Output Regulation of General Linear Systems with Saturating Actuators[1]

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Abstract

This paper studies the classical problem of output regulation for linear systems subject to actuator saturation. The asymptotically regulatable region, the set of all initial conditions of the plant and the exosystem for which output regulation is possible, is characterized in terms of the null controllable region of the anti-stable subsystem of the plant. Feedback laws are constructed that achieve regulation on the asymptotically regulatable region. These feedback laws are constructed from the stabilizing feedback laws in such a way that a stabilizing feedback law that achieves a larger domain of attraction leads to a feedback law that achieves output regulation on a larger subset of the asymptotically regulatable region and, a stabilizing feedback law on the entire asymptotically null controllable region leads to a feedback law that achieves output regulation on the entire asymptotically regulatable region.

1 Introduction

There has been considerable research on the problem of stabilization and output regulation of linear systems subject to actuator saturation. The problem of stabilization involves the issues of the characterization of the asymptotically null controllable region $\mathcal{C}^a$, the set of all initial conditions that can be driven to the origin by the saturating actuators asymptotically, and the construction of feedback laws that achieve stabilization on the entire or a large portion of $\mathcal{C}^a$. Recent years have witnessed extensive research that addresses these issues. In particular, for an open loop system that is stabilizable and has all its poles in the closed left-half plane, it was established in [2, 11] that $\mathcal{C}^a$ is the entire state space. For this reason, a linear system that is stabilizable in the usual linear sense and has all its poles in the closed left-half plane is referred to as asymptotically null controllable with bounded controls, or ANCBC. For ANCBC systems subject to actuator saturation, various feedback laws that achieve global or semi-global stabilization on $\mathcal{C}^a$ have been constructed (see, for example, [13, 12, 8, 7]). Here by semi-global stabilization on $\mathcal{C}^a$ we mean the construction of a feedback law that achieves a domain of attraction large enough to include any *a priori* given (arbitrarily large) bounded set in $\mathcal{C}^a$. For an exponentially unstable open-loop system subject to actuator saturation, $\mathcal{C}^a$ was recently characterized in [3]. Also in [3], simple feedback laws were constructed that achieve semi-global stabilization on $\mathcal{C}^a$ for linear systems with only two exponentially unstable poles. More recently, feedback laws have been constructed that achieve semi-global stabilization on $\mathcal{C}^a$ for general linear systems subject to actuator saturation[4].

In comparison with the problem of stabilization, the problem of output regulation for linear systems subject to actuator saturation, however, has received relatively less attention. The few works that have motivated our current research are [10], [9] and [13]. In [9, 13], the problem of output regulation was studied for ANCBC systems subject to actuator saturation. Necessary and sufficient conditions on the plant/exosystem and their initial conditions were derived under which output regulation can be achieved. Under these conditions, feedback laws that achieve output regulation were constructed based on the semi-global stabilizing feedback laws of [8]. The recent work [10] made an attempt to address the problem of output regulation for exponentially unstable linear systems subject to actuator saturation. The attempt was to enlarge the set of initial conditions of the plant and the exosystem under which output regulation can be achieved. In particular, for plants with only one positive pole and exosystems that contain only one frequency component, feedback laws were constructed that achieve output regulation on what we will characterize in this paper as the asymptotically regulatable region.

The objective of this paper is to systematically study the problem of output regulation for general linear systems subject to actuator saturation. In particular, we will first characterize the asymptotically regulatable region $\mathcal{R}_g^a$, the set of plant and exosystem initial conditions for which output regulation is possible with the saturating actuators. It turns out that $\mathcal{R}_g^a$ can be characterized in terms of the null controllable region of the anti-stable subsystem of the plant. We then construct feedback laws that achieve regulation on $\mathcal{R}_g^a$.

The remainder of this paper is organized as follows. In Section 2, we precisely formulate the problem of output regulation for linear systems with saturating actuators. Section 3 characterizes the asymptotically regulatable region $\mathcal{R}_g^a$. Sections 4 and 5 respectively construct state feedback and error feedback laws that achieve output regulation on $\mathcal{R}_g^a$. Finally, Section 6 gives a brief

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concluding remark.

Throughout the paper, we will use standard notation. For a vector $u \in \mathbf{R}^m$, we use $|u|_\infty$ and $|u|_2$ to denote the vector ∞-norm and the 2-norm. For a measurable function $u : [0,\infty) \to \mathbf{R}^m$, we define $\|u\|_\infty = \sup_{t\in[0,\infty]} |u(t)|_\infty$. We use $\mathrm{sat}(\cdot)$ to denote the standard (vector) saturation function. The proofs are sketched or omitted due to space limitation.

2 Preliminaries and Problem Statement

In this section, we first recall from [1, 6] the classical formulation and results on the problem of output regulation for linear systems. This brief review will motivate our formulation as well as the solution to the problem of output regulation for linear systems subject to actuator saturation.

