Dynamic Feedback Control of Bifurcations

Hua O. Wang* Dong S. Chen Linda G. Bushnell†

Department of Electrical and Computer Engineering
Duke University, Durham, NC 27708 USA

Abstract

Bifurcation control deals with the modification of the bifurcation characteristics of a parameterized nonlinear system by a judiciously designed control input. In this paper, we consider the problem of dynamic feedback control of bifurcations. In particular, previous results on the control of bifurcations using washout filters are extended to some general forms of dynamic feedback controllers. It is shown that high-pass filters such as washout filters can be represented by a special form of dynamic feedback controller. The control effect on bifurcations can be readily assessed by analytical formulae. These dynamic feedback controllers offer more flexibility over the original controller in bifurcation control. The results are viable for the design and analysis of nonlinear control systems involving bifurcations.

1 Bifurcation and Bifurcation Control

Bifurcation denotes for a change in the number of candidate operating conditions of a nonlinear system when a parameter is quasistatically varied [3]. The candidate operating condition is either an equilibrium point, a periodic solution, or other invariant subset of its limit set, without regard to its stability properties. The parameter being varied is referred to as the bifurcation parameter. A nonlinear dynamical system can exhibit many different kinds of bifurcations as one or more parameters are varied. The complex behavior associated with bifurcations can be understood through bifurcation analysis.

Consider a general one-parameter finite-dimensional continuous-time system:

$$\dot{x}(t) = F\big(x(t); \mu\big). \qquad (1)$$

Here $x(t) \in \mathbb{R}^n$ is the state vector, $\mu \in \mathbb{R}$ is the bifurcation parameter, the vector field F is smooth in x and μ, and $F(0;\mu) = 0$, i.e., $x = 0$ is always a fixed point of the system. Suppose the *Jacobian* $L_0 = \frac{\partial F}{\partial x}\Big|_{\substack{x=0\\ \mu=0}}$ is singular at $\mu = 0$, and the fixed point $x = 0$ loses stability as μ changes its sign, then we can find a bifurcation occurring at $\mu = 0$. As the fixed point loses stability, the system either gives rise to a new branch of fixed points or converge to an oscillatory limit sets nearby. Bifurcations in (1) can be classified in many different ways. An important class of bifurcation is Hopf bifurcation, by which the equilibrium point bifurcates to a period orbit.

For one-parameter finite-dimensional discrete-time system

$$x_{k+1} = F(x_k; \mu), \qquad (2)$$

assume again, $F(0;\mu) = 0$ and $x = 0$ is always a fixed point of the system. Bifurcation occurs when one eigenvalue of the Jacobian matrix L_0 crosses the unit circle with the change of μ. We assume that this happens at $\mu = 0$. An important type of bifurcation in discrete-time system (2) is period doubling bifurcation, in which the output oscillates with a period 2, i.e., $x_{k+2} = x_k$ but $x_{k+1} \neq x_k$.

One important attribute of a bifurcation is the direction, or stability, of the bifurcation. Supercritical bifurcations permit smooth transition of system states; while subcritical, transcritical and saddle-node bifurcations normally lead to hysteresis and "jump" behaviors, which are undesirable. Therefore a typical bifurcation control objective is to delay and/or stabilize an existing non-superciritcal bifurcation, that is, convert the bifurcation to supercritical or eliminate it [1, 3].

In general, bifurcation control deals with the modification of the bifurcation characteristics of a parameterized nonlinear system by a control input. The control can be a static or dynamic feedback, or an open-loop control law. The objective of control can be stabilization and/or delay of a given bifurcation, reduction of the amplitude of bifurcated solutions, optimization of a performance index near bifurcation, re-shaping of a bifurcation diagram, or a combination of these. The work has been applied to control problems in high incidence flight, stall of compression system in jet engines,

*Corresponding author. E-mail: hua@ee.duke.edu; Tel: (919) 660-5273; Fax: (919) 660-5293. Dr. Wang is also with the Center for Nonlinear and Complex Systems and the Dept. of Control Science and Engineering, Huazhong University of Science and Technology, Wuhan, China.

†Dr. Bushnell is also with the Dept. of Electrical Engineering, University of Washington, Box 352500, Seattle, Washington 98195-2500.

voltage collapse in power systems and oscillatory behavior of tethered satellites (see [3, 7] and references therein).

