On the Control of Hopf Bifurcations

B. Hamzi	W. Kang	J.-P. Barbot
LSS/CNRS, Supélec,	Mathematics Department,	ECS, ENSEA,
Plateau de Moulon,	Naval Postgraduate School,	6, avenue du Ponceau,
91192 Gif sur Yvette, France.	Monterey, CA 93943, USA.	95014 Cergy, France.
hamzi@lss.supelec.fr	wkang@math.nps.navy.mil	barbot@ensea.fr

1 Introduction

This paper addresses the problem of control of the Hopf bifurcation for nonlinear systems with uncontrollable modes. The controller design for systems with bifurcations has fundamental differences between systems with a controllable or uncontrollable linearization. For linearly controllable systems, the bifurcation can be delayed or stabilized by a linear feedback ([7]). However, nonlinear feedback is essential for systems with uncontrollable bifurcation modes. The case of more than one uncontrollable mode in the linearized system requires special attention. Aeyels [3], Abed and Fu [1, 2], Colonius and Kliemann [8] have studied the problem of systems with one and/or two uncontrollable modes. Using normal forms, Kang [12] developed the analysis and control design algorithm for systems with one uncontrollable mode.

In this paper, control systems with two uncontrollable modes on imaginary axis are studied. We derive the normal form for nonlinear control systems with two complex conjugate uncontrollable modes. In this paper, "bifurcation control" means the control of the orientation of the periodic solution and the stabilization of the periodic solution. Based on the normal form, we find the center manifold orientation that is achievable by state feedback. Finally, using the same normal form approach, we derive sufficient conditions for the tuning of a nonlinear control law to make the Hopf bifurcation supercritical. The normal form approach adopted in this paper generalizes the Poincaré's normal form method, commonly used in dynamical systems theory, to the area of control systems. It was first introduced in [13, 11]. The results proved in this paper indicate that the linear control determines the orientation of the periodic solution close to the origin, and the quadratic feedback is critical to stabilize the periodic solution. The analysis is based on the center manifold theorem [6] and the Poincaré-Andronov-Hopf theorem [15].

In section §2, the relation between the feedback and the orientation of the center manifold at the bifurcation point is derived. Section 3 is devoted to the determination of quadratic normal form. The coefficients of the quadratic terms (or invariants) are then computed. Based on the invariants, the quadratic center manifold is derived. In §4, sufficient conditions on stabilizability of the periodic solution is found.

Due to space limitations, the proofs cannot be inclused. They can be found in the longer version of this paper.

2 The Center Manifold Orientation at the Origin

Consider the following nonlinear system Σ_H

$$\dot{\zeta} = f(\zeta,\mu) + g(\zeta,\mu)\, v \tag{2.1}$$

the variable $\zeta \in \mathbb{R}^n$ is the state, $v \in \mathbb{R}$ is the input variable, and the parameter $\mu \in \mathbb{R}$. The vector fields $f(\zeta,\mu)$ and $g(\zeta,\mu)$ are assumed to be C^k for some sufficiently large k.

Assume $f(0,0) = 0$, $g(0,0) \neq 0$ and suppose that the linearization of the system at the origin is (A, B)

$$A = \frac{\partial f}{\partial \zeta}(0,0), \quad B = g(0,0)$$

with

$$\text{rank}([B\ AB\ A^2B\ \cdots\ A^{n-1}B]) = n-2 \tag{2.2}$$

So, the system is not linearly controllable at the origin. If the uncontrollable modes have nonzero real parts, the stabilizability is determined by the real part of the uncontrollable modes. However, if the real part of the uncontrollable modes are zero, bifurcation occurs even with feedback control. In this paper, we assume that $\pm i\,\omega$ are the uncontrollable modes.

Assumption 1: The linearization of Σ_H has two uncontrollable modes, $\pm i\,\omega$, at the origin.