2.1 Output Regulation for Linear Systems

Consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw, \\ \dot{w} &= Sw, \\ e &= Cx + Qw. \end{aligned} \tag{1}$$

The first equation of this system describes a plant, with state $x \in \mathbf{R}^n$ and input $u \in \mathbf{R}^m$, subject to the effect of a disturbance represented by Pw. The third equation defines the error $e \in \mathbf{R}^q$ between the actual plant output Cx and a reference signal $-Qw$ that the plant output is required to track. The second equation describes an autonomous system, often called the exosystem, with state $w \in \mathbf{R}^r$. The exosystem models the class of disturbances and references taken into consideration.

The objective of the output regulation problem is to achieve internal stability and output regulation by state feedback or error feedback. Internal stability means that if we disconnect the exosystem and set w equal to zero then the closed-loop system is asymptotically stable. Output regulation means that for any initial conditions of the closed-loop system, we have that $e(t) \to 0$ as $t \to \infty$.

The solution to the output regulation problem is based on the following three assumptions.

A1. The eigenvalues of S have nonnegative real parts;
A2. The pair (A, B) is stabilizable;
A3. The pair $\left(\begin{bmatrix} C & Q \end{bmatrix}, \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \right)$ is detectable.

The following results, due to Francis [1], describe necessary and sufficient conditions for the existence of solutions to the output regulation problem.

Proposition 1 *Suppose Assumptions A1 and A2 hold. Then, the problem of output regulation is solvable by state feedback if and only if there exist matrices* Π *and* Γ *that solve the linear matrix equations*

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + P, \\ C\Pi + Q &= 0. \end{aligned} \tag{2}$$

Moreover, if in addition Assumption A3 also holds, the solvability of the above linear matrix equations is also a necessary and sufficient condition for the solvability of the output regulation problem by error feedback.

2.2 Output Regulation for Linear Systems Subject to Actuator Saturation

Motivated by the classical formulation of output regulation for linear systems, we consider the following linear system subject to actuator saturation and the exosystem,

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw, \\ \dot{w} &= Sw, \\ e &= Cx + Qw, \end{aligned} \tag{3}$$

where $u \in \mathbf{R}^m$ is the output of the saturating actuators and, without loss of generality, can be assumed to be measurable and satisfy the bound $\|u\|_\infty \le 1$. A control u that satisfies these assumptions will be referred to as an *admissible control*. Because of the bound on the control input, both the plant and the exosystem cannot operate in the entire state space. For this reason, we assume that $(x_0, w_0) \in \mathcal{Y}_0$ for some $\mathcal{Y}_0 \in \mathbf{R}^n \times \mathbf{R}^r$. Let $\mathcal{X}_0 = \{x_0 \in \mathbf{R}^n : \ (x_0, 0) \in \mathcal{Y}_0\}$.

The problems to be addressed in this paper are the following:

Problem 1 *The problem of output regulation by state feedback for the system (3) is to find a feedback law* $u = \phi(x, w)$*, with* $|\phi(x,w)|_\infty \le 1$ *and* $\phi(0,0) = 0$*, such that*

1. the equilibrium $x = 0$ *of the system* $\dot{x} = Ax + B\phi(x, 0)$ *is asymptotically stable with* $\mathcal{X}_0$ *contained in its domain of attraction;*

2. for all $(x_0, w_0) \in \mathcal{Y}_0$*, the interconnection of (3) and the feedback law* $u = \phi(x,w)$ *results in* $\lim_{t\to\infty} e(t) = 0$ *and that* $x(t)$ *is bounded.*

Problem 2 *The problem of output regulation by error feedback for the system (3) is to find an error feedback law of the form*

$$\begin{aligned} \dot{\xi} &= \psi(\xi, e), \quad \xi_0 \in \mathcal{Y}_0, \\ u &= \phi(\xi), \end{aligned}$$

with $\psi(0,0) = 0$, $\phi(0) = 0$ *and* $|\phi(\xi)|_\infty \le 1$*, such that*

1. the equilibrium $(x, \xi) = (0, 0)$ *of the system*

$$\begin{aligned} \dot{x} &= Ax + B\phi(\xi), \\ \dot{\xi} &= \psi(\xi, Cx) \end{aligned} \tag{4}$$

is asymptotically stable with $\mathcal{X}_0 \times \mathcal{Y}_0$ *contained in its domain of attraction;*

2. for all $(x_0, w_0, \xi_0) \in \mathcal{Y}_0 \times \mathcal{Y}_0$*, the interconnection of (3) and (4) results in* $\lim_{t\to\infty} e(t) = 0$ *and that* $x(t)$ *and* $\xi(t)$ *are bounded.*

Our objective is to characterize the maximal set of initial conditions (x_0, w_0) on which the above two problems are solvable and to explicitly construct feedback laws that actually solve the problems.