Control of bifurcations by state feedback was investigated by Abed and Fu in [1, 2]. In [12], Wang and Abed applied washout filter-aided dynamic feedback controller to the problem of bifurcation control. In [11], Wang and Abed further extended the washout filter control to discrete-time systems. Recently, Kang [10] investigated the problem of bifurcation control with normal form and invariants. The effects of nonlinear control on stationary bifurcations are discussed from the differential geometry point of view. Gu et al [8] extended the state feedback controller to output feedback case and showed some interesting results on the equivalence of linear and nonlinear output feedback control of bifurcations. In this paper, we consider the problem of dynamic feedback control of bifurcations. In particular, the washout filter method is extended to a higher dimensional filter to render general dynamic feedback controllers. The control effects of this extended washout filter are computed analytically.

2 Bifurcation Theorems

The attributes of bifurcation such as location and direction can be determined analytically. Here we present the theorems for two kinds of most important bifurcations: Hopf bifurcation and period-doubling bifurcation. These background materials are necessary for bifurcation analysis and controller design in the sequel.

2.1 Hopf Bifurcation Theorem

Using Taylor series, the right hand side of (1) can be expanded to,

$$\dot{x} = Ax + \mu Lx + Q(x,x) + C(x,x,x) + o(x^3)$$

In this equation, A and L are constant matrices; $Q(x,x)$ is a quadratic function of x generated by a symmetric bilinear form; $C(x,x,x)$ is a cubic function generated by a symmetric trilinear form.

Assume that the linear part $A(\mu)$ at the origin has a pair of eigenvalues $\lambda_{1,2}(\mu) = \alpha(\mu) \pm i\omega(\mu)$ with $\alpha(\mu_0) = 0$ and $\omega_c = \omega(\mu_0) \neq 0$. Let l and r denote the normalized left and right eigenvector of $A(0)$ associated with $\lambda = i\omega_c$, i.e., $l(i\omega_c I - A) = (i\omega_c I - A)r = 0$ and $l \cdot r = 1$.

Further suppose that the pair of eigenvalues cross the imaginary axis with nonzero speed[1], i.e.,

$$\frac{\partial \alpha(\mu_0)}{\partial \mu} \neq 0, \tag{3}$$

[1] This condition is known as the *transversality* condition for the crossing of the eigenloci at the imaginary axis.

Then in any neighborhood U of the point x_0 and for any given $\delta > 0$, there is a $\bar{\mu}$ with $|\bar{\mu} - \mu_0| < \delta$ such that the system has a nontrivial periodic orbit in U. That is, the system is undergoing a Hopf bifurcation at the bifurcation point (x_0, μ_0) [9].

Denotes ϵ for the amplitude of the bifurcated solution, then the period of this periodic orbit is $T = 2\pi/\omega_c + o(\epsilon)$. The stability of the bifurcated periodic solution is determined by the critical eigenvalue of the periodic orbit. This eigenvalue is given by a series of ϵ as

$$\beta(\epsilon) = \beta_1 \epsilon + \beta_2 \epsilon^2 + \cdots \tag{4}$$

and the leading coefficients can be obtained by the following formulae [1]:

$$\beta_1 = 0$$
$$\beta_2 = 2\mathrm{Re}\{2lQ(r,a) + lQ(\bar{r},b) + \tfrac{3}{4}lC(r,r,\bar{r})\}$$

where $\bar{r}$ denotes for the complex conjugation of r. The parameters a and b are defined by

$$a \triangleq -\tfrac{1}{2}A^{-1}Q(r,\bar{r}),$$
$$b \triangleq \tfrac{1}{2}(2i\omega_c I - A)^{-1}Q(r,r).$$

If $\beta_2 = 0$, the stability of the period orbit is determined by higher terms of $\beta(\epsilon)$. In this case, the bifurcation is called degenerated. Generically, we assume that $\beta_2 \neq 0$. Under this assumption, the sign of β_2 determines the direction of bifurcation. The above discussions are summarized by the following theorem:

Theorem 2.1 (Hopf Bifurcation Theorem)
If a continuous-time system (1) satisfies all the conditions stated above, then there exists one Hopf bifurcation at $\mu = 0$. The bifurcation is supercritical if $\beta_2 < 0$ but is subcritical if $\beta_2 > 0$.