The first step of the analysis is to simplify the linear part of the system (i.e. determination of the linear normal form). There exists a linear change of coordinates and a feedback independent of μ transforming the system (2.1) into

$$\begin{array}{ll} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} & = A_1 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \Gamma_1\mu + O(x,z,\mu,u)^2 \\ \dot{x} & = A_2 x + B_2 u + \Gamma_2\mu + O(x,z,\mu,u)^2 \end{array} \quad (2.3)$$

with $\Gamma_1 = \begin{bmatrix} \gamma_{1\,1} & \gamma_{1\,2} \end{bmatrix}^T$, $z \in \mathbb{R}^2$, $x \in \mathbb{R}^{n-2}$, $A_2 \in \mathbb{R}^{(n-2)\times(n-2)}$, and $B_2 \in \mathbb{R}^{(n-2)\times 1}$.

$$A_1 = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \Gamma_2 = \begin{bmatrix} \gamma_{2\,1} \\ \gamma_{2\,2} \\ \vdots \\ \gamma_{2\,n-2} \end{bmatrix}.$$

The system can be simplified further. The $\mu-$terms can be cancelled by the change of coordinates: $\bar{z}_1 = z_1 + \frac{\gamma_{1\,2}}{\omega}\mu$, $\bar{z}_2 = z_2 - \frac{\gamma_{1\,1}}{\omega}\mu$, $\bar{x}_1 = x_1$, $\bar{x}_i = x_i + \gamma_{2\,i-1}\mu$ for $i = 2, \cdots, n$ and a feedback $\bar{u} = u + \gamma_{2\,n}\mu$. The linear normal form of Σ_H is summarized in the following lemma

Lemma 2.1 *There exists a linear change of coordinates and feedback which transforms (2.1) into*

$$\begin{array}{ll} \dot{z} & = A_1 z + f_1^{[2]}(z,x,\mu) + g_1^{[1]}(z,x,\mu)u + O(\cdot)^3 \\ \dot{x} & = A_2 x + B_2 u + f_2^{[2]}(z,x,\mu) + g_2^{[1]}(z,x,\mu)\,u + O(\cdot)^3 \end{array} \quad (2.4)$$

where $O(\cdot)^3$ stands for the remaining cubic terms in x, z, μ, u. ◇

2.1 The Linear Center Manifold

Now, let us determine the linear part of the center manifold. Consider

$$\begin{array}{lll} \dot{z} & = & A_1 z + O(x,z,\mu,u)^2 \\ \dot{x} & = & A_2 x + B_2 u + O(x,z,\mu,u)^2 \end{array} \quad (2.5)$$

and the feedback

$$u = F_1 z_1 + F_2 z_2 + F_3\mu + \sum_{i=1}^{n-2} a_i x_i + O(x,z,\mu)^2. \quad (2.6)$$

To stabilize the system around the bifurcation point, the controllable part has to be stable. So, we assume

Assumption 2: The matrix

$$A_2 + B_2 \begin{bmatrix} a_1 & \cdots & a_{n-2} \end{bmatrix}$$

is Hurwitz.

Suppose that the center manifold is

$$x = \Pi(z,\mu)$$

Suppose that the linear part of Π is

$$\Pi^{[1]}(z,\mu) = \Pi^{[1]} \begin{bmatrix} z \\ \mu \end{bmatrix}, \quad (2.7)$$

with

$$\Pi^{[1]} = \begin{bmatrix} \Pi_{11}^{[1]} & \Pi_{12}^{[1]} & \Pi_{13}^{[1]} \\ \vdots & \vdots & \vdots \\ \Pi_{n-2,1}^{[1]} & \Pi_{n-2,2}^{[1]} & \Pi_{n-2,3}^{[1]} \end{bmatrix} \in \mathbb{R}^{(n-2)\times 3}.$$

Lemma 2.2 *Given any feedback (2.6) satisfying Assumption 2, the linear part of the center manifold $x = \Pi^{[1]}(z,\mu)$ is uniquely determined by*

$$\left\{ \begin{array}{l} \Pi_1^{[1]} = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix} \begin{bmatrix} P(A_1)^{-1} & 0 \\ 0 & -\frac{1}{a_1} \end{bmatrix} \\ \Pi_{i+1}^{[1]} = \Pi_1^{[1]} \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}^i. \end{array} \right. \quad (2.8)$$

with $P(\lambda)$ the characteristic polynomial of $A_2 + B_2 \begin{bmatrix} a_1 & \cdots & a_{n-2} \end{bmatrix}$, i.e.

$$P(\lambda) = \lambda^{n-2} - \sum_{i=1}^{n-2} a_i \lambda^{i-1} \quad (2.9)$$

Let us note that formula (2.8) holds only if $P(A_1)$ is inevitable, and $a_1 \neq 0$. This is always true. Since we assume Assumption 2, the eigenvalues of $A_2 + B_2[a_1 \;\; a_2 \;\; \cdots a_{n-2}]$ are not on the imaginary axis. Therefore, the roots of the characteristic polynomial $P(\lambda)$ is not on the imaginary axis. Therefore, $P(\pm i\omega) \neq 0$. On the other hand, the eigenvalues of $P(A_2)$ are $P(\pm i\omega)$, which are nonzero. Therefore, the matrix $P(A_2)$ is inevitable. Furthermore, the value $(-1)^{n-1}a_1$ equals the multiplication of all eigenvalues of $A_2 + B_2[a_1 \; a_2 \; \cdots a_{n-2}]$. If all the eigenvalues are on the left half plane, it is easy to check that $a_1 < 0$.