We will assume that (A, B) is stabilizable and that S has all its eigenvalues on the imaginary axis and are simple. These are necessary or without loss of generality.

3 The Regulatable Region

In this section, we will characterize the set of all initial states of the plant and the exosystem on which the problems of output regulation are solvable under the restriction that $\|u\|_\infty \le 1$. We will refer to this set as the asymptotically regulatable region.

To begin with, we observe from the classical output regulation theory that for these two problems to be solvable, there must exist matrices $\Pi \in \mathbf{R}^{n\times r}$ and $\Gamma \in \mathbf{R}^{m\times r}$ that solve the following matrix equations,

$$\begin{array}{rcl} \Pi S & = & A\Pi + B\Gamma + P, \\ 0 & = & C\Pi + Q. \end{array} \tag{5}$$

Given the matrices Π and Γ, we define a new state $z = x - \Pi w$ and rewrite the system equations as

$$\begin{array}{rcl} \dot{z} & = & Az + Bu - B\Gamma w, \\ \dot{w} & = & Sw, \\ e & = & Cz. \end{array} \tag{6}$$

From these new equations, it is clear that $e(t)$ goes to zero asymptotically if $z(t)$ goes to zero asymptotically. This is possible only if (see [9]) $\sup_{t\ge 0} \left|\Gamma e^{St} w_0\right|_\infty < 1$. For this reason, we will restrict our attention to exosystem initial conditions in the following compact set

$$\mathcal{W}_0 = \left\{ w_0 \in \mathbf{R}^r : |\Gamma w(t)|_\infty = |\Gamma e^{St} w_0|_\infty \le \rho, \forall t \ge 0 \right\},$$

for some $\rho \in [0,1)$. For later use, we also denote $\delta = 1 - \rho$. We note that the compactness of $\mathcal{W}_0$ can be guaranteed by the observability of (Γ, S). We can now precisely define the notion of asymptotically regulatable region as follows.

Definition 1 *Given $T > 0$, a pair $(z_0, w_0) \in \mathbf{R}^n \times \mathcal{W}_0$ is regulatable in time T if there exists an admissible control $u(\cdot)$, such that the response of (6) satisfies $z(T) = 0$; A pair (z_0, w_0) is regulatable if there exist a finite $T > 0$ and an admissible control $u(\cdot)$ such that $z(T) = 0$. The set of all (z_0, w_0) regulatable in time T is denoted as $\mathcal{R}_g(T)$ and the set of all regulatable (z_0, w_0) is referred to as the regulatable region and is denoted as $\mathcal{R}_g$. The set of all (z_0, w_0) for which there exists an admissible control $u(\cdot)$ such that the response of (6) satisfies $\lim_{t\to\infty} z(t) = 0$ is referred to as the asymptotically regulatable region and is denoted as $\mathcal{R}_g^a$.*

We will describe $\mathcal{R}_g(T)$, $\mathcal{R}_g$ and $\mathcal{R}_g^a$ in terms of the asymptotically null controllable region of the plant $\dot{v} = Av + Bu, \|u\|_\infty \le 1$, which is defined as follows.

Definition 2 *The asymptotically null controllable region, denoted as $\mathcal{C}^a$, is the set of all v_0 that can be driven to the origin asymptotically by an admissible control. The null controllable region at time T, denoted as $\mathcal{C}(T)$, is*

$$\mathcal{C}(T) = \left\{ \int_0^T e^{-A\tau} Bu(\tau) d\tau : \ \|u\|_\infty \le 1 \right\}; \tag{7}$$

and the null controllable region, denoted as $\mathcal{C}$, is $\mathcal{C} = \bigcup_{T\in[0,\infty)} \mathcal{C}(T)$.

Simple methods to describe $\mathcal{C}$ and $\mathcal{C}^a$ were given in [3]. To simplify the characterization of $\mathcal{R}_g$ and $\mathcal{R}_g^a$, and without loss of generality, let us assume that $z = \left[\begin{array}{cc} z_1^{\mathrm{T}} & z_2^{\mathrm{T}} \end{array}\right]^{\mathrm{T}}, z_1 \in \mathbf{R}^{n_1}, z_2 \in \mathbf{R}^{n_2}$ and

$$A = \left[\begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array}\right], \quad B = \left[\begin{array}{c} B_1 \\ B_2 \end{array}\right], \tag{8}$$

where $A_1 \in \mathbf{R}^{n_1\times n_1}$ is semi-stable (i.e., all its eigenvalues are in the closed left-half plane) and $A_2 \in \mathbf{R}^{n_2\times n_2}$ is anti-stable (i.e., all its eigenvalues are in the open right-half plane). The anti-stable subsystem

$$\begin{array}{rcl} \dot{z}_2 & = & A_2 z_2 + B_2 u - B_2 \Gamma w, \\ \dot{w} & = & Sw, \end{array} \tag{9}$$

