

# A Geometric Perspective on Bifurcation Control<sup>1</sup>

Yong Wang<sup>2</sup>  
yongwang@cds.caltech.edu

Richard M. Murray  
murray@cds.caltech.edu

Control and Dynamical Systems  
California Institute of Technology  
Pasadena, California 91125

## Abstract

In this paper, we analyze the problem of bifurcation control from a geometric perspective. Our goal is to provide coordinate free, geometric conditions under which control can be used to alter the bifurcation properties of a nonlinear control system. These insights are expected to be useful in understanding the role that magnitude and rate limits play in bifurcation control, as well as giving deeper understanding of the types of control inputs that are required to alter the nonlinear dynamics of bifurcating systems. We also use a model from active control of rotating stall in axial compression systems to illustrate the geometric sufficient conditions of stabilizability.

## 1 Introduction

Traditional approaches to feedback control of nonlinear systems focus on (potentially global) stabilization of an equilibrium point or a trajectory using smooth feedback. For systems whose linearization is controllable (or at least stabilizable), these techniques are guaranteed to work at least in a neighborhood of the point or trajectory.

We consider a class of system of the form

$$\dot{x} = f(x, u, \mu) \quad (1)$$

where  $x \in \mathbb{R}^n$  is the state,  $u \in \mathbb{R}$  is the input and  $\mu \in \mathbb{R}$  is a parameter, where the linearization of the system around an equilibrium point  $x_0$  is not stabilizable. Such problems arise in a number of contexts and are quite common in fluid systems, where stabilization of a steady flow field many not be achievable.

A specific example of this phenomena is stabilization of rotating stall and surge in compression systems us-

ing low spatial authority actuation schemes. It can be shown that for a variety of different actuation mechanisms, the linearization of the system is unstabilizable at a certain parameter value and hence it is not possible to extend the operating envelope of the system through simple stabilization. Rotating stall is a particularly detrimental instability because moving past the instability point results in a large amplitude limit cycle that is stable even if the parameter is reduced to its original value (hysteresis behavior).

Motivated by this system, Liaw and Abed demonstrated that it was possible to change the bifurcation behavior of the system using a simple model for the dynamics [3]. They showed that through the use of nonlinear feedback it was possible to “bend over” the bifurcation and convert a subcritical Hopf bifurcation (which results in a large amplitude limit cycle) to a supercritical Hopf bifurcation (which gives a small amplitude limit cycle and eliminates the hysteresis loop).

Several authors have built on the results of Liaw and Abed to explore bifurcation control of nonlinear systems (see, for example, Kang [2] and the references therein). Our previous results in the area have given necessary and sufficient conditions for modifying the criticality of the bifurcation [7, 8] as well as studied the effects of magnitude and rate saturations on control of bifurcations [5, 6].

To date, the conditions for bifurcation control of nonlinear systems have been predominantly algebraic in nature and have basically relied on the conditions of the Hopf theorem, which gives conditions under which the bifurcation is sub- or supercritical. In this paper we explore a more geometric description of these results and attempt to provide additional insight into control of bifurcations through nonlinear control techniques.

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<sup>2</sup>Currently in Division of Applied Mathematics at Brown University.

## 2 Linear time-invariant systems

In this section we give a brief review of linear stabilizability and introduce notations that will be used throughout this paper.

Consider the following linear time-invariant (LTI) system

$$\dot{x} = Ax + Bu := f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad (2)$$

which can be viewed as the linearization of a general nonlinear system around an equilibrium. Assume  $A$  has a simple pair of pure imaginary eigenvalues  $\pm i\omega$ , with left and right eigenvectors  $l, l^*, r, r^*$  satisfying

$$\begin{bmatrix} l \\ l^* \\ \tilde{l} \end{bmatrix} A \begin{bmatrix} r & r^* & \tilde{r} \end{bmatrix} = \begin{bmatrix} i\omega & 0 & 0 \\ 0 & -i\omega & 0 \\ 0 & 0 & A_0 \end{bmatrix}, \quad (3)$$

and

$$\begin{bmatrix} l \\ l^* \\ \tilde{l} \end{bmatrix} \begin{bmatrix} r & r^* & \tilde{r} \end{bmatrix} = I. \quad (4)$$