2.2 Period Doubling Bifurcation Theorem

An important type of bifurcation in a discrete-time system is period doubling bifurcation. Before stating the conditions of period-doubling bifurcation, we make the following assumption for Eq. (2):

(P) The map F of Eq. (2) is sufficiently smooth and has a fixed point at $x = 0$ for all $\mu = 0$. The linearization of (2) along the fixed point which is the continuous extension of the origin possesses an eigenvalue $\lambda_1(\mu)$ with $\lambda_1(0) = -1$ and $\lambda_1'(0) \neq 0$. All remaining eigenvalues of the linearization have magnitude less than unity.

Expanding the map F into Taylor series,

$$x_{k+1} = Ax_k + \mu Lx_k + Q(x_k, x_k) + C(x_k, x_k, x_k) + o(x_k^3).$$

The matrices A, Q, L, C are the same as with Hopf bifurcation.

Under hypothesis (P), a period-2 orbit should bifurcate from $x = 0$ at $\mu = 0$. Let l and r denote the normalized left and right eigenvectors of $A(0)$ associated with the eigenvalue -1, i.e. $l(I+A) = (I+A)r = 0$ and $l \cdot r = 1$.

To analyze the stability of period-2 orbit, we should consider $x_{k+2} = \tilde{F}(x_k) = F\big(F(x_k)\big)$. The following notations are from Taylor expansion of $\tilde{F} - I$:

$$\begin{aligned}
\tilde{A}(\mu) &\triangleq A^2(\mu) - I \\
\tilde{Q}(x,x) &\triangleq A(0)Q(x,x) + Q(A(0)x, A(0)x) \\
\tilde{C}(x,x,x) &\triangleq A(0)C(x,x,x) + 2Q(A(0)x, Q(x,x)) \\
&\quad + C(A(0)x, A(0)x, A(0)x)
\end{aligned}$$

The stability of a period doubling orbit is determined by the Floquet exponent β [9]. Similar to Hopf bifurcation case, it can be represented by a Taylor series in amplitude parameter ϵ:

$$\beta(\epsilon) = \beta_1 \epsilon + \beta_2 \epsilon^2 + \cdots , \tag{5}$$

the leading coefficients can be obtained by the following formulae:

$$\begin{aligned}
\beta_1 &= 0, \\
\beta_2 &= 2l[\tilde{C}(r,r,r) - 2\tilde{Q}(r, \tilde{A}^{-}\tilde{Q}(r,r))],
\end{aligned}$$

where

$$A^{-} \triangleq (A^T A + l^T l)^{-1} A^T.$$

Similar to the continuous case, $\beta_2 = 0$ only in degenerative case. Generically, we assume $\beta_2 \neq 0$. Theorem 2.2 summarizes the condition for the existence and stability of a period doubling bifurcation.

Theorem 2.2 (Period Doubling Bifurcation Theorem) *If (P) holds, then a period-doubled orbit bifurcates from the origin of (2) at $\mu = 0$. The period-doubled orbit is supercritical and stable if $\beta_2 < 0$ but is subcritical and unstable if $\beta_2 > 0$.*

3 Washout Filter and its Extension

In our previous designs of bifurcation controller, we normally apply a washout filter to facilitate the design [11, 12, 13, 5, 6, 4] to achieve certain objectives. Washout filter is a specific one-order high-pass filter. In continuous-time system, it can be represented by the transfer function

$$T(s) = \frac{s}{s+d}, \tag{6}$$

where $d > 0$ is the damping coefficient of this filter. In discrete-time system, the corresponding filter assumes the z-transformed form

$$T(z) = \frac{z-1}{z+(d-1)}, \tag{7}$$

where the damping coefficient d should satisfy condition $0 < d < 2$ to ensure the stability of this filter.

The filter is included in the feedback path so that the slow components do not generate any control action. Therefore the equilibrium points of the system do not change after controller is applied. This washout-filter-aided controller, which preserves the fixed-point structure of the system bifurcation diagram, simplifies the bifurcation parameter calculation and can be designed more easily and directly. Considering its successfulness, several questions emerge naturally: Can any other high-order filter be used? How will it affect the bifurcation? The washout filter used in [11] is already a high-order one, but all the states are fed back with *identical* washout filters. We expect that some variants to this filter structure may also offer the ability to control the bifurcation.