is of crucial importance. Denote its regulatable regions as $\mathcal{R}_{g2}(T)$ and $\mathcal{R}_{g2}$, and the null controllable regions for the system $\dot{v}_2 = A_2 v_2 + B_2 u$ as $\mathcal{C}_2(T)$ and $\mathcal{C}_2$. Then, the asymptotically null controllable region for the system $\dot{v} = Av + Bu$ is given by $\mathcal{C}^a = \mathbf{R}^{n_1} \times \mathcal{C}_2$ [2], where $\mathcal{C}_2$ is a bounded set. Denote the closure of $\mathcal{C}_2$ as $\bar{\mathcal{C}}_2$, then

$$\bar{\mathcal{C}}_2 = \left\{ \int_0^\infty e^{-A_2\tau} B_2 u(\tau) d\tau : \ \|u\|_\infty \le 1 \right\}.$$

Theorem 1 *Let $V_2 \in \mathbf{R}^{n_2\times r}$ be the unique solution to the matrix equation*

$$-A_2 V_2 + V_2 S = -B_2 \Gamma \tag{10}$$

and let $V(T) = V_2 - e^{-A_2 T} V_2 e^{ST}$. Then

1) $\mathcal{R}_{g2}(T) = \{(z_2, w) \in \mathbf{R}^{n_2} \times \mathcal{W}_0 : z_2 - V(T)w \in \mathcal{C}_2(T)\}$;
2) $\mathcal{R}_{g2} = \{(z_2, w) \in \mathbf{R}^{n_2} \times \mathcal{W}_0 : z_2 - V_2 w \in \mathcal{C}_2\}$;
3) $\mathcal{R}_g^a = R^{n_1} \times \mathcal{R}_{g2}$.

Proof. 1) Given $(z_{20}, w_0) \in \mathbf{R}^{n_2} \times \mathcal{W}_0$ and an admissible control u, the solution of (9) at $t = T$ is,

$$z_2(T) = e^{A_2 T}\left(z_{20} + \int_0^T e^{-A_2\tau} B_2 \left(u(\tau) - \Gamma e^{S\tau} w_0\right) d\tau \right).$$

From (10), we have

$$-\int_0^T e^{-A_2\tau} B_2 \Gamma e^{S\tau} d\tau = \int_0^T d e^{-A_2\tau} V_2 e^{S\tau} = -V(T). \tag{11}$$

Thus

$$e^{-A_2 T} z_2(T) = z_{20} - V(T) w_0 + \int_0^T e^{-A_2\tau} B_2 u(\tau) d\tau.$$

By setting $z_2(T) = 0$, we immediately obtain 1) from the definitions of $\mathcal{R}_g(T)$ and (7).

2) Since A_2 is anti-stable and S is stable, we have that $\lim_{T\to\infty} V(T) = V_2$. 2) follows from 1) and the fact that $\mathcal{R}_{g2}(T)$ and $\mathcal{C}_2(T)$ approach $\mathcal{R}_{g2}$ and $\mathcal{C}_2$ respectively, as T goes to infinity.

3) It is easy to see that $\mathcal{R}_g^a \subset \mathbf{R}^{n_1} \times \mathcal{R}_{g2}$. We need to show that $\mathbf{R}^{n_1} \times \mathcal{R}_{g2} \subset \mathcal{R}_g^a$. Suppose that $(z_0, w_0) =$

$(z_{10}, z_{20}, w_0) \in \mathbf{R}^{n_1} \times \mathcal{R}_{g_2}$. There exist a $T \geq 0$ and an admissible $u(\cdot)$ such that $z_2(T) = 0$. For $t \geq T$, let $u = \Gamma w + u_\delta$, with $\|u_\delta\|_\infty \leq \delta = 1 - \rho$. Since $w_0 \in \mathcal{W}_0$, we have $\|\Gamma w\|_\infty \leq \rho$ and hence, $\|u\|_\infty \leq 1$ and

$$\dot{z} = Az + Bu - B\Gamma w = Az + B\, u_\delta.$$

Since $\begin{bmatrix} z_1(T) \\ 0 \end{bmatrix}$ is inside the asymptotically null controllable region for the above system under the constraint $\|u_\delta\|_\infty \leq \delta$ [2], there exists a $u_\delta(\cdot)$ such that $\lim_{t\to\infty} z(t) = 0$. Hence $(z_0, w_0) \in \mathcal{R}_g^a$. This establishes that $\mathbf{R}^{n_1} \times \mathcal{R}_{g_2} \subset \mathcal{R}_g^a$ and hence $\mathbf{R}^{n_1} \times \mathcal{R}_{g_2} = \mathcal{R}_g^a$. □

4 State Feedback Controller

In this section, we will construct a feedback law that solves the problem of output regulation by state feedback for linear systems subject to actuator saturation. We will assume that a stabilizing feedback $u = f(v), |f(v)|_\infty \leq 1$ for all $v \in \mathbf{R}^n$, has been designed and the equilibrium $v = 0$ of the closed-loop system

$$\dot{v} = Av + Bf(v) \tag{12}$$

has a domain of attraction $\mathcal{S} \subset \mathcal{C}^a$. We will construct our feedback law from this stabilizing feedback law.