Any point  $x$  in the state space  $\mathbb{R}^n$  can be uniquely expressed as

$$x = rz + r^*z^* + \tilde{r}\tilde{x},$$

where  $z = lx$ ,  $z^* = l^*x$ , and  $\tilde{x} = \tilde{l}x$ . In other words,  $\mathbb{R}^n$  is uniquely decomposed as  $\mathcal{M}_o \oplus \tilde{\mathcal{M}}_o$ , where

$$\mathcal{M}_o = \{x \in \mathbb{R}^n \mid x = rz + r^*z^*, \quad z \in \mathbb{C}\}, \quad (5)$$

$$\tilde{\mathcal{M}}_o = \{x \in \mathbb{R}^n \mid x = \tilde{r}\tilde{x}, \quad \tilde{x} \in \mathbb{R}^{n-2}\}, \quad (6)$$

where both  $\mathcal{M}_o$  and  $\tilde{\mathcal{M}}_o$  are  $A$ -invariant, and  $\mathcal{M}_o$  is called a linear center manifold, which is foliated by circles

$$\{z \in \mathbb{C} \mid z = Ce^{i\omega t}, \quad C \in \mathbb{C}, \quad t \in \mathbb{R}\}.$$

$\tilde{\mathcal{M}}_o$  is the maximal  $A$ -invariant subspace in  $\mathbb{R}^n$  that does not contain  $\mathcal{M}_o$ . Suppose all other eigenvalues of  $A$  have negative real parts, the PBH test [1] says that the system is unstabilizable if and only if the projection of control vector  $B$  to the linear center manifold  $\mathcal{M}_o$  along  $\tilde{\mathcal{M}}_o$  is zero, i.e.,  $lB = l^*B = 0$ . Since  $B = \frac{\partial f}{\partial u}$ , the unstabilizability conditions can be written as

$$\mathbf{P}_{\mathcal{M}_o} \frac{\partial f}{\partial u} = 0,$$

where  $\mathbf{P}_{\mathcal{M}_o}$  is the projection operator along  $\tilde{\mathcal{M}}_o$ . For a nonlinear system with critical uncontrollable linearization, we expect that if the derivatives of the projection of the control vector along certain directions is nonzero, then the system is stabilizable, which will be explored in the next section.

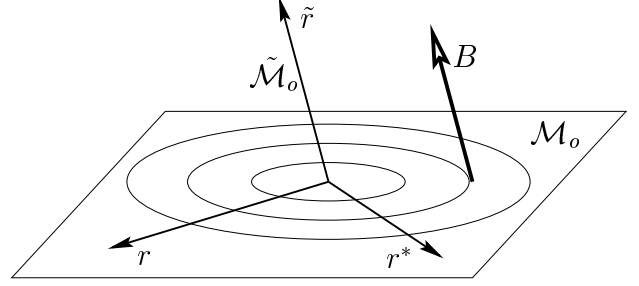


Figure 1: Geometry of linear unstabilizability.

## 3 Nonlinear systems

Consider a system of the form of equation (1) that undergoes a simple Hopf bifurcation at  $\mu = \mu^*$ . We assume that the linearization of the system at the bifurcation point has a single pair of imaginary eigenvalues and that all other eigenvalues have negative real part. Furthermore, we assume that the Hopf bifurcation is subcritical, so that the limit cycle created at the bifurcation point is unstable. Finally, we assume that the system is linearly unstabilizable through the input  $u$ , so that the stability of the linearization is not affected by the control law.

We wish to find a control law,  $u = u(x)$ , such that the closed loop system

$$\dot{x} = f(x, u(x), \mu)$$

undergoes a *super*-critical Hopf bifurcation at  $\mu^*$ . In order to study the geometry associated with bifurcation control, we begin by simplifying the problem to the bifurcation point  $\mu^*$ . Criticality of Hopf bifurcation is determined by nonlinear stability of the system at the bifurcation point and hence it suffices to nonlinearly stabilize the system at  $\mu^*$ .

Consider the system

$$\dot{x} = f(x, u), \quad f(0, 0) = 0, \quad (7)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ , and  $f$  is smooth. Define

$$A := \frac{\partial f}{\partial x}(0, 0), \quad B := \frac{\partial f}{\partial u}(0, 0).$$

Assume  $A$  has a pair of pure imaginary eigenvalues  $\pm i\omega$  whose eigenspace is linearly unstabilizable, i.e.,  $lB = 0$ . We assume  $l, l^*, \tilde{l}, r, r^*$ , and  $\tilde{r}$  satisfy (3) and (4).