To maintain the equilibrium-preserving property, we only consider high-pass filters. The following lemma shows that almost all of the stable high-pass filters can be represented by a form similar to the original washout filter. This structure can facilitate the bifurcation analysis later on.

Lemma 3.1 *Assume a stable system $G(s) \in s^{p\times q}$ can be represented by a rational proper transfer function matrix of s with degree n. If $G(s)$ is a high-pass filter, i.e., $\lim_{s\to 0} G(s) = 0$, then $\exists K \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{n\times n}$ such that the LTI system*

$$\dot{z} = Cu - Dz, \tag{8a}$$

$$y = K(Cu - Dz) \tag{8b}$$

has the transfer function $G(s)$, where $z \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^p$ is the input vector, and $y \in \mathbb{R}^q$ is the output vector.

Proof: Since $G(s)$ is stable, all the poles of $G(s)$ are located in the right half-plane. Define $P(s) = \frac{1}{s}G(s)$, then $\lim_{s\to 0} P(s) < \infty$. $s = 0$ is not a pole of $P(s)$. Therefore $P(s)$ is also stable. Because $G(s)$ is proper, $P(s)$ is strict proper. Using the minimal realization theorem, $P(s)$ can be represented by a stable LTI system

$$\begin{aligned}
\dot{z} &= -Dz + Cu, \\
y &= Kz,
\end{aligned}$$

and $P(s) = K(sI+D)^{-1}C$. The transfer-function matrix of (8) is $K(C - D(sI+D)^{-1}C) = sK(sI+D)^{-1}C = sP(s) = G(s)$, therefore (8) is a realization of $G(s)$. ∎

On the other hand, we can easily see that for any C, D, and K, filter $G(s)$ is a high-pass filter. The equilibrium point structure is not changed after $G(s)$ is applied to the feedback loop. And the controller design can be performed in a way similar to what we did with washout filters. We refer this filter to an extended washout filter.

With the extended washout filter, the controller can be written as following:

$$\dot{z} = Cx - Dz, \tag{9a}$$
$$y = Cx - Dz, \tag{9b}$$
$$u = g(y). \tag{9c}$$

The gain matrix K is now a part of g. From bifurcation theory [9], only the quadratic and cubic terms affect the stability of a bifurcation, the higher-order terms make no difference generically in terms of stability and are ignored in the controller design for simplicity. Higher-order can and do affect other aspects of a bifurcation. The control function g is assumed to be

$$g(y) = Q_u(y, y) + C_u(y, y, y), \tag{10}$$

which can represent all quadratic and cubic functions.

For discrete-time system (2), we have a similar argument and a similar filter can also be used,

$$z_{k+1} = Cx_k - (D - I)z_k, \tag{11a}$$
$$y_k = Cx_k - DZ_k. \tag{11b}$$
$$u_k = g(y_k). \tag{11c}$$

Here C is the output measurement matrix and D determines the dynamic properties of the controller.

As with the original washout filter, we expect these filters to have similar properties for control of bifurcations. In the following sections, we confirm their utility for control of bifurcations under some similar non-degenerative conditions. Their effects on the stability of bifurcations can also be determined analytically.

4 Dynamic Control of Hopf Bifurcation

In this section, we consider the continuous-time system (1). The state feedback controller is used first. Add control $u = u(x)$, then

$$\dot{x} = F(x, u; \mu) \tag{12a}$$
$$= Ax + Q(x, x) + C(x, x, x) + o(x^3) \tag{12b}$$
$$+ Bu + Q_B(x, u) + o(xu) + \mu Lx + o(\mu x). \tag{12c}$$

If u is constant or linear, A will change after applying the controller, and the control system will be hard to analyze. Hence we only consider nonlinear controller here. Suppose the nonlinear state feedback controller can be written as

$$u(x) = Q_u(x, x) + C_u(x, x, x), \tag{13}$$

the system equation is then transformed to

$$\begin{aligned}\dot{x} =& Ax + [Q + BQ_u](x, x) + [C + BC_u](x, x, x) \\ &+ Q_B(x, Q_u(x, x)) + \mu Lx + o(\mu x) + O(x^4).\end{aligned}$$

Define $Q^* = Q + BQ_u$, $Q_{Bu}(x, y, z) = Q_B(x, Q_u(y, z))$, and $C^* = C + BC_u + Q_{Bu}$. The new system has the same form as (2.1) except for adding * to Q and C.