Given a feedback $u = g(z, w), |g(z, w)|_\infty \leq 1$ for all $(z, w) \in \mathbf{R}^n \times \mathcal{W}_0$, we have the closed-loop system

$$\begin{aligned} \dot{z} &= Az + B\, g(z, w) - B\Gamma w, \\ \dot{w} &= Sw. \end{aligned} \tag{13}$$

Denote the time response of $z(t)$ to the initial state (z_0, w_0) as $z(t, z_0, w_0)$ and define

$$\mathcal{S}_{zw} := \left\{ (z_0, w_0) \in \mathbf{R}^n \times \mathcal{W}_0 : \lim_{t\to\infty} z(t, z_0, w_0) = 0 \right\}.$$

Clearly by definition, we have $\mathcal{S}_{zw} \subset \mathcal{R}_g^a$. Our objective is to design a control law $u = g(z, w)$ such that $\mathcal{S}_{zw}$ is as large as possible, or as close to $\mathcal{R}_g^a$ as possible.

Firstly we need a mild assumption which will be removed latter. Assume that there exists a matrix $V \in \mathbf{R}^{n\times r}$ such that

$$-AV + VS = -B\Gamma. \tag{14}$$

This will be the case if A and S have no common eigenvalues. Denote

$$D_{zw} := \{(z, w) \in \mathbf{R}^n \times \mathcal{W}_0 : z - Vw \in \mathcal{S}\}, \tag{15}$$

on which the following observation can be made.

Observation 1 *a) The set D_{zw} increases as $\mathcal{S}$ increases, and if $\mathcal{S} = \mathcal{C}^a$, then $D_{zw} = \mathcal{R}_g^a$;*

b) In the absence of w, $x_0 \in \mathcal{S} \Rightarrow (z_0, 0) \in D_{zw}$.

With this observation, we see that our objective in Problem 1 is simply to design a feedback law such that $D_{zw} \subset \mathcal{S}_{zw}$. We will reach this objective through a series of lemmas.

Lemma 1 *Let $u = f(z - Vw)$. Consider the closed-loop system*

$$\begin{aligned} \dot{z} &= Az + Bf(z - Vw) - B\Gamma w, \\ \dot{w} &= Sw. \end{aligned} \tag{16}$$

For this system, D_{zw} is an invariant set and for all $(z_0, w_0) \in D_{zw}$, $\lim_{t\to\infty}(z(t) - Vw(t)) = 0$.

Proof. Substitute (14) into system (16), we obtain

$$\dot{z} = A(z - Vw) + Bf(z - Vw) + V\dot{w}.$$

Define the new state $v := z - Vw$, we have

$$\dot{v} = Av + Bf(v),$$

which has a domain of attraction $\mathcal{S}$. The result of the lemma follows immediately. □

Lemma 1 says that, in the presence of w, the simple feedback $u = f(z - Vw)$ will cause $z(t)$ to approach $Vw(t)$, which is bounded. Our next step is to construct a finite sequence of controllers $u = f_k(z, w, \alpha)$, $k = 0, 1, 2, \cdots, N$, all parameterized in $\alpha \in (0, 1)$. By judiciously switching between these controllers, we can cause $z(t)$ to approach $\alpha^k Vw(t)$ for any k. By choosing N large enough, $z(t)$ will become arbitrarily small. Once $z(t)$ becomes small enough, we will use the controller $u = \Gamma w + \delta\, \mathrm{sat}(\frac{Fz}{\delta})$ (F to be specified latter) to make $z(t)$ converge to the origin.

Let $F \in \mathbf{R}^{m\times n}$ be such that

$$\dot{v} = Av + B\, \mathrm{sat}(Fv), \tag{17}$$

is asymptotically stable. Let $X > 0$ be such that $(A + BF)^{\mathrm{T}} X + X(A + BF) < 0$ and the ellipsoid $\mathcal{E} := \{v \in \mathbf{R}^n : v^{\mathrm{T}} Xv \leq 1\}$ is in the linear region of the saturation function, i.e., $|Fv|_\infty \leq 1$ for all $v \in \mathcal{E}$. Then $\mathcal{E}$ is an invariant set and is in the domain of attraction for the closed-loop system (17).

