On a Minimally Restrictive Supervisory Policy that Enforces Liveness in Partially Controlled Free Choice Petri Nets

Ramavarapu S. Sreenivas[1]
Coordinated Science Laboratory & Department of General Engineering,
University of Illinois at Urbana-Champaign
Urbana, IL 61801, U.S.A
rsree@uiuc.edu

Abstract

A *Petri Net* (PN) is said to be *live* if it is possible to fire any transition from every reachable marking, although not necessarily immediately. Under appropriate conditions, a non-live PN can be made live via supervision. Under this paradigm an external-agent, the supervisor, prevents the firing of certain transitions at each reachable marking so as to enforce liveness. A PN is *partially controlled* if the supervisor can prevent the firing of only a subset of transitions. A *Free Choice Petri net* (FCPN) is a PN where every arc from a place to a transition is either the unique output arc from that place, or, it is the unique input arc to the transition. In this paper we show for each partially controlled FCPN that can be made live via supervision, any marking that is reachable under a policy that enforces liveness should cover (with respect to the standard partial ordering of vectors) a member of a specific, finite set of minimal-markings. Assuming this set is readily available, this observation is used to (i) test the existence, and (ii) to synthesize a minimally restrictive supervisory policy that enforces liveness in a partially controllable FCPN. We suggest investigations into the computation of this specific, finite set of minimal markings as a future research topic.

1 Introduction

A large class of systems can be modeled as systems with independent, interacting, concurrent components. Typically, each independent process is split into several operations; the execution of each operation is conditioned on the satisfaction of a set of logical preconditions. Upon the execution of any such operation, a new set of logical conditions is created that inhibit the execution of some operations and enables the execution of others in the system. The supervisory control of such systems requires an external agent to regulate, or limit, the operations of each component so as to guarantee a common objective. In this paper we concern ourselves with a stronger version of deadlock avoidance called *liveness*. From any reachable state of a *live* system, it should be possible for any of the components to execute any of its operations, although not necessarily immediately.

Petri nets (PNs) [5] are an ideal choice of the modeling of such systems as they allow easy representation of the logical preconditions as the marking, and the operations can be represented as transitions. A PN is said to be *live*, if from any reachable marking it is possible to fire any transition, although not necessarily immediately. A *Free-Choice Petri net* (FCPN) is a restricted class of Petri nets (PNs) where every arc from a place to a transition is either the unique output arc from that place, or, it is the unique input arc