The closed-loop system has the same linear part as the open-loop system, therefore A, l, r are the same. As stated in the Hopf bifurcation theorem, the bifurcation parameter of the closed-loop system is

$$\beta_2^* = 2\mathsf{Re}\{2lQ^*(r, a_1) + lQ^*(\bar{r}, b_1) + \tfrac{3}{4} lC^*(r, r, \bar{r})\}.$$

The parameters a^* and b^* are decided by

$$\begin{aligned}a^* &= a - \tfrac{1}{2} A^{-1} BQ_u(r, \bar{r}) = a - \Delta_a, \\ b^* &= b + \tfrac{1}{2}(2i\omega_c I - A)^{-1} BQ_u(r, r) = b + \Delta_b.\end{aligned}$$

β_2^* can be written as $\beta_2^* = \beta_2 + \Delta$ where β_2 is the bifurcation coefficient of the open loop system and

$$\begin{aligned}\Delta =& 2\mathsf{Re}\{lB\{Q_u(r, a_1) + Q_u(\bar{r}, b_1) + \tfrac{3}{4} C_u(r, r, \bar{r})\} \\ &+ 2lQ(r, \Delta_a) + lQ(\bar{r}, \Delta_b) + \tfrac{3}{4} lQ_{Bu}(r, r, \bar{r})\}.\end{aligned}$$

Now we proceed to general dynamic feedback controller. Combine the controller (9) to the uncontrolled system (12), we get

$$\begin{aligned}\dot{x} &= Ax + Bu + Q(x, x) + Q_B(x, u) + f(x, u; \mu), \\ \dot{z} &= y = Cx - Dz, \\ u &= g(y),\end{aligned}$$

where f denotes for the higher order terms in the uncontrolled system.

Denote $\xi(t) \triangleq \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}$, then $x = \begin{pmatrix} I, & 0 \end{pmatrix} \xi \triangleq I_0 \xi$ and $y = \begin{pmatrix} C, & -D \end{pmatrix} \xi \triangleq D_1 \xi$. The extended system can be write in terms of ξ and u

$$\dot{\xi} = A_1 \xi + B_1 u + Q_1(\xi, \xi) + Q_{B1}(\xi, u) + o(\xi^2), \tag{15a}$$
$$u = Q_{u1}(\xi, \xi) + C_{u1}(\xi, \xi, \xi). \tag{15b}$$

where

$$A_1 = \begin{pmatrix} A & 0 \\ C & -D \end{pmatrix}, \qquad B_1 = \begin{pmatrix} B \\ 0 \end{pmatrix},$$
$$Q_1 = \begin{pmatrix} Q(I_0\xi, I_0\xi) \\ 0 \end{pmatrix}, \quad Q_{B1} = \begin{pmatrix} Q(I_0\xi, u) \\ 0 \end{pmatrix},$$
$$Q_{u1} = Q_u(D_1\xi, D_1\xi), \qquad C_{u1} = C_u(D_1\xi, D_1\xi, D_1\xi).$$

Let l_1 and r_1 to be the left and right eigenvectors of A_1 associated with eigenvalue $i\omega_c$. They are found to be a extension of l and r: $l_1 = (l, 0)$ and $r_1 = \begin{pmatrix} r \\ (D + i\omega_c I)^{-1}C \end{pmatrix}$. l_1 and r_1 are also normalized since $l_1 \cdot r_1 = l \cdot r = 1$. Also define

$$r_2 \triangleq D_1 r_1 = Cr - D(D + i\omega_c I)^{-1}C.$$

In (15), the dynamic feedback controller of the original system has become a state feedback control of the extended system. So we can apply the results from state feedback control.