2.2 Orientation

The orientation of a center manifold at a given point is a set of vectors. They are orthogonal to the tangent space of the manifold, linearly independent and they generate a complement subspace of the manifold. In other words, the orientation of a manifold at a given point P is a basis of the orthogonal complement subspace of the tangent space at P.

Theorem 2.1 *Given any $(n-2)\times(n+1)$ matrix of the form*

$$[\mathcal{M}_{(n-2)\times 3} \ \ \mathcal{N}_{(n-2)\times(n-2)}]$$

Then, its row vectors define the center manifold orientation at the origin for (2.5), (2.6) if and only if $\mathcal{N}^{-1}$ exists and $\Pi^{[1]} = -\mathcal{N}^{-1}\mathcal{M}$ satisfies (2.8).

Proof. Suppose that $[\mathcal{M}_{(n-2)\times 3} \ \ \mathcal{N}_{(n-2)\times(n-2)}]$ defines the orientation of a center manifold. Then, it is orthogonal to the tangent space of the center manifold. It is known that the tangent space of the center manifold is given by its linear part,

$$x - \Pi^{[1]}\begin{bmatrix} z \\ \mu \end{bmatrix} = 0,$$

where $\Pi^{[1]}$ satisfies (2.8). In the (z,μ,x)-space, a set of orthogonal vectors of the tangent space is the row vectors of $[-\Pi^{[1]} \ I]$. Therefore, $[-\Pi^{[1]} \ I]$ and $[\mathcal{M}_{(n-2)\times 3} \ \ \mathcal{N}_{(n-2)\times(n-2)}]$ generates the same space, which is orthogonal to the tangent space of the center manifold. Therefore,

$$[-\Pi^{[1]} \ I] = \mathcal{N}^{-1}[\mathcal{M}_{(n-2)\times 3} \ \ \mathcal{N}_{(n-2)\times(n-2)}].$$

So, $\Pi^{[1]} = -\mathcal{N}^{-1}\mathcal{M}$ and it satisfies (2.8).

On the other hand, suppose $-\mathcal{N}^{-1}\mathcal{M}$ satisfies (2.8). By Lemma 2.2, the linear space

$$\mathcal{N}^{-1}\mathcal{M}\begin{bmatrix} z \\ \mu \end{bmatrix} + x = 0$$

represents the linear part of the center manifold. It is the tangent space of the center manifold. Therefore, $[\mathcal{N}^{-1}\mathcal{M} \ \ I]$, the row vectors in the coefficient matrix of this equation, forms a basis of the orthogonal space. It is easy to check that the row vectors of $[\mathcal{M} \ \mathcal{N}]$ and $[\mathcal{N}^{-1}\mathcal{M} \ \ I]$ generates the same vector space. Therefore, $[\mathcal{M} \ \ \mathcal{N}]$ defines the orientation of the center manifold. ◃

3 Quadratic Center Manifold

The following quadratic transformations are employed to simplify the quadratic part of a system into its normal form while leaving the linear part invariant.

$$\begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{x} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ x \end{bmatrix} + \phi^{[2]}(z,x,\mu) \qquad (3.10)$$

$$\bar{u} = u + \alpha^{[2]}(z,x,\mu) + \beta^{[1]}(z,x,\mu)u. \qquad (3.11)$$

The normal form is given in the following theorem.

