementioned hypotheses lead to a SLAM problem in which the estimate of the position of the robot and the landmarks can be expressed in terms of bounded sets.

Set Membership SLAM Problem: Let $\Xi(0) \subset \mathbb{R}^{2(n+1)}$ be a set containing the initial position of the vehicle and the landmarks X(0). Given the dynamic model (2) and the measurement equations (3)-(4), find at each time $k = 1, 2, \ldots$, the set $\Xi(k|k)$ of state vectors $X(k)$ which are compatible with the robot dynamics, the assumptions (5)-(7) on the disturbances, and the measurements collected up to time k.

Adopting standard terminology from set membership estimation theory [9], the set $\Xi(k|k)$ is the *feasible state set* at time k, based on the information available at time k. The solution of the above Set Membership SLAM Problem can be obtained by the following recursion

$$\Xi(0|0) = \Xi(0), \quad (8)$$
$$\Xi(k|k-1) = \Xi(k-1|k-1) + E_2 m(k) + +(E_2 G(k)\mathrm{Diag}\{\epsilon^w(k)\})\mathcal{B}_\infty, \quad (9)$$
$$\Xi(k|k) = \Xi(k|k-1) \bigcap \mathcal{M}(k), \quad (10)$$

where $\mathcal{B}_\infty$ is the unit ball in the ℓ_∞ norm, the *measurement set* $\mathcal{M}(k)$ is defined by

$$\mathcal{M}(k) = \bigcap_{i=1}^{n} \mathcal{M}_i(k), \quad (11)$$
$$\mathcal{M}_i(k) = \Big\{X \in \mathbb{R}^{2(n+1)} : \ |\Delta_i(k) - d_i(X)| \le \epsilon_i^{v_d}(k) \text{ and } |\Theta_i(k) - \theta_i(X)| \le \epsilon_i^{v_\theta}(k)\Big\}, \quad (12)$$

and the functions $d_i(\cdot)$ and $\theta_i(\cdot)$ are given by (4). Notice that, as all measurements are relative, we are allowed to choose an arbitrary reference system. Without loss of generality, we will set the origin of the reference system in the initial position of the robot.
Unfortunately, the exact computation of the sets Ξ in (9)-(10) is generally a prohibitive task. Gridding techniques for the evaluation of the sets would be prohibitively time consuming, and in addition they would not guarantee that the estimated region contains the true position of the robot and the landmarks. In the next section, efficient algorithms for on-line approximations of these sets will be introduced.

3 Approximations of the feasible position sets

In order to obtain computationally tractable solutions of the Set Membership SLAM problem, we look for simple approximations of the sets Ξ in (8)-(10). The approximating regions $\mathcal{R}$ must be simple enough to be recursively updated via efficient algorithms, suitable for on-line implementations. Moreover, at each time instant k, the approximating regions $\mathcal{R}(k|k-1)$, $\mathcal{R}(k|k)$ must contain the corresponding exact sets $\Xi(k|k-1)$, $\Xi(k|k)$, so that the true state vector $X(k)$ is guaranteed to belong to the approximating set.
To satisfy the above requirements, approximations are introduced at different stages of the SLAM algorithm:

1. Decomposition of the state vector $X(k)$ into subsets of state variables and set membership estimation of each subset.
2. Guaranteed approximations of the true feasible sets through classes of simple regions.

We discuss the role of these approximations separately.

3.1 State decomposition

As a first simplifying step, we decompose the state vector $X(k)$ into two different subsets of variables: robot position $p(k)$ and landmarks position $L(k) = [L_1'(k) \ \ldots \ L_n'(k)]'$. The two state subsets are considered separately in the measurement update step (10). First, we will process all the measurements in order to get a set of robot positions that are compatible with all the available information about landmarks and the measurements. During this step we do not update landmarks position. In the second step, we reprocess the same measurements to (possibly) tighten the uncertainty set of each landmark. This allows us to simplify the measurement update process, but it also introduces an approximation because information about correlation between robot and landmark position is lost.
Let us analyze how splitting of the state vector modifies the feasible set recursive updating outlined in (8)-(12). Let Ξ_p denote the feasible robot position set, Ξ_{L_i} denote the feasible ith landmark position set, and let $\Xi_p(0)$, $\Xi_{L_i}(0)$ be the corresponding initial sets (they can be easily obtained by projecting $\Xi(0)$ onto the subspaces defined by p and L_i). Eqns. (8) and (9) clearly split into

$$\Xi_p(0|0) = \Xi_p(0); \quad \Xi_{L_i}(0|0) = \Xi_{L_i}(0), \quad (13)$$
$$\Xi_p(k|k-1) = \Xi_p(k-1|k-1) + m(k) + +G(k)\mathrm{Diag}\{\epsilon^w(k)\}\mathcal{B}_\infty, \quad (14)$$
$$\Xi_{L_i}(k|k-1) = \Xi_{L_i}(k-1|k-1); \ i = 1, \ldots, n. \quad (15)$$