For the uncontrolled system

$$\dot{x} = f(x, 0),$$

let  $\mathcal{M}_o$  be the linear center manifold given by (5). Since  $\mathcal{M}_o$  is uncontrollable, we have

$$\mathbf{P}_{\mathcal{M}_o} \frac{\partial f}{\partial u} \Big|_{(x,u)=(0,0)} = 0$$

according to the PBH test. The linear center manifold  $\mathcal{M}_o$  is the tangent space of the center manifold  $\mathcal{M}$  at  $x = 0$ , i.e.,  $\mathcal{M}_o = T_0\mathcal{M}$ .

Suppose the feedback is given by

$$u = F(z, z^*),$$

where  $z := lx$  and  $F$  satisfies  $F(0, 0) = 0$ ,  $\frac{\partial F}{\partial z}(0, 0) = 0$ , and  $\frac{\partial F}{\partial z^*}(0, 0) = 0$ , i.e., we only consider feedback with vanishing linear part. It is clear that the linear center manifold  $\mathcal{M}_o$  remains unchanged by the feedback, but the shape of center manifold  $\mathcal{M}$  changes with the feedback. In the  $(z, \tilde{x})$  coordinates, the system is given by

$$\begin{aligned}\dot{z} &= f_1(z, z^*, \tilde{x}, u), \\ \dot{\tilde{x}} &= f_2(z, z^*, \tilde{x}, u).\end{aligned}$$

Assume the center manifold  $\mathcal{M}$  for the closed loop system is parameterized by

$$\mathcal{M} = \{x_{\mathcal{M}} = (z, \tilde{x}) \mid \tilde{x} = \beta(u; z, z^*)\}, \quad (8)$$

which implies that  $x_{\mathcal{M}}$  is an *algebraic* function of  $u$ . Note this is generally not true because the dependence of  $x_{\mathcal{M}}$  on  $u$  is described by a partial differential equation. But let us assume (8) is true and let  $f = (f_1, f_2)$ , then by using the chain rule, we define  $\frac{df}{du}\Big|_{\mathcal{M}}^{u=0}$  as

$$\frac{df}{du}\Big|_{\mathcal{M}}^{u=0} := \frac{\partial f}{\partial \tilde{x}}(\tilde{x}, 0) \frac{\partial \beta}{\partial u}(0; z, z^*) + \frac{\partial f}{\partial u}(\tilde{x}, 0),$$

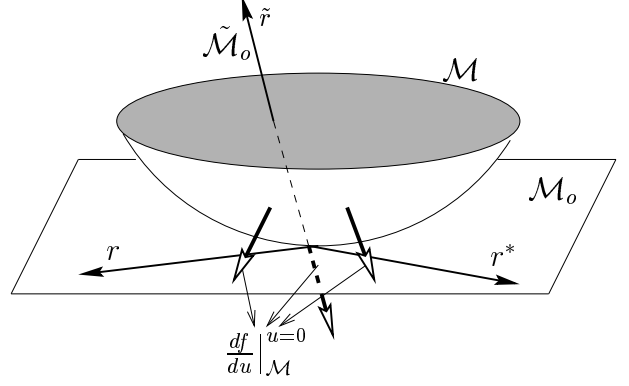
where  $\tilde{x} = (z, \beta(0; z, z^*)) \in \mathcal{M}$ . In the  $(z, \tilde{x})$  coordinates,  $\frac{df}{du}\Big|_{\mathcal{M}}^{u=0}$  is given by

$$\frac{df}{du}\Big|_{\mathcal{M}}^{u=0} = \begin{bmatrix} \frac{\partial f_1}{\partial \tilde{x}}(\tilde{x}, 0) \frac{\partial \beta}{\partial u}(0; z, z^*) \\ \frac{\partial f_2}{\partial \tilde{x}}(\tilde{x}, 0) \frac{\partial \beta}{\partial u}(0; z, z^*) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial u}(\tilde{x}, 0) \\ \frac{\partial f_2}{\partial u}(\tilde{x}, 0) \end{bmatrix}.$$

The geometric meaning of  $\frac{df}{du}\Big|_{\mathcal{M}}^{u=0}$  is the control vector evaluated on the controlled center manifold  $\mathcal{M}$ , which is different from the control vector evaluated at a fixed point in the state space denoted by  $\frac{\partial f}{\partial u}(x, 0)$ . A pictorial illustration of  $\mathcal{M}_o$ ,  $\mathcal{M}$ , and the control vector  $\frac{df}{du}\Big|_{\mathcal{M}}^{u=0}$  is given in Figure 2.

The following theorem provides geometric sufficient conditions of stabilizability of system (7) when the linear center manifold  $\mathcal{M}_o$  is linearly unstabilizable.