$$\begin{aligned}
a_1 =& \begin{pmatrix} a, & 0 \end{pmatrix}^T - \Delta_{a1}, \qquad b_1 = \begin{pmatrix} b, & 0 \end{pmatrix}^T - \Delta_{b1}, \\
\Delta_{a1} =& \tfrac{1}{2} A_1^{-1} B_1 Q_{u1}(r_1, \bar{r}_1) \\
=& \begin{pmatrix} \frac{1}{2}A^{-1} \\ \frac{1}{2}D^{-1}CA^{-1} \end{pmatrix} BQ_u(r_2, \bar{r}_2), \\
\Delta_{b1} =& \tfrac{1}{2}(2i\omega_c I_1 - A_1)^{-1} B_1 Q_{u1}(r_1, r_1) \\
=& \begin{pmatrix} \frac{1}{2}(2i\omega_c I - A)^{-1} \\ -\frac{1}{2}(2i\omega_c I_2 - D)^{-1}C(2i\omega_c I - A)^{-1} \end{pmatrix} BQ_u(r_2, r_2), \\
\Delta =& 2\mathsf{Re}\{lBQ_u(r_2, a_2) + Q_u(\bar{r}_2, b_2) + \tfrac{3}{4}C_u(r_2, r_2, \bar{r}_2)\} \\
&+ 2lQ(r, \Delta_{a1}^0) + lQ(\bar{r}, \Delta_{b1}^0) + \tfrac{3}{4}lQ_{Bu}(r, r_2, \bar{r}_2),
\end{aligned}$$

where $a_2 = D_1 a_1$, $b_2 = D_1 b_1$, and Δ_x^0 denotes for the upper parts of Δ_x. I_1 and I_2 are identity matrices with the same dimension as A and D.

If the system is linearly controllable ($lB \neq 0$), we can let $Q_u = 0$, then

$$\Delta = \tfrac{3}{2}\mathsf{Re}\{lBC_u(r_2, r_2, \bar{r}_2)\}.$$

For the uncontrollable case ($lB = 0$), only the quadratic term takes effects.

$$\Delta = 2\mathsf{Re}\{2lQ(r, \Delta_{a1}^0) + lQ(\bar{r}, \Delta_{b10}) + \tfrac{3}{4}lQ_{Bu}(r, r_2, \bar{r}_2)\}$$

5 Dynamic Control of Period Doubling Bifurcation

The analysis and controller design in discrete-time system (2) is similar to last section. The state feedback control case is studied first. By introducing a scalar control variable u into the system equation and perform Taylor expansion, we get

$$\begin{aligned}
x_{k+1} =& F(x_k, u; \mu) \\
=& Ax_k + \mu Lx_k + uL_u x_k + ub + Q(x_k, x_k) \qquad (16) \\
&+ uQ_u(x_k, x_k) + C(x_k, x_k, x_k) + \cdots
\end{aligned}$$

Similar to the continuous-time case, linear controller can change A and so is hard to analyze. Only the nonlinear controller is considered here. Assume u to have the following form

$$u(x_k) = x_k^T Q_u x_k + C_u(x_k, x_k, x_k) \qquad (17)$$

Computing the stability coefficients gives

$$\beta_1^* = 0, \qquad \beta_2^* = \beta_2 + \Delta$$

The expression of Δ is long and complex (See [11]). It is omitted here.

Similar to the continuous-time system, we combine the uncontrolled system dynamics with controller dynamics to analyze the system behavior. Denote $\zeta_k \triangleq \begin{pmatrix} x_k \\ z_k \end{pmatrix}$, then the extended system (plant dynamics plus controller dynamics) becomes

$$\begin{aligned}
\zeta_{k+1} =& A_1\zeta_k + \mu\bar{L}_1\zeta_k + u\bar{L}_{u1}\zeta_k + u_k b_1 + Q_1(\zeta_k, \zeta_k) \\
&+ u_k Q_{u1}(\zeta_k, \zeta_k) + C_1(\zeta_k, \zeta_k, \zeta_k) + \cdots,
\end{aligned}$$

Where

$$\begin{aligned}
A_1 =& \begin{pmatrix} A & 0 \\ C & I - D \end{pmatrix}, \qquad b_1 = \begin{pmatrix} b \\ 0 \end{pmatrix}, \\
L_{u1} =& \begin{pmatrix} L_{u1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q_1(\zeta_k, \zeta_k) = \begin{pmatrix} Q(x_k, x_k) \\ 0 \end{pmatrix}, \\
Q_{u1}(\zeta_k, \zeta_k) =& \begin{pmatrix} Q_u(x_k, x_k) \\ 0 \end{pmatrix}, \\
C_1(\zeta_k, \zeta_k, \zeta_k) =& \begin{pmatrix} C(x_k, x_k, x_k) \\ 0 \end{pmatrix}.
\end{aligned}$$