As said above, the measurement update (10) is performed in two steps. First, let us consider robot position. Since each distance and orientation measurement is relative, it is clear that landmark L_i "sees" the robot under the angle $\theta(p(k), L_i(k)) + \pi$. It turns out that, using the measurement taken with respect to the ith landmark, the position of the vehicle can be written as

$$\begin{cases} x(k) = x_{L_i}(k) - d_i(X(k)) \cos\theta_i(X(k)) \\ y(k) = y_{L_i}(k) - d_i(X(k)) \sin\theta_i(X(k)) \end{cases} . \quad (16)$$

Since $(x_{L_i}, y_{L_i}) \in \Xi_{L_i}(k)$, and noises affecting measurements of $d_i(X(k))$ and $\theta_i(X(k))$ are bounded, it turns out that each of the measurements provides a feasible set $\mathcal{C}_{R_i}(k)$ where the robot position must lie. Moreover, since the sources of uncertainty in (16) are independent, the set of feasible robot positions generated by the ith measurement at time k is given by

$$\mathcal{C}_{R_i}(k) = \Xi_{L_i}(k|k-1) + \mathcal{M}_{R_i}(k), \quad (17)$$

where

$$\mathcal{M}_{R_i}(k) = \{p \in \mathbb{R}^2 : \ |\Delta_i(k) - d(p,0)| \le \epsilon_i^{v_d}(k) \text{ and } |\Theta_i(k) - \theta(p,0)| \le \epsilon_i^{v_\theta}(k)\} \quad (18)$$

is the robot uncertainty set relative to the ith measurement, for a landmark placed at the origin of the reference system.
Since this process can be repeated for every landmark, the robot position is constrained to lie in the set

$$\mathcal{C}_R(k) = \bigcap_{i=1}^{n} \mathcal{C}_{R_i}(k). \quad (19)$$

Consequently, it turns out that measurement update for robot position can be performed as

$$\Xi_p(k|k) = \Xi_p(k|k-1) \bigcap \mathcal{C}_R(k). \quad (20)$$

As a second step, we reconsider each measurement performed at time k, this time trying to refine our knowledge on the landmarks. Using an approach similar to the one presented for the robot position, we can state that, due to the measurement at time k, the ith landmark will lie in the set defined by

$$\mathcal{C}_{L_i}(k) = \Xi_p(k|k) + \mathcal{M}_{L_i}(k), \tag{21}$$

where

$$\begin{aligned}\mathcal{M}_{L_i}(k) \quad = \quad & \{L \in \mathbb{R}^2 : \ |\Delta_i(k) - d(0,L)| \leq \epsilon_i^{v_d}(k) \\ & \text{and} \ \ |\Theta_i(k) - \theta(0,L)| \leq \epsilon_i^{v_\theta}(k)\}\end{aligned} \tag{22}$$

is the ith landmark uncertainty set, for a robot placed at the origin of the reference system. Consequently, measurement update of the ith landmark position can be performed as

$$\Xi_{L_i}(k|k) = \Xi_{L_i}(k|k-1) \bigcap \mathcal{C}_{L_i}(k). \tag{23}$$

Notice that the sets $\mathcal{M}_{R_i}(k)$ and $\mathcal{M}_{L_i}(k)$ are sectors of corona; some examples of $\mathcal{M}_{L_i}(k)$ are shown in Fig 1.

3.2 Set approximations

Computing exact sums and intersections of nonconvex regions bounded by nonlinear curves, as required by eqns. (14) and (17)-(23) is still a prohibitive task from the computational burden point of view. Hence, we pursue outer approximations of the sets $\Xi_p(k|k)$ and $\Xi_{L_i}(k|k)$ employing classes of simple structure sets. At each time k, minimum area sets in the chosen class, containing $\Xi_p(k|k-1)$, $\Xi_p(k|k)$, $\Xi_{L_i}(k|k-1)$ and $\Xi_{L_i}(k|k)$, will be selected.