Theorem 3.1 *Consider system Σ_H. Suppose that its linearization is given by (2.4). Then, there exists a quadratic change of coordinates (3.10) and feedback (3.11) that transform the system into*

$$\begin{aligned}\dot{z} &= A_1 z + f_1^{[2,0]}(z,\mu) + f_1^{[1,1]}(z,\mu,x) + f_1^{[0,2]}(x) + O(z,\mu,x)^3 \\ \dot{x} &= A_2 x + B_2 u + f_2^{[0,2]}(x) + O(z,\mu,x,u)^3\end{aligned} \qquad (3.12)$$

where

$$\begin{aligned} f_1^{[2,0]}(z,\mu) &= \sum_{i=1}^{2}\sum_{j=1}^{2}\beta_i^j e_1^i\, z_j\, z_3, \\ f_1^{[1,1]}(z,\mu,x_1) &= \sum_{i=1}^{2}\sum_{j=1}^{3}\gamma_i^j e_1^i\, z_j\, x_1, \\ f_1^{[0,2]}(x) &= \sum_{i=1}^{2}\sum_{j=1}^{n-2}\delta_i^j e_1^i\, x_j^2, \end{aligned}$$

$$f_2^{[0,2]}(x) = \sum_{i=1}^{n-2}\sum_{j=i+2}^{n-2}\rho_i^j e_2^i\, x_j^2$$

In this notations, $z_3 = \mu$, $\beta_1^1 = \beta_2^2$, $\beta_1^2 = -\beta_2^1$. We will continue to use both z_3 and μ in the rest of this paper. The notation e_1^i is the ith unit vector in z space. The vector e_2^i is the i^{th} unit vector in x space.

3.1 Quadratic Center Manifold

Now let us determine the quadratic part of the center manifold. In the following, $Ad_X(Y)$ represents the Lie bracket of two matrices X and Y, i.e.

$$Ad_X(Y) = XY - YX.$$

Theorem 3.2 *Consider the normal form (3.12). Under any feedback*

$$u = \sum_{i=1}^{n-2} a_i x_i + \begin{bmatrix} z & \mu \end{bmatrix} Q \begin{bmatrix} z \\ \mu \end{bmatrix} + O(z,\mu,x)^3, \qquad (3.13)$$

the quadratic part of the center manifold satisfies

$$\Pi_i^{[2]}(z,\mu) = \begin{bmatrix} z & \mu \end{bmatrix} (-1)^{i-1} Ad^{i-1}_{\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}}(Q_1) \begin{bmatrix} z \\ \mu \end{bmatrix}, \qquad (3.14)$$

where $Q = P(-Ad_{\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}})(Q_1)$, $P(\lambda)$ *is the characteristic polynomial of* $A_2 + B_2K$ *for* $K = \begin{bmatrix} a_1 & \cdots & a_{n-2} \end{bmatrix}$.

Remark. Theorem 3.2 implies that, for any given matrix Q_1, there always exits

$$Q = P(-Ad_{\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}})(Q_1) \tag{3.15}$$

so that the feedback (3.13) yields a center manifold satisfying (3.14). In the next section it will be proved that the stability of the Hopf bifurcation is determined by Q_1 and the invariants. Theorem 3.2 guarantees that, if a Q_1 stabilizes a Hopf bifurcation, it is always achievable by a suitable quadratic feedback. Thus, the problem of finding the stabilizing feedback is converted to the problem of maneuvering the quadratic part of the center manifold. ◁

Remark. The spectrum of the operator $Ad_{\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}}$ consists of $\{\pm i\omega, \pm 2i\omega, 0\}$, all on the imaginary axis. The spectrum of $P(-Ad_{\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}})$ are $P(\pm i\omega)$, $P(\pm 2i\omega)$ and $P(0)$. Since the roots of $P(\lambda)$ are all in the left half plane, the spectrum of $P(-Ad_{\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}})$ do not contain zero. So, $P(-Ad_{\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}})$ is an inevitable linear operator. ◁

In Theorem 3.2, we assume that the linear feedback is independent of (z, μ). However, it does not mean to lose any generality. The next theorem shows that if a closed-loop system has nonzero F_i terms, it can be transformed into a system in which the linear part of the controllable system is not explicitly a function of (z, μ). Then, the formulae in Theorem 3.2 is applicable to the new system.