**Theorem 3.1** *Suppose the uncontrolled system (7) is unstable, then the system is stabilizable if one of the following two conditions holds.*



**Figure 2:** Geometry of nonlinear stabilizability.

1. For  $u = K|z|^2$ , where  $K \in \mathbb{R}^m$  and  $z = lx$ , we have

$$\operatorname{Re} L_r \left\langle l, \frac{df}{du}\Big|_{\mathcal{M}_c}^{u=0} \right\rangle = \operatorname{Re} L_r \mathbf{P}_r \frac{df}{du}\Big|_{\mathcal{M}}^{u=0} \neq 0$$

in a neighborhood of  $x = 0$ . Here  $\mathcal{M}$  is the center manifold,  $\mathbf{P}_r$  is the projection operator to  $r$  along  $\tilde{\mathcal{M}}_o \oplus \operatorname{span}\{r^*\}$ , and  $L_r$  is the Lie derivative along  $r$ . The stabilizing feedback is in the form of  $u = K|z|^2$ .

2. For  $u = Kz^2$ , where  $K \in \mathbb{C}^m$ , we have

$$L_{r^*} \left\langle l, \frac{df}{du}\Big|_{\mathcal{M}_c}^{u=0} \right\rangle = L_{r^*} \mathbf{P}_r \frac{df}{du}\Big|_{\mathcal{M}}^{u=0} \neq 0$$

in a neighborhood of  $x = 0$ . If this is satisfied, the stabilizing feedback is in the form of  $u = Kz^2 + K^*z^{*2}$ .

In the following, we use the term “the projection” to denote  $\mathbf{P}_r \frac{df}{du}\Big|_{\mathcal{M}}^{u=0}$ , i.e., the projection onto  $r$  along  $\tilde{\mathcal{M}}_o \oplus \operatorname{span}\{r^*\}$  for the control vectors evaluated on  $\mathcal{M}$ . The first statement says that if the real part of the derivative of “the projection” along  $r$  is nonzero, then the system is stabilizable, with the stabilizing feedback in the form  $u = K|z|^2$ , where  $K \in \mathbb{R}^m$ . The second statement says that if the derivative of “the projection” along  $r^*$  is nonzero, then the system is stabilizable, with the stabilizing feedback in the form  $u = Kz^2 + K^*z^{*2}$ , where  $K \in \mathbb{C}^m$ .

**Proof:** In the coordinates  $(z, \tilde{x})$ , the system (7) is given by

$$\begin{aligned}\dot{z} &= i\omega z + q(z, z^*, \tilde{x}, u) + c(z, z^*, \tilde{x}, u) + \text{h.o.t.}, \\ \dot{\tilde{x}} &= A_0 \tilde{x} + B_0 u + \tilde{q}(z, z^*, \tilde{x}, u) + \text{h.o.t.},\end{aligned}$$

where  $q$  and  $\tilde{q}$  denote the quadratic terms,  $c$  denotes the cubic terms, and h.o.t. denote “higher order terms”.

We assign the indices 1, 2, 3 and 4 to  $z$ ,  $z^*$ ,  $\tilde{x}$  and  $u$ , respectively. So  $q_{11}$  is the coefficient of  $z^2$  in the first equation,  $\tilde{q}_{12}$  denotes the coefficient of  $|z|^2$  in the second equation, etc.

Now substitute the feedback law  $u = K|z|^2$  into the equations, and assume the center manifold  $\mathcal{M}$  is parameterized by

$$\mathcal{M} = \{x_{\mathcal{M}} = (z, \tilde{x}) \mid \tilde{x} = \beta_1 z^2 + \beta_2 |z|^2 + \beta_3 z^{*2} + \text{h.o.t.}, z \in \mathbb{C}\}.$$

By differentiating (9) with respect to time and utilizing the system dynamics, we get

$$\begin{aligned} \beta_1 &= (2\omega i - A_0)^{-1} \tilde{q}_{11}, \\ \beta_2 &= (-A_0)^{-1} (B_0 K + \tilde{q}_{12}), \\ \beta_3 &= (-2\omega i - A_0)^{-1} \tilde{q}_{22}, \end{aligned}$$

Letting  $x = (z, \tilde{x})$ , and using the fact that  $u = K|z|^2$ , we have

$$\left. \frac{dx_{\mathcal{M}}}{du} \right|_{\mathcal{M}}^{u=0} = \begin{bmatrix} 0 \\ (-A_0)^{-1} B_0 + \text{h.o.t.} \end{bmatrix}.$$