Lemma 2 *Suppose that $D \subset \mathbf{R}^n$ is an invariant set in the domain of attraction for the system $\dot{v} = Av + Bf(v)$, then for any $\alpha > 0$, αD is an invariant set in the domain of attraction for the system $\dot{v} = Av + \alpha Bf(v/\alpha)$.*

For any $\alpha \in (0, 1)$, there exists a positive integer N such that

$$\alpha^N |X^{\frac{1}{2}} Vw|_2 < \delta, \quad \forall\, w \in \mathcal{W}_0. \tag{18}$$

Define a sequence of subsets in $\mathbf{R}^n \times \mathcal{W}_0$ as,

$$\begin{aligned} D_{zw}^k &= \left\{(z, w) \in \mathbf{R}^n \times \mathcal{W}_0 : z - \alpha^k Vw \in \alpha^k \mathcal{E}\right\}, \\ & \qquad\qquad k = 0, 1, \cdots, N, \\ D_{zw}^{N+1} &= \{(z, w) \in \mathbf{R}^n \times \mathcal{W}_0 : z \in \delta\mathcal{E}\}, \end{aligned}$$

and, on each of these sets, define a state feedback law as follows,

$$\begin{aligned} & f_k(z, w, \alpha) = (1 - \alpha^k)\Gamma w + \alpha^k\, \mathrm{sat}\left(F(z - \alpha^k Vw)/\alpha^k\right), \\ & f_{N+1}(z, w) = \Gamma w + \delta\, \mathrm{sat}\left(Fz/\delta)\right). \end{aligned}$$

It can be verified that, for any $k = 0$ to $N + 1$, $|f_k(z, w, \alpha)|_\infty \leq 1$ for all $(z, w) \in \mathbf{R}^n \times \mathcal{W}_0$.

Lemma 3 *Let $u = f_k(z, w, \alpha)$. Consider the closed-loop system*

$$\begin{array}{rcl} \dot{z} &=& Az + Bf_k(z,w,\alpha) - B\Gamma w, \\ \dot{w} &=& Sw. \end{array} \tag{19}$$

For this system, D^k_{zw} is an invariant set. Moreover, if $k = 0, 1, \cdots, N$, then for all $(z_0, w_0) \in D^k_{zw}$, $\lim_{t\to\infty}(z(t) - \alpha^k V w(t)) = 0$; if $k = N+1$, then, for all $(z_0, w_0) \in D^{N+1}_{zw}$, $\lim_{t\to\infty} z(t) = 0$.

Proof. For the case $k = 0, 1, \cdots, N$, we have

$$\dot{z} = Az + \alpha^k B\text{sat}\left(F(z - \alpha^k V w)/\alpha^k\right) - \alpha^k B\Gamma w.$$

Let $v_k = z - \alpha^k V w$, then by (14),

$$\dot{v}_k = Av_k + \alpha^k B\text{sat}\left(Fv_k/\alpha^k\right). \tag{20}$$

The result of the lemma follows from Lemma 2. The case where $k = N+1$ is similar. □

Based on the lemmas established above, we construct our final state feedback law as follows,

$$u = g(z,w,\alpha,N) = \begin{cases} f_{N+1}(z,w), \text{ if } (z,w) \in \Omega^{N+1} := D^{N+1}_{zw}, \\ f_k(z,w,\alpha), \text{ if } (z,w) \in \Omega^k := D^k_{zw} \setminus \cup_{j=k+1}^{N+1} D^j_{zw}, \\ f(z - Vw), \text{ if } (z,w) \in \Omega := \mathbf{R}^n \times \mathcal{W}_0 \setminus \cup_{j=0}^{N+1} D^j_{zw}. \end{cases}$$

What remains to be shown is that this controller will accomplish our objective if the parameter α is properly chosen. Let

$$\alpha_0 = \max_{w \in \mathcal{W}_0} \frac{|X^{\frac{1}{2}} V w|_2}{|X^{\frac{1}{2}} V w|_2 + 1}.$$

It is obvious that $\alpha_0 \in (0, 1)$.

Theorem 2 *Choose any $\alpha \in (\alpha_0, 1)$ and let N be specified as in (18). Then for all $(z_0, w_0) \in D_{zw}$, the solution of the closed-loop system*

$$\begin{array}{rcl} \dot{z} &=& Az + B\,g(z,w,\alpha,N) - B\Gamma w, \\ \dot{w} &=& Sw \end{array} \tag{21}$$

satisfies $\lim_{t\to\infty} z(t) = 0$, i.e., $D_{zw} \subset \mathcal{S}_{zw}$.

Proof. The control $u = g(z,w,\alpha,N)$ is executed by choosing one from $f_k(z,w,\alpha), k = 0, 1, \cdots, N+1$, and $f(z - Vw)$. The crucial point is to guarantee that no chattering will occur and that (z, w) will move successively from Ω, to $\Omega^0, \Omega^1, \cdots$, finally entering Ω^{N+1}, in which $z(t)$ will converge to the origin. This can be shown by combining Lemmas 1, 2 and 3. □

In what follows, we will deal with the case that there is no V that satisfies $-AV + VS = -B\Gamma$.