Proposition 3.1 *Given a system*

$$\begin{aligned} \dot{z} &= A_1 z + O(z,x)^2 \\ \dot{x} &= A_2 x + B_2(\sum_{i=1}^{3} F_i z_i + \sum_{i=1}^{n-2} a_i x_i) + O(z,x)^2 \end{aligned} \tag{3.16}$$

There exists a linear change of coordinates to transform the system into the following form

$$\begin{aligned} \dot{\tilde{z}} &= A_1 \tilde{z} + O(\tilde{z},\tilde{x})^2 \\ \dot{\tilde{x}} &= A_2 \tilde{x} + B_2(\sum_{i=1}^{n-2} a_i \tilde{x}_i) + O(\tilde{z},\tilde{x})^2 \end{aligned} \tag{3.17}$$

4 Control of the Hopf Bifurcation

The center manifold has dimension two. To determine its stability we use the Poincaré-Andronov-Hopf theorem. Let us recall this result

Theorem 4.1 *[9] Consider the following system*

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \Psi(z_1, z_2, \mu) \\ \tilde{\Psi}(z_1, z_2, \mu) \end{bmatrix} \tag{4.18}$$

Then if

$$\begin{aligned} &\Psi_{\mu z_1} + \tilde{\Psi}_{\mu z_2} \neq 0 \\ &\breve{a} \neq 0 \end{aligned}$$

where $\breve{a}$ *is a constant defined below, a curve of periodic solutions bifurcates from the origin into* $\mu < 0$ *if* $\breve{a}(\Psi_{\mu z_1} + \tilde{\Psi}_{\mu z_2}) > 0$ *or* $\mu > 0$ *if* $\breve{a}(\Psi_{\mu z_1} + \tilde{\Psi}_{\mu z_2}) < 0$.

The periodic solution is stable if $\breve{a} < 0$. *The periodic solution is unstable if* $\breve{a} > 0$. *The origin is stable for* $\mu > 0$ *(resp.* $\mu < 0$*) and unstable for* $\mu < 0$ *(resp.* $\mu > 0$*) if* $\Psi_{\mu z_1} + \tilde{\Psi}_{\mu z_2} < 0$ *(resp.* $\Psi_{\mu z_1} + \tilde{\Psi}_{\mu z_2} > 0$*).*

The coefficient $\breve{a}$ is a constant involving partial derivatives evaluated at the bifurcation point, i.e. $(z_1, z_2, \mu) = (0, 0, 0)$. It is given by

$$\begin{aligned} \breve{a} = &\frac{1}{16}(\Psi_{z_1 z_1 z_1} + \tilde{\Psi}_{z_1 z_1 z_2} + \Psi_{z_1 z_2 z_2} + \tilde{\Psi}_{z_2 z_2 z_2}) \\ &+ \frac{1}{16\omega}\Big(\Psi_{z_1 z_2}(\Psi_{z_1 z_1} + \Psi_{z_2 z_2}) + \Psi_{z_2 z_2} \tilde{\Psi}_{z_2 z_2} \\ &- \tilde{\Psi}_{z_1 z_2}(\tilde{\Psi}_{z_1 z_1} + \tilde{\Psi}_{z_2 z_2}) - \Psi_{z_1 z_1} \tilde{\Psi}_{z_1 z_1}\Big) \end{aligned} \tag{4.19}$$

Remark. For $(\Psi_{\mu z_1} + \tilde{\Psi}_{\mu z_2}) > 0$. If $\breve{a} < 0$ then a stable periodic orbit of amplitude approximately

$$R = \Big(\frac{(\Psi_{\mu z_1} + \tilde{\Psi}_{\mu z_2})\mu}{|\breve{a}|}\Big)^{\frac{1}{2}} \tag{4.20}$$

bifurcates from the origin into $\mu > 0$ as μ passes through zero. The origin itself is a stable focus if $\mu < 0$. If $\breve{a} > 0$ then an unstable periodic orbit of amplitude

$$R = \Big(\frac{(\Psi_{\mu z_1} + \tilde{\Psi}_{\mu z_2})\mu}{\breve{a}}\Big)^{\frac{1}{2}} \tag{4.21}$$

bifurcates into $\mu < 0$, where the origin is a stable focus, and there are no periodic orbits in a small neighborhood of the origin of $\mu > 0$. The case where $(\Psi_{\mu z_1} + \tilde{\Psi}_{\mu z_2}) < 0$ is similar with the sign of μ changed. ◁

Remark. If $\breve{a} < 0$, stable period solution attracts local trajectories when the origin is an unstable equilibrium point. So, the system stays around the origin even if the origin is not a stable point. We call a Hopf bifurcation supercritical if $\breve{a} < 0$. ◁