Now letting

$$f(x, u) = \begin{bmatrix} i\omega z + q(z, z^*, \tilde{x}, u) + c(z, z^*, \tilde{x}, u) \\ A_0 \tilde{x} + B_0 u + \tilde{q}(z, z^*, \tilde{x}, u) \end{bmatrix} + \text{h.o.t.},$$

we have

$$\begin{aligned} \left. \frac{df}{du} \right|_{\mathcal{M}}^{u=0} &= \frac{\partial f}{\partial u} + \frac{\partial f}{\partial x} \left. \frac{dx_{\mathcal{M}}}{du} \right|_{\mathcal{M}}^{u=0} \\ &= \begin{bmatrix} q_{14}z + q_{24}z^* + q_{34}\tilde{x} + \text{h.o.t.} \\ \tilde{q}_{14}z + \tilde{q}_{24}z^* + \tilde{q}_{34}\tilde{x} + \text{h.o.t.} \end{bmatrix} + \\ &\quad \begin{bmatrix} i\omega + \text{h.o.t.} \\ 2\tilde{q}_{11}z + \tilde{q}_{12}z^* + \tilde{q}_{13}\tilde{x} + \text{h.o.t.} \\ q_{13}z + q_{23}z^* + 2q_{33}\tilde{x} + \text{h.o.t.} \\ A_0 + \text{h.o.t.} \end{bmatrix}. \\ &\quad \begin{bmatrix} 0 \\ (-A_0)^{-1} B_0 + \text{h.o.t.} \end{bmatrix}. \end{aligned}$$

Since the system is already in the coordinates  $(z, \tilde{x})$ , we have  $l = [1 \ 0]$ , and  $r = [1 \ 0]^T$ , so  $L_r = \frac{\partial}{\partial z}$ , and straightforward calculations show

$$L_r \left\langle l, \left. \frac{df}{du} \right|_{\mathcal{M}}^{u=0} \right\rangle = q_{14} + q_{13}(-A_0)^{-1} B_0 + \text{h.o.t.} \quad (9)$$

One the other hand, by substituting the center manifold expression (9) into the first equation of the system, we get the dynamics on the center manifold  $\mathcal{M}$ :

$$\dot{z} = i\omega z + q_{11}z^2 + q_{12}|z|^2 + q_{22}z^2 + C|z|^2 z + \text{o.c.t.} + \text{h.o.t.}, \quad (10)$$

where o.c.t. denotes ‘‘other cubic terms’’ of  $z$  and  $z^*$ , and  $C$  is given by

$$\begin{aligned} C &= c_{112} + q_{13}(-A_0)^{-1} \tilde{q}_{12} + q_{23}(2\omega i - A_0)^{-1} \tilde{q}_{11} \\ &\quad + (q_{14} + q_{13}(-A_0)^{-1} B_0) K. \end{aligned}$$

A normal form of (10) is given by

$$\dot{\zeta} = i\omega \zeta + \tilde{\alpha} |\zeta|^2 \zeta + \text{h.o.t.},$$

where  $\tilde{\alpha}$  is given by

$$\tilde{\alpha} = -\frac{q_{11}q_{12}}{i\omega} + C + \text{p.i.t.},$$

where p.i.t. denotes ‘‘pure imaginary terms’’. Define  $\alpha = \text{Re } \tilde{\alpha}$ , then we have

$$\alpha = \alpha_0 + \Theta_1 K,$$

where

$$\begin{aligned} \alpha_0 &= \text{Re} \left\{ -\frac{q_{11}q_{12}}{i\omega} + c_{112} + q_{13}(-A_0)^{-1} \tilde{q}_{12} \right. \\ &\quad \left. + q_{23}(2\omega i - A_0)^{-1} \tilde{q}_{11} \right\}, \\ \Theta_1 &= \text{Re} \{ q_{14} + q_{13}(-A_0)^{-1} B_0 \}. \end{aligned} \quad (11)$$

Since the uncontrolled system is not asymptotically stable, we have  $\alpha_0 \geq 0$ . It is straightforward from the normal form equation that the closed loop system is asymptotically stable if  $\alpha < 0$ . This can be achieved if  $\Theta_1 \neq 0$ . By comparing (9) and (11), we get

$$\Theta_1 = \text{Re } L_r \left\langle l, \left. \frac{df}{du} \right|_{\mathcal{M}}^{u=0} \right\rangle \Big|_{x=0}.$$

By smoothness of the vector fields, we conclude  $\Theta_1 \neq 0$  is equivalent to  $\text{Re } L_r \left\langle l, \left. \frac{df}{du} \right|_{\mathcal{M}}^{u=0} \right\rangle \neq 0$  in a neighborhood of  $x = 0$ .