Let us consider a class of regions $\mathcal{R}$ of fixed structure, and let us denote by $\bar{\mathcal{R}}(\mathcal{S})$ the minimum area set in the class $\mathcal{R}$, containing the set $\mathcal{S}$. It is easy to check that the desired approximation is obtained through the following recursion

$$\begin{aligned}
\mathcal{R}_p(0|0) &= \bar{\mathcal{R}}\left(\Xi_p(0)\right), && (24)\\
\mathcal{R}_{L_i}(0|0) &= \bar{\mathcal{R}}\left(\Xi_{L_i}(0)\right); \quad i = 1,\ldots,n && (25)\\
\mathcal{R}_p(k|k-1) &= \bar{\mathcal{R}}\left(\mathcal{R}_p(k-1|k-1) + m(k) + \right.\\
&\quad \left. + G(k)\mathrm{Diag}\{\epsilon^w(k)\}\mathcal{B}_\infty\right), && (26)\\
\mathcal{R}_{L_i}(k|k-1) &= \mathcal{R}_{L_i}(k-1|k-1), && (27)\\
\mathcal{R}_p(k|k) &= \bar{\mathcal{R}}\left(\mathcal{R}_p(k|k-1) \bigcap \mathcal{C}_R(k)\right), && (28)\\
\mathcal{R}_{L_i}(k|k) &= \bar{\mathcal{R}}\left(\mathcal{R}_{L_i}(k|k-1) \bigcap \mathcal{C}_{L_i}(k)\right). && (29)
\end{aligned}$$

In the following, we will consider axis-aligned boxes (or *orthotopes*) as approximating sets $\mathcal{R}$. An axis-aligned box is defined as

$$\mathcal{B} = \mathcal{B}(b,c) = \{q : \ q = c + \mathrm{Diag}\{b\}\alpha, \ \|\alpha\|_\infty \leq 1\},$$

where c is the center of the box and the absolute values of the elements of b represent the size of the edges. We observe that premultiplication of a box by a (nonsingular) square matrix gives a parallelotope, which is defined as $\mathcal{P} = \mathcal{P}(T,c) = \{q : \ q = c + T\alpha, \ \|\alpha\|_\infty \leq 1\}$, where c is the center and T is a (nonsingular) matrix whose column vectors represent the edges of the parallelotope.

According to recursion (26)-(29), one must solve the following three set approximation problems:

B1) compute the minimum area box containing the vector sum of a box and a parallelotope (eqn. (26));

B2) compute the minimum area box containing the intersection of a box and the robot measurement set $\mathcal{C}_R(k)$ (eqn. (28));

B3) compute the minimum area box containing the intersection of a box and the ith landmark measurement set $\mathcal{C}_{L_i}(k)$ (eqn. (29)).

Optimal solution to problem B1 has been presented in [14], and can be resumed by the following proposition. Let $\bar{\mathcal{B}}(\mathcal{S})$ denote the minimum area box containing the set $\mathcal{S}$ and e_i be the ith column of the identity matrix.

Proposition 1 *Let $\mathcal{R}(k-1|k-1) = \mathcal{B}(b,c)$. Then*

$$\bar{\mathcal{B}}(\mathcal{B}(b,c) + m(k) + G(k)\mathrm{Diag}\{\epsilon^w(k)\}\mathcal{B}_\infty) = \mathcal{B}(\bar{b},\bar{c}),$$

where

$$\begin{aligned}\bar{c} &= c + m(k);\\ \bar{b}_i &= \|[\mathrm{Diag}\{b\} \ G(k)\mathrm{Diag}\{\epsilon^w\}]'e_i\|_1 \quad i = 1,2.\end{aligned}$$

Now, let us consider problems B2 and B3. In order to simplify the computations, we introduce some further conservativeness by performing first the approximation of the sets $\mathcal{C}_R(k)$ and $\mathcal{C}_{L_i}(k)$, and then that of the sets defined by the intersections in eqns. (28) and (29). In other words, we replace the equations (28) and (29) with the following ones

$$\begin{aligned}\tilde{\mathcal{R}}_p(k|k) &= \bar{\mathcal{R}}\left(\mathcal{R}_p(k|k-1) \bigcap \bar{\mathcal{R}}\left(\mathcal{C}_R(k)\right)\right),\\ \tilde{\mathcal{R}}_{L_i}(k|k) &= \bar{\mathcal{R}}\left(\mathcal{R}_{L_i}(k|k-1) \bigcap \bar{\mathcal{R}}\left(\mathcal{C}_{L_i}(k)\right)\right).\end{aligned}$$