Using the expressions of the center manifold (3.14) and the normal form (3.12), we can determine the dynamics of (2.1) on the center manifold. A straightforward application of Theorem 4.1 and (4.19) to the reduced dynamical system yields the formula of $\breve{a}$ for normal form (3.12). Based on this formula, feedback coefficients can be computed to stabilize the Hopf bifurcation. Substituting (3.14) into the $\dot{z}$ dynamics of the normal form (3.12), the critical coefficients in the reduced system can be found. In this case,

$$\breve{a} = \frac{Q_{11}(3\gamma_1^1 + \gamma_2^2) + 2Q_{12}(\gamma_1^2 + \gamma_2^1) + Q_{22}(3\gamma_2^2 + \gamma_1^1)}{8} + \frac{1}{16}\left(\frac{\partial^3 f_{11}^{[3]}}{\partial z_1^3} + \frac{\partial^3 f_{12}^{[3]}}{\partial z_1^2 \partial z_2} + \frac{\partial^3 f_{11}^{[3]}}{\partial z_1 \partial z_2^2} + \frac{\partial^3 f_{12}^{[3]}}{\partial z_2^3}\right) \quad (4.22)$$

$$\Psi_{\mu z_1} + \tilde{\Psi}_{\mu z_2} = 2\beta_1^1 \quad (4.23)$$

where Q_{ij} are elements in the i^{th}−row and the j^{th}−column of Q_1, the matrix of the quadratic center manifold, and

$$f_1^{[3]} = \begin{bmatrix} f_{11}^{[3]} \\ f_{12}^{[3]} \end{bmatrix}$$

represents the cubic part of the $\dot{z}$ equation in (3.12). The next theorem of bifurcation control is a straightforward corollary of Theorem 4.1 and (4.22).

Theorem 4.2 *Given a system in the normal form (3.12), Suppose $\beta_1^1 \neq 0$. If one of the following conditions is satisfied,*

1) $3\gamma_1^1 + \gamma_2^2 \neq 0$,

2) $\gamma_1^2 + \gamma_2^1 \neq 0$,

3) $3\gamma_2^2 + \gamma_1^1 \neq 0$,

then, there always exists a nonlinear feedback (3.13) that renders the Hopf bifurcation supercritical. The feedback coefficient Q_{fb} is determined by (3.15), in which Q_1 is any symmetric matrix satisfying $\beta_1^1 \cdot \breve{a} < 0$.

5 Invariants

Given a system in the form of (2.4), how to find its normal form? In this section, a set of numbers associated with (2.4) is found. The numbers are invariant under any quadratic change of coordinates and feedback. They are called the quadratic invariants. It is also proved that these invariants equal the coefficients in the normal form. Two systems are equivalent under a quadratic change of coordinates and feedback if and only if the invariants of the systems are equal. Therefore, the set of quadratic invariants completely characterizes the quadratic part of a nonlinear control system. For a given system (2.4), the values of the invariants are the coefficients in the normal form.

Consider a system in the form of (2.4). Denote by C_x, C_z the following row vector and matrix,

$$C_x = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}_{1\times n},$$
$$C_z = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}_{2\times n}.$$

Given two vector fields $X(\xi)$ and $Y(\xi)$ defined in $\mathbb{R}^n$, the operator ad_X is defined by

$$ad_X(Y) = [X, Y] = \frac{\partial Y}{\partial \xi}X - \frac{\partial X}{\partial \xi}Y$$

The Lie operator L_X is defined by

$$L_X(\kappa(\xi)) = \frac{\partial \kappa(\xi)}{\partial \xi}X$$

for C^1 functions defined in $\mathbb{R}^n$.

Definition 5.1 *Given a system (2.4), the quadratic invariants are defined by*

$$\rho_t^{n-r-1} = \tfrac{1}{2}C_x A^{t-1}\left[ad_f^r(g), ad_f^{r-1}(g)\right]|_{z=0,x=0,\mu=0},$$

$$\begin{bmatrix} \delta_1^{n-r-1} \\ \delta_2^{n-r-1} \end{bmatrix} = \tfrac{1}{2}C_z\left[ad_f^r(g), ad_f^{r-1}(g)\right]|_{z=0,x=0,\mu=0},$$

$$\begin{bmatrix} \gamma_1^j \\ \gamma_2^j \end{bmatrix} = (-1)^n C_z \frac{\partial}{\partial z_j} ad_f^{n-2}(g)\Big|_{x=0,\, z=0,\, \mu=0},$$

$$\begin{bmatrix} \beta_1^1 \\ \beta_1^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{\partial^2 f_{11}}{\partial z_3 \partial z_1} + \frac{\partial^2 f_{12}}{\partial z_3 \partial z_2} \\ \frac{\partial^2 f_{11}}{\partial z_3 \partial z_2} - \frac{\partial^2 f_{12}}{\partial z_3 \partial z_1} \end{bmatrix}\Big|_{x=0,\, z=0,\, \mu=0} \quad (5.1)$$

With $1 \leq r \leq n-4$, $1 \leq t \leq n-r-3$, $1 \leq r \leq n-2$, $j = 1, 2, 3$, where f and g are the right hand side of (2.4). The vector $f_1 = [f_{11} \;\; f_{12}]^T$ represents the vector field of $\dot{z}$ in (2.4).