A similar argument can be applied to the second statement in the theorem.  $\blacksquare$

As a special case, we consider the the following control affine system:

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0, \quad (12)$$

where  $x \in \mathbb{R}^n$ , and  $u \in \mathbb{R}^m$ . We assume  $g(0) = 0$  and  $A := \left. \frac{df}{dx} \right|_0(0)$  has a pair of pure imaginary eigenvalues  $\pm i\omega$ . Suppose  $l, l^*, \tilde{l}, r, r^*$  and  $\tilde{r}$  satisfy (3) and (4). Suppose all other eigenvalues of  $A$  have negative real parts. We also assume the uncontrolled system is not asymptotically stable at  $x = 0$ . Letting  $r = r_R + ir_I$ , then we have

$$Ar_R = -\omega r_I, \quad Ar_I = \omega r_R,$$

and  $A$  acts on the linear center manifold  $\mathcal{M}_o := \text{span}\{r_R, r_I\}$  by a rotation. We have the following corollary.

**Corollary 3.1** *The system is stabilizable if there exists a vector  $a \in \text{span}\{r_R, r_I\}$  such that  $L_a \mathbf{P}_a g \neq 0$  in a neighborhood of  $x = 0$ . Here  $\mathbf{P}_a$  is the projection operator onto  $\text{span}\{a\}$  along  $\tilde{\mathcal{M}}_o \oplus \text{span}\{b\}$ , where  $b$  satisfies  $a + bi = \kappa r$  for some  $\kappa \in \mathbb{C}$ , and  $\tilde{\mathcal{M}}_o$  is  $A$ -invariant and satisfies  $\tilde{\mathcal{M}}_o \oplus \mathcal{M}_o = \mathbb{R}^n$ .*

**Proof:** By using the same argument as in the proof of Theorem 3.1 and using the fact that  $g(0) = 0$ , we have

$$\begin{aligned} \text{Re } L_r \left\langle l, \frac{df}{du} \Big|_{\mathcal{M}_c} \Big|_{x=0}^{u=0} \right\rangle &= \text{Re } L_r \left\langle l, \frac{\partial f}{\partial u} \Big|_{x=0}^{u=0} \right\rangle \\ &= \text{Re } L_r \langle l, g \rangle_{x=0} \end{aligned}$$

and

$$\begin{aligned} L_{r^*} \left\langle l, \frac{df}{du} \Big|_{\mathcal{M}_c} \Big|_{x=0}^{u=0} \right\rangle &= L_{r^*} \left\langle l, \frac{\partial f}{\partial u} \Big|_{x=0}^{u=0} \right\rangle \\ &= L_{r^*} \langle l, g \rangle_{x=0} \end{aligned}$$

By Theorem 3.1, if either  $\text{Re } L_r \langle l, g \rangle_{x=0} \neq 0$ , or  $L_{r^*} \langle l, g \rangle_{x=0} \neq 0$ , the system is stabilizable.

In the following we prove that the fact there exists  $a \in \text{span}\{r_R, r_I\}$  such that  $L_a \mathbf{P}_a g|_{x=0} \neq 0$  implies either  $\text{Re } L_r \langle l, g \rangle_{x=0} \neq 0$ , or  $L_{r^*} \langle l, g \rangle_{x=0} \neq 0$ .

Any  $a \in \text{span}\{r_R, r_I\}$  can be written as

$$a = \kappa_R r_R - \kappa_I r_I = \frac{1}{2}(\kappa r + \kappa^* r^*) = \text{Re } \kappa r,$$

where  $\kappa = \kappa_R + i\kappa_I$ , and  $\kappa_R, \kappa_I \in \mathbb{R}$ . Define  $b \in \text{span}\{r_R, r_I\}$  as

$$b = \kappa_I r_R + \kappa_R r_I = \frac{1}{2i}(\kappa r - \kappa^* r^*) = \text{Im } \kappa r,$$

and define  $\xi$  as

$$\xi = \frac{1}{\kappa} l + \frac{1}{\kappa^*} l^*.$$

It is easy to verify that

$$\langle \xi, a \rangle = 1, \quad \langle \xi, b \rangle = 0, \quad \langle \xi, \tilde{r} \rangle = 0.$$