Take the control u to be of the form

$$u = y_k^T Q_u y_k + C_u(y_k, y_k, y_k), \qquad (18)$$

where Q_u is a real symmetric $n \times n$ matrix, C_u is a cubic form function. Transform u into ζ coordinates, we have

$$u = \zeta_k^T Q_{u1}\zeta_k + C_{u1}(\zeta_k, \zeta_k, \zeta_k), \qquad (19)$$

where

$$\begin{aligned}
Q_{u1} =& \begin{pmatrix} C^T \\ -D^T \end{pmatrix} Q_u \begin{pmatrix} C & -D \end{pmatrix} = \begin{pmatrix} C^T Q_u C & -C^T Q_u D \\ -D^T Q_u C & D^T Q_u D \end{pmatrix}, \\
C_{u1}(\zeta_k, \zeta_k, \zeta_k) =& C_u(Cx_k - Dz_k, Cx_k - Dz_k, Cx_k - Dz_k).
\end{aligned}$$

The normalized eigenvectors of A_1 is found out to be $l_1 = (l, 0)$ and $r_1 = \begin{pmatrix} r \\ (D - 2I)^{-1}C \end{pmatrix}$.

Now the dynamic feedback controller of the original system becomes a state feedback controller of the extended system. So the results of state feedback can be employed.

Similar to the previous results, we have

$$\beta_{21} = \beta_{20} + \Delta_1,$$

where β_{20} is the bifurcation stability coefficient of the extended system without control, which is identical to β_2 of the original system, and

$$\begin{aligned}\Delta_1 = & -4l_1 C_{u1}(r_1, r_1, r_1) b_1 \\ & - 4l_1[(r_1^T Q_{u1} r_1) L_{u1} r_1 + r_1^T Q_{u1}((r_1^T Q_{u1} r_1) b_1 \\ & + Q_1(r_1, r_1)) b_1 + Q_1(r_1, (r_1^T Q_{u1} r_1) b_1)] \\ & + 4l_1[Q_1(r_1, \tilde{A}_1^-(A_1 + I)(r_1^T Q_{u1} r_1) b_1) + (r_1^T Q_{u1} \tilde{A}_1^- \\ & ((A_1 + I)(r_1^T Q_{u1} r_1) b_1 + (A_1 + I) Q_1(r_1, r_1))) b_1] \\ & + 4l_1[Q_1(r_1, A_1 \tilde{A}_1^-((A_1 + I)(r_1^T Q_{u1} r_1) b_1) + r_1^T Q_{u1} \\ & (A_1 \tilde{A}_1^-((A_1 + I)(r_1^T Q_{u1} r_1) b_1 + (A_1 + I) Q_1(r_1, r_1))) b_1]\end{aligned}$$

If $lb \neq 0$, i.e., the critical mode is linearly controllable, a cubic feedback control can be designed to stabilize the bifurcation. This can be seen by setting $Q_u = 0$ and considering the effect of the cubic terms in the feedback control. Then Δ_1 reduces to

$$\begin{aligned}\Delta_1 = & -4C_{u1}(r_1, r_1, r_1) l_1 b_1 \\ = & -4C_u(Cr - D(D - 2I)^{-1} Cr, r_1, r_1) lb\end{aligned}$$

The case $lb = 0$ corresponds to the linearly uncontrollable critical mode. Define

$$\begin{aligned}H \triangleq & l_1 L_{u1} r_1 + l_1 Q_1(r_1, b_1) - l_1 Q_1(r_1, \tilde{A}_1^-(A_1 + I) b_1) \\ & - l_1 Q_1(r_1, A_1 \tilde{A}_1^-(A_1 + I) b_1)\end{aligned}$$

as the measure of non-degeneracy of the plant. Generically, $H \neq 0$. The bifurcation can then controlled by a quadratic controller Q_u, and the expression for Δ_1 is

$$\Delta_1 = -4r^T Q_u r H.$$

6 Discussion

In this paper, it is shown that all the stable rational high-pass filters can be represented as extended washout filters. It is further shown that bifurcations can be effectively controlled by dynamic feedback controllers incorporating the extended washout filters. As a result one can utilize high-pass filters other than washout filters for dynamic control of bifurcations. This offers considerably more flexibility in the design of bifurcation control laws. In addition, the extended washout filter is a specific form of the general dynamic feedback control. Analysis in this paper could be extended further to the analysis of bifurcation control with a general dynamic feedback controller.

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