The notations such as ρ_i^j, δ_i^i, γ_i^j and β_i^j are used for both the invariants and the coefficients in the normal form (3.12). In the following, it is proved that they are actually equal to each other.

Theorem 5.1 *Consider a system in the form (2.4).*

(i) The quadratic transformation (3.10)-(3.11) does not change the value of quadratic invariants.

(ii) For a system in normal form (3.12), its quadratic invariants (5.1) are equal to the coefficients of the quadratic terms in the normal form.

(iii) Given two systems in the form of (2.4) with the same linearization (same ω), the quadratic part of one system can be transformed into that of another system by a suitable transformation (3.10)-(3.11) if and only if they have the same quadratic invariants.

$\diamond$

6 Conclusion

In this paper, linear and quadratic normal forms of nonlinear systems with a pair of imaginary uncontrollable modes are derived. Based on the normal form, formulae of feedbacks are found to control the bifurcation of the system. The Hopf bifurcation can not be removed from the closed-loop system, because the imaginary eigenvalues are uncontrollable. However, is it proved that both the orientation and the stability of the periodic solution can be controlled by state feedback. It is proved in this paper that a linear feedback determines the orientation of the periodic solution around the bifurcation point, and the quadratic feedback controls the stability of the periodic solution. The explicit relation between the feedback and the performance of the periodic solution, such as the orientation and stability, is derived.

References

[1] Abed, E. H. and J.-H. Fu (1986). Local Feedback stabilization and bifurcation control, part I. Hopf Bifurcation. *Systems and Control Letters*, **7**, 11-17.

[2] Abed, E. H. and J.-H. Fu (1987). Local Feedback stabilization and bifurcation control, part II. Stationary Bifurcation. *Systems and Control Letters*, **8**, 467-473.

[3] Aeyels, D. (1985). Stabilization of a class of nonlinear systems by a smooth feedback control. *Systems and Control Letters*, **5**, 289-294.

[4] Arnold, V.I. (1988). *Geometrical Methods in the Theory of Ordinary Differential Equations, second edition.* Springer-Verlag.

[5] Behtash, S. and S. Sastry (1988). Stabilization of Nonlinear Systems with Uncontrollable Linearization. *IEEE Trans. Automatic Control*, **33**, 585-590.

[6] Carr, J. (1981). *Application of Center Manifold Theory.* Springer.

[7] Chen, G. and X. Dong (1998). *From chaos to order.* Ed. World Scientific.

[8] Colonius, F., and W. Kliemann (1995). Controllability and stabilization of one-dimensional systems near bifurcation points.*Systems and Control Letters*, **24**, 87-95.

[9] Glendinning, P. (1994). *Stability, Instability and Chaos: an introduction to the theory of nonlinear differential equations.* Cambridge University Press.

[10] Gu, G., X. Chen, A. G. Sparks and S. S. Banda (1999). Bifurcation Stabilization with Local Output Feedback. *Siam J. Control and Optimization*, **37**, 934-956.

[11] Kang, W. and A.J. Krener (1992). Extended Quadratic Controller Normal Form and Dynamic State Feedback Linearization of Nonlinear Systems. *Siam J. Control and Optimization*, **30**, 1319-1337.

[12] Kang, W. (1998a). Bifurcation and Normal Form of Nonlinear Control Systems-part I; part II. *Siam J. Control and Optimization*, **36**:193-212;213-232.

[13] Krener, A. J. (1984). Approximate linearization by state feedback and coordinate change. *Systems and Control Letters*, **5**, 181-185.

[14] Wang, Y. and R. M. Murray (1998). Feedback Stabilization of Steady-State Feedback and Hopf Bifurcations. *Proc. of the 37th IEEE CDC*, 2431-2437.

[15] Wiggins, S. (1990). *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Texts in Applied Mathematics 2, Springer.