Suppose  $\kappa \neq 0$ , then any vector  $v \in \mathbb{R}^n$  can be expressed by the basis  $a, b$  and  $\tilde{r}$  as

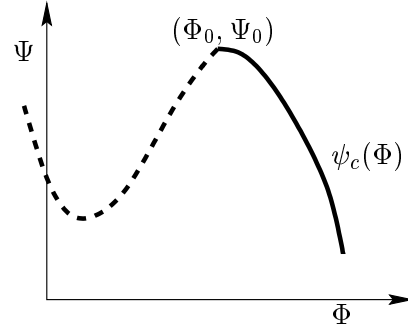
$$v = ac_1 + bc_2 + \tilde{r}c_3,$$

where  $c_1 = \langle \xi, v \rangle$ . So we have

$$\mathbf{P}_a g(x) = \langle \xi, g(x) \rangle,$$

and

$$\begin{aligned} L_a \mathbf{P}_a g(x) &= L_a \langle \xi, g \rangle \\ &= L_{\frac{1}{2}(\kappa r + \kappa^* r^*)} \left\langle \frac{1}{\kappa} l + \frac{1}{\kappa^*} l^*, g \right\rangle \\ &= \text{Re } L_r \langle l, g \rangle + \text{Re } \frac{\kappa^*}{\kappa} L_{r^*} \langle l, g \rangle. \end{aligned}$$



**Figure 3:** A typical compressor characteristic  $\psi_c(\cdot)$ .

From this expression it is clear that if  $\text{Re } L_r \langle l, g \rangle_{x=0} = L_{r^*} \langle l, g \rangle_{x=0} = 0$ , then we have  $L_a \mathbf{P}_a g(x)|_{x=0} = 0$  for any  $0 \neq \kappa \in \mathbb{C}$ . In other words,  $L_a \mathbf{P}_a g(x)|_{x=0} \neq 0$  for some  $a \in \text{span}\{r_R, r_I\}$  implies either  $\text{Re } L_r \langle l, g \rangle_{x=0} \neq 0$ , or  $L_{r^*} \langle l, g \rangle_{x=0} \neq 0$ . ■

The important assumption in the corollary is that  $g(0) = 0$ . With this assumption any nonlinear control law with vanishing linear part will not change the center manifold up to the second order, so we have

$$\frac{df}{du} \Big|_{\mathcal{M}}^{u=0} = \frac{\partial f}{\partial u} + \text{q.t.} = g(x) + \text{q.t.},$$

where q.t. denotes quadratic terms. In this case, we can use  $\frac{\partial f}{\partial u}$  in Theorem 3.1 instead of  $\frac{df}{du} \Big|_{\mathcal{M}}^{u=0}$ .

## 4 An example

As an example, we consider the Moore-Greitzer model for axial compression systems (see [4]):

$$\begin{aligned} \dot{z} &= \frac{1}{m + \mu} \left[ (\psi'_c(\Phi) - i\lambda) z + \frac{1}{2} \psi'''_c(\Phi) |z|^2 z \right] + \text{h.o.t.}, \\ \dot{\Phi} &= \frac{1}{l_c} [-\Psi + \psi_c(\Phi) + \psi''_c(\Phi) |z|^2] + \text{h.o.t.}, \\ \dot{\Psi} &= \frac{1}{4B^2 l_c} [\Phi - (\gamma + u) \sqrt{\Psi}], \end{aligned}$$

where  $z \in \mathbb{C}$  denotes the amplitude and phase of the first Fourier mode of rotating stall around the compressor face,  $\Phi, \Psi \in \mathbb{R}$  are annulus averaged flow and pressure-rise coefficient,  $u$  is the control input of bleed valves downstream of the compressor,  $\gamma$  is the throttle coefficient relating to the loading to the compressor,  $B$  is the Greitzer  $B$  parameter, and  $l_c, m, \lambda$  and  $\mu$  are compressor parameters.  $\psi_c(\cdot)$  is called the compressor characteristic. It is a nonlinear curve with a peak point  $(\Phi_0, \Psi_0)$  (see Figure 3). If  $u = 0$ , then for any  $\gamma$ , the nominal equilibrium of the system is  $(0, \Phi_e(\gamma), \Psi_e(\gamma))$ .