When axis-aligned boxes are used as approximating sets, the above equations take on the simpler form

$$\begin{aligned}\tilde{\mathcal{R}}_p(k|k) &= \mathcal{R}_p(k|k-1) \bigcap \bar{\mathcal{B}}\left(\mathcal{C}_R(k)\right), && (30)\\ \tilde{\mathcal{R}}_{L_i}(k|k) &= \mathcal{R}_{L_i}(k|k-1) \bigcap \bar{\mathcal{B}}\left(\mathcal{C}_{L_i}(k)\right). && (31)\end{aligned}$$

It is important to point out that sets $\tilde{\mathcal{R}}_p(k|k)$ and $\tilde{\mathcal{R}}_{L_i}(k|k)$ always contain sets $\mathcal{R}_p(k|k)$ and $\mathcal{R}_{L_i}(k|k)$, respectively. Moreover, we note that the conservativeness introduced by using (30)-(31) in place of (28)-(29) depends on the shape of sets $\mathcal{C}_R(k)$ and $\mathcal{C}_{L_i}(k)$. If these sets do not stretch along the diagonals of the $x-y$ plane, the approximation introduced is quite reasonable.

Due to the above simplification, the approximation problems to be solved are the following:

B2') compute the minimum area box containing the robot measurement set $\mathcal{C}_R(k)$ (eqn. (19));

B3') compute the minimum area box containing the ith landmark measurement set $\mathcal{C}_{L_i}(k)$ (eqn. (21)).

First of all, we observe that problem B3' is just a special case of problem B2', in which the set $\mathcal{C}_R$ is generated by only one measurement. In order to reduce the computational complexity of problem B2', a suboptimal solution based on recursive approximation is pursued: for every i, the minimum area box containing $\mathcal{C}_{R_i}$ is computed, and then all the approximating boxes are intersected. In other words, we replace $\bar{\mathcal{R}}(\mathcal{C}_R(k))$ in (30) by $\cap_{i=1}^n \bar{\mathcal{R}}(\mathcal{C}_{R_i}(k))$. Hence, problem B2' boils down to solving n problems of the same type of problem B3'. Thus, we concentrate on problem B3', and observe that the set $\mathcal{C}_{L_i}(k)$ arises from the sum of the current landmark feasible set and the sector of corona $\mathcal{M}_{L_i}(k)$ (see eqn. (21)). Then, we exploit the following result.

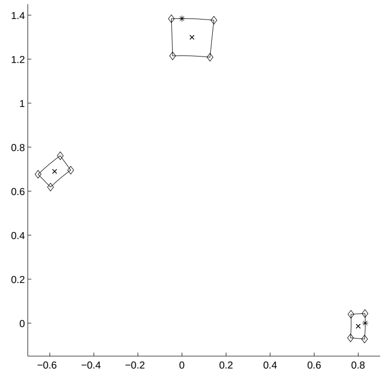


Figure 1: The points in $\mathcal{V}(\mathcal{M}_{L_i})$ and $\mathcal{T}(\mathcal{M}_{L_i})$ for three typical landmark uncertainty set $\mathcal{M}_{L_i}$ ($\diamond$: points in $\mathcal{V}(\mathcal{M}_{L_i})$; $*$: points in $\mathcal{T}(\mathcal{M}_{L_i})$).

Proposition 2 *Let $\mathcal{B}(k|k)$ be a given box containing $\Xi_p(k|k)$ and $\mathcal{M}_{L_i}(k)$ be defined as in (22). Then,*

$$\mathcal{C}_{L_i}(k) \subset \bar{\mathcal{B}}\left(\mathcal{B}(k|k) + \mathcal{M}_{L_i}(k)\right) = \mathcal{B}(k|k) + \bar{\mathcal{B}}\left(\mathcal{M}_{L_i}(k)\right).$$

Proof. It immediately follows from (21) and $\bar{\mathcal{B}}(\mathcal{B}+\mathcal{S}) = \mathcal{B}+\bar{\mathcal{B}}(\mathcal{S})$, which holds for any generic set $\mathcal{S}$ and box $\mathcal{B}$.

Let us denote by $\mathcal{V}(\mathcal{M}_{L_i})$ the set of vertices of $\mathcal{M}_{L_i}$. Moreover denote by $\mathcal{T}(\mathcal{M}_{L_i})$ the set of points of the convex arc bounding $\mathcal{M}_{L_i}$ (the one corresponding to the maximum distance $\Delta_i + \epsilon_i^{v_d}$), in which the tangent to the arc is parallel to one of the coordinate axes (see points denoted by asterisks in Fig. 1). The following result holds.