Suppose that the z-system (6) has the following form,

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u - \begin{bmatrix} B_1\Gamma \\ B_2\Gamma \end{bmatrix} w \tag{22}$$

with $A_1 \in \mathbf{R}^{n_1 \times n_1}$ semi-stable and $A_2 \in \mathbf{R}^{n_2 \times n_2}$ anti-stable. Also suppose that there is a known function $f(v_2), |f(v_2)|_\infty \le 1$ for all $v_2 \in \mathbf{R}^{n_2}$ such that the origin of the system $\dot{v}_2 = A_2 v_2 + B_2 f(v_2)$ has a domain of attraction $\mathcal{S}_2$, which is a bounded set. Then by Lemma 2, the system $\dot{v}_2 = A_2 v_2 + \delta B_2 f(v_2/\delta)$ has a domain of attraction $\delta\mathcal{S}_2$; and by [14] there exists a control $u = \delta\,\text{sat}(h(v))$ such that the origin of

$$\dot{v} = Av + \delta B\text{sat}(h(v))$$

has a domain of attraction $\mathcal{S}_\delta = \mathbf{R}^{n_1} \times \delta\mathcal{S}_2$.

Since there is a V_2 satisfying $-A_2V_2 + V_2S = -B_2\Gamma$, by the foregoing development of Theorem 2, there exists a controller $u = g(z_2, w, \alpha, N)$ such that for any (z_0, w_0) satisfying $z_{20} - V_2 w_0 \in \mathcal{S}_2$, $w_0 \in \mathcal{W}_0$, the response of the closed-loop system satisfies $\lim_{t\to\infty} z_2(t) = 0$. Hence there is a finite time $t_1 > 0$ such that $z(t_1) \in \mathbf{R}^{n_1} \times \delta\mathcal{S}_2$. Let the controller be,

$$u = \begin{cases} g(z_2, w, \alpha, N), & \text{if } z_2 \in \mathbf{R}^{n_2} \setminus \delta\mathcal{S}_2 , \\ \Gamma w + \delta\,\text{sat}(h(z)), & \text{if } z_2 \in \delta\mathcal{S}_2. \end{cases}$$

We conclude that under this control, the set $\{(z,w) \in \mathbf{R}^n \times \mathcal{W}_0 : z_2 - V_2 w \in \mathcal{S}_2\} \subset \mathcal{S}_{zw}$.

5 Error Feedback

Consider the same open-loop system as (6). Here in this section, we assume that only the error $e = Cz$ is available for feedback. Also, without loss of generality, assume that the pair $(\bar{C}, \bar{A}) = \left([C \;\; 0], \begin{bmatrix} A & -B\Gamma \\ 0 & S \end{bmatrix}\right)$ is observable.

We use the following observer to reconstruct the state z and w,

$$\begin{array}{rcl} \dot{\bar{z}} &=& A\bar{z} + Bu - B\Gamma\bar{w} - L_1(e - C\bar{z}), \\ \dot{\bar{w}} &=& S\bar{w} - L_2(e - C\bar{z}). \end{array} \tag{23}$$

Let $\tilde{z} = z - \bar{z}, \tilde{w} = w - \bar{w}$, the composite system is

$$\begin{array}{rl} \dot{z} &= Az + Bu - B\Gamma w, \\ \dot{w} &= Sw, \\ \begin{bmatrix} \dot{\tilde{z}} \\ \dot{\tilde{w}} \end{bmatrix} &= \begin{bmatrix} A + L_1C & -B\Gamma \\ L_2C & S \end{bmatrix} \begin{bmatrix} \tilde{z} \\ \tilde{w} \end{bmatrix}. \end{array} \tag{24}$$

Now we have to use $(\bar{z}, \bar{w})$ instead of (z, w) to construct a feedback controller. Since $(\bar{C}, \bar{A})$ is observable, we can choose $L = \begin{bmatrix} L_1^{\mathrm{T}} & L_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ appropriately such that the estimation error $(\tilde{z}, \tilde{w})$ decays arbitrarily fast. Nevertheless, we need an additional assumption on the existing stabilizing controller $f(v)$ so that it can tolerate some class of disturbances.

Consider the system

$$\dot{v} = Av + Bf(v + \eta),$$

where η stands for the disturbance arising from the observer error. Assume that $|f(v)|_\infty \le 1$ for all $v \in \mathbf{R}^n$ and that there exist a set $D_0 \in \mathbf{R}^n$ and positive numbers γ and d_0 such that the solution of the system satisfies

$$\|v\|_\infty \le \gamma \max(|v_0|_\infty, \|\eta\|_\infty), \quad \|v\|_a \le \gamma\|\eta\|_a,$$

for all $v_0 \in D_0$, $\|\eta\|_\infty \le d_0$, where $\|v\|_a = \limsup_{t\to\infty} |v(t)|_\infty$. This system is said to satisfy an asymptotic bound from D_0 with gain γ and restriction d_0 [14]. In [5], a saturated linear feedback $u = f(v) = \text{sat}(F_0 v)$ with such property is constructed for second-order anti-stable systems. Moreover the set D_0 can be made arbitrarily close to the null controllable region.