It is easy to check that the linearization of the system around the nominal equilibria is uncontrollable, and the uncontrollable mode is the stall mode  $z$ . Also, the stall mode is stable if  $\psi'_c(\Phi_e) > 0$  and unstable if  $\psi'_c(\Phi_e) < 0$ .  $B$  and  $\gamma$  are the bifurcation parameters in the model. If  $B$  is small enough, then the dynamics on the invariant plane  $z = 0$  is stable. When  $\gamma$  decreases past  $\gamma_0 = \frac{\Phi_0}{\sqrt{\Psi_0}}$ , a Hopf bifurcation occurs. It has been shown in [5] that if

$$\alpha := \frac{1}{4(m + \mu)} \left( \psi_c'''(\Phi_0) + \frac{\gamma_0 \psi_c''(\Phi_0)^2}{\sqrt{\Psi_0}} \right) > 0,$$

then the Hopf bifurcation is subcritical.

Letting  $x := (z, \Phi, \Psi)$ , and  $x_0 = (0, \Phi_0, \Psi_0)$ , and assuming  $B$  is small, we rewrite the system as

$$\dot{x} = f(x, u).$$

The left eigenvector of  $\frac{\partial f}{\partial x}(x_0)$  associated with the eigenvalue  $\frac{\psi'_c(\Phi_e) - i\lambda}{m + \mu}$  is  $l = [1 \ 0 \ 0]$ , and the right eigenvector is  $r = l^T$ . Following Theorem 3.1, we use the control law  $u = K|z|^2$ , then straightforward calculations show that the center manifold  $\mathcal{M}$  around the peak of the compressor characteristic is parameterized by  $x_{\mathcal{M}} = (z, \Phi_{\mathcal{M}}, \Psi_{\mathcal{M}})$ , where

$$\Phi_{\mathcal{M}} = \Phi_0 + \sqrt{\Psi_0}(\gamma - \gamma_0 + u) + \frac{\gamma_0 \psi_c''^2}{8\sqrt{\Psi_0}}|z|^2 + \text{h.o.t.}$$

$$\Psi_{\mathcal{M}} = \Psi_0 + \frac{\psi_c''}{4}|z|^2 + \text{h.o.t.},$$

so we have

$$\left. \frac{dx_{\mathcal{M}}}{du} \right|_{\mathcal{M}}^{u=0} = [ \ 0 \ \sqrt{\Psi_0} + \text{h.o.t.v.} \ \text{h.o.t.v.} ]^T.$$

Here h.o.t.v. denotes terms that vanish if evaluated at  $x = 0$ . Since  $l = [1 \ 0 \ 0]$  and  $r = l^T$ , we have  $L_r = \frac{\partial}{\partial z}$ , and  $\langle l, f \rangle = f_1$ . So

$$\begin{aligned} L_r \left\langle l, \left. \frac{df}{du} \right|_{\mathcal{M}}^{u=0} \right\rangle &= \frac{\partial}{\partial z} \left( \frac{\partial f_1}{\partial x} \left. \frac{dx_{\mathcal{M}}}{du} \right|_{\mathcal{M}} + \frac{\partial f_1}{\partial u} \right)_{u=0} \\ &= \sqrt{\Psi_0} \frac{\partial^2 f_1}{\partial z \partial \Phi} + \text{h.o.t.} \end{aligned}$$

By calculating the partial derivatives and evaluating the above expression at the peak of the characteristic, we get

$$\text{Re } L_r \left\langle l, \left. \frac{df}{du} \right|_{\mathcal{M}}^{u=0} \right\rangle \Big|_{x=0} = \frac{\sqrt{\Psi_0} \psi_c''(\Phi_0)}{m + \mu}.$$

By Theorem 3.1, if  $\psi_c''(\Phi_0) \neq 0$ , then the peak of the compressor characteristic can be stabilized by a nonlinear control law  $u = K|z|^2$ , which was constructed by Liaw and Abed in [3].

## 5 Summary

In this paper we have provided a geometric insight into the problem of feedback stabilization of nonlinear systems whose only linearly unstabilizable eigenvalues are a pair of pure imaginary numbers. More specifically, we have interpreted some of the sufficient algebraic conditions in our previous papers [7, 8] in a geometric framework. The key result here is that if the Lie derivative of the projection of the control vector along the linearly unstabilizable eigen-directions is nonzero, then the system is stabilizable via a nonlinear feedback. These conditions can be viewed as a generalization of the PBH stabilizability test for LTI systems to the nonlinear setting.

It should be pointed out that more work needs to be done for the more complex cases when the geometric sufficient conditions in this paper fail. We have algebraic conditions in the previous papers [7, 8] to deal with some of these scenarios, but the geometry of those conditions needs to be investigated.

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