Proposition 3 *The minimum area box containing $\mathcal{M}_{L_i}(k)$ is the one containing the set*

$$\mathcal{VT} = \mathcal{V}(\mathcal{M}_{L_i}(k)) + \mathcal{T}(\mathcal{M}_{L_i}(k)),$$

i.e., $\bar{\mathcal{B}}(\mathcal{M}_{L_i}(k)) = \mathcal{B}(\bar{c}, \bar{b})$, where

$$\bar{c}_1 = (\overline{x} + \underline{x})/2, \quad \bar{c}_2 = (\overline{y} + \underline{y})/2$$

$$\bar{b}_1 = (\overline{x} - \underline{x})/2, \quad \bar{b}_2 = (\overline{y} - \underline{y})/2$$

and

$$\overline{x} = \max_{p \in \mathcal{VT}} x, \quad \underline{x} = \min_{p \in \mathcal{VT}} x,$$

$$\overline{y} = \max_{p \in \mathcal{VT}} y, \quad \underline{y} = \min_{p \in \mathcal{VT}} y.$$

Proof. Let co$\{\mathcal{S}\}$ denote the convex hull of the set $\mathcal{S}$. We have $\bar{\mathcal{B}}(\mathcal{M}_{L_i}(k)) = \bar{\mathcal{B}}(\text{co}\{\mathcal{M}_{L_i}(k)\})$. The boundary of co$\{\mathcal{M}_{L_i}(k)\}$ is made of three segments and one arc of circumference (the one corresponding to the maximum distance $\Delta_i(k) + \epsilon_i^{v_d}(k)$). Hence, it is clear that the only points that must be considered besides the vertices $\mathcal{V}(\mathcal{M}_{L_i}(k))$ are those (if any) in $\mathcal{T}(\mathcal{M}_{L_i}(k))$.

Propositions 2 and 3 provide the exact solution of problem B3', and hence an approximate solution of problem B2', in the sense explained above. It is then possible to compute a suboptimal solution of problems B2 and B3, as suggested by eqns. (30)-(31) and the previous discussion.
The kth step of the overall recursive approximating procedure is summarized in Table 1.

Let $\mathcal{B}_p(k-1\|k-1)$ and $\mathcal{B}_{L_i}(k-1\|k-1)$, $i = 1, \ldots, n$ be given.
- $\mathcal{B}_p(k\|k-1) = \bar{\mathcal{B}}(\mathcal{B}_p(k-1\|k-1) + m(k) + G(k)\text{Diag}\{\epsilon^w(k)\}\mathcal{B}_\infty)$ [see Prop. 1];
- $\mathcal{B}_{L_i}(k\|k-1) = \mathcal{B}_{L_i}(k-1\|k-1)$, $i = 1, \ldots, n$;
- $\mathcal{B}_{R_i} = \bar{\mathcal{B}}(\mathcal{C}_{R_i})$, $i = 1, \ldots, n$ [Prop. 2 and 3];
- $\mathcal{B}_R = \bigcap_{i=1}^{n} \mathcal{B}_{R_i}$;
- $\mathcal{B}_{L_i} = \bar{\mathcal{B}}(\mathcal{C}_{L_i})$, $i = 1, \ldots, n$ [Prop. 2 and 3];
- $\mathcal{B}_p(k\|k) = \mathcal{B}_p(k\|k-1) \cap \mathcal{B}_R$;
- $\mathcal{B}_{L_i}(k\|k) = \mathcal{B}_{L_i}(k\|k-1) \cap \mathcal{B}_{L_i}$, $i = 1, \ldots, n$;

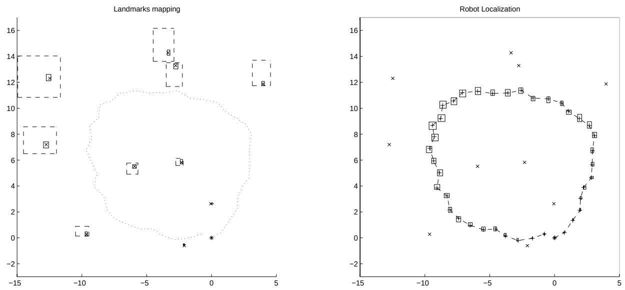
Table 1: The kth step of the orthotope-based recursive SLAM algorithm.