Let $D \in \mathbf{R}^n$ be in the interior of D_0. Given a positive number M, denote

$$D_M = \{(z, w, \tilde{z}, \tilde{w}) : z - Vw \in D, |[\tilde{z}^{\mathrm{T}} \ \ \tilde{w}^{\mathrm{T}}]^{\mathrm{T}}|_2 \leq M\}.$$

Lemma 4 *There exists an $L \in \mathbf{R}^{(n+r)\times q}$, such that under the control $u = f(\bar{z} - V\bar{w})$, the solution of the system (24) satisfies $\lim_{t\to\infty}(z(t) - Vw(t)) = 0$ for all $(z_0, w_0, \tilde{z}_0, \tilde{w}_0) \in D_M$.*

Lemma 4 means that we can keep $z(t)$ bounded if $(z_0, w_0, \tilde{z}_0, \tilde{w}_0) \in D_M$. Just as the state feedback case, we want to move $z(t)$ to the origin by making $z(t) - \alpha^k Vw(t)$ small with increased k. Due to the switching nature of the final controller and that the feedback has to be based on $(\bar{z}, \bar{w})$, we need to construct a sequence of sets which are invariant with respect to $(\bar{z}, \bar{w})$ rather than (z, w) under the corresponding controllers.

Using linear system theory, it is easy to design an $F \in \mathbf{R}^{m\times n}$, along with a matrix $X > 0$ such that $A + BF$ is Hurwitz and the set $\mathcal{E} = \{v \in \mathbf{R}^n : v^{\mathrm{T}} X v \leq 1\}$ is inside the linear region of the saturation function $\text{sat}(Fv)$ and that for some positive number d_1, $\mathcal{E}$ is invariant for the system

$$\dot{v} = Av + B\text{sat}(Fv) - \eta, \quad |\eta|_\infty \leq d_1. \qquad (25)$$

Let α and N be determined from X in the same way as with the state feedback controller. With $F \in \mathbf{R}^{m\times n}$, we form a sequence of controllers,

$$u = f_k(\bar{z}, \bar{w}, \alpha) = (1-\alpha^k)\Gamma\bar{w} + \alpha^k \text{sat}\left(F(\bar{z} - \alpha^k V\bar{w})/\alpha^k\right), \quad k = 0, 1, 2, \cdots, N,$$

$$u = f_{N+1}(\bar{z}, \bar{w}) = \Gamma\bar{w} + \delta\, \text{sat}\,(F\bar{z}/\delta).$$

Define a sequence of sets in $\mathbf{R}^n \times \mathbf{R}^r$:

$$D^k_{\bar{z}\bar{w}} = \left\{(\bar{z}, \bar{w}) \in \mathbf{R}^n \times \mathbf{R}^r : \bar{z} - \alpha^k V\bar{w} \in \alpha^k \mathcal{E}\right\}, \quad k = 0, 1, 2, \cdots, N,$$

$$D^{N+1}_{\bar{z}\bar{w}} = \left\{(\bar{z}, \bar{w}) \in \mathbf{R}^n \times \mathbf{R}^r : \bar{z} \in \delta\mathcal{E}\right\}.$$

Our final error feedback law is

$$u = \begin{cases} f_{N+1}(\bar{z}, \bar{w}), & \text{if } (\bar{z}, \bar{w}) \in \Omega^{N+1}, \\ f_k(\bar{z}, \bar{w}, \alpha), & \text{if } (\bar{z}, \bar{w}) \in \Omega^k \\ f(\bar{z} - V\bar{w}), & \text{if } (\bar{z}, \bar{w}) \in \Omega, \end{cases} \qquad (26)$$

where $\Omega^k = D^k_{\bar{z}\bar{w}} \setminus \cup_{j=k+1}^{N+1} D^j_{\bar{z}\bar{w}}$, $k = 0, 1, \ldots, N$, $\Omega^{N+1} = D^{N+1}_{\bar{z}\bar{w}}$ and $\Omega = \mathbf{R}^n \times \mathbf{R}^r \setminus \cup_{j=0}^{N+1} D^j_{\bar{z}\bar{w}}$.

Theorem 3 *Consider the feedback system (24) with (26). Suppose that F is chosen such that $\mathcal{E}$ is invariant for the system (25). There exists an $L \in \mathbf{R}^{(n+r)\times q}$ such that $\lim_{t\to\infty} z(t) = 0$ for all $(z_0, w_0, \tilde{z}_0, \tilde{w}_0) \in D_M$.*

6 Conclusions

In this paper, we have systematically studied the problem of output regulation for linear systems subject to actuator saturation. The plants considered here are general and can be exponentially unstable. We first characterized the regulatable region, the set of initial conditions of the plant and the exosystem for which output regulation can be achieved. We then constructed state feedback and error feedback laws that achieve output regulation on the regulatable region.

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