3.3 Initialization and computational complexity

The proposed set membership algorithm based on box approximations, must be initialized by selecting the boxes $\mathcal{R}_p(0|0)$, $\mathcal{R}_{L_i}(0|0)$, to start recursion (24)-(29). As observed in section 2.3, we can set the origin of the reference system at the initial position of the robot without loss of generality, so that $\mathcal{R}_p(0|0) = \mathcal{B}(0, 0)$. If the SLAM experiment is performed in a bounded environment of rectangular shape, one may select the entire environment as initial estimate $\mathcal{R}_{L_i}(0|0)$, $\forall i$. Otherwise, if no knowledge at all is available, the initial set estimate for landmark position may be obtained from the first set of measurements at time $t = 0$, i.e. $\mathcal{R}_{L_i}(0|0) = \bar{\mathcal{B}}(\mathcal{M}_{L_i}(0))$, $\forall i$.
The proposed set membership SLAM strategy can be easily implemented in real-time problems, due to its low computational complexity. In fact, the most demanding tasks are the computation of the sets $\mathcal{R}_p(k|k)$ and of the n sets $\mathcal{R}_{L_i}(k|k)$, in (28) and (29), respectively. It turns out that each problem, when solved via the proposed recursive suboptimal solution, requires $O(n)$ operations, and also the memory requirements to store all the information about the map are proportional to the number of landmarks. For these reasons, the proposed approach can handle also quite wide (or feature-dense) environments.

4 Numerical simulations

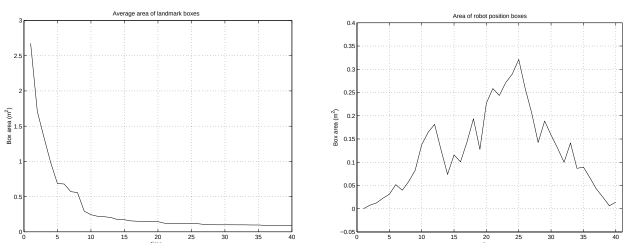
In this section, some simulation results of the set membership localization and mapping algorithm are reported. In all the simulations the robot covers a rough circle of about 35 metres, in a square area of 20 metres side. On this area, 10 landmarks are spread randomly. The vehicle moves and activates the SLAM algorithm once every metre covered. The simulations do not consider any problem of landmark visibility, and assume that at each iteration a scan of the whole horizon is available. The robot performs distance and angle measurements, corrupted by additive noises $v_{d_i}(k)$ and $v_{\theta_i}(k)$, generated as i.u.d. signals satisfying eqns. (6) and (7), with constant bounds for the



a b

Figure 2: Examples of the results provided by the algorithm: a) Landmark map (true landmark position: ×; estimates after the first iteration: dashed box; final estimates: solid box). b) Robot position estimates (solid boxes).

latter ($\epsilon_i^{v_\theta}(k) = \epsilon^{v_\theta},\ \forall i, k$), while the bounds on the former depend quadratically on the distance measured ($\epsilon_i^{v_d}(k) = \kappa_d d_i^2(X(k))$) [15]. In order to take into account the error accumulation occurring in odometric integration, disturbance $w_i(k)$ in eq. (2) is generated as a nonstationary i.u.d. signal, with mean value proportional to the distance covered during the last robot move. Figure 2 reports the result given by the algorithm. During this experiment, the odometry error bound is set to 5% of the distance covered, uncertainty bound on angle measurements is 3°, while the constant for the bound on distance measurements is $\kappa_d = 0.005$. Notice the remarkable uncertainty reduction for landmarks that are far from the robot initial position. Figure 3 shows some relevant quantities for a typical run. The robot uncertainty box generally grows when the robot moves in regions far from its initial position, where landmark localization is less accurate. Choosing the center of the box as current estimate of the robot position, the maximum localization error is less then 0.1 m, its mean value being 0.048 m.



a b

Figure 3: For the same experiment shown in fig. 2: a) Average area of landmark boxes at each step; b) Area of uncertainty box for the robot position at each step.

5 Conclusions

In this paper, an approach to the simultaneous localization and mapping for a mobile robot exploring an unknown environment has been presented. A set membership framework has been adopted to deal with the uncertainties affecting all the measurements available to the robot, thus requiring no strong assumptions on the measurement errors. The proposed technique provides recursive estimates for both robot and landmark uncertainty sets. Experimental integration, using landmarks detected from stereovision data in natural outdoor environments is currently in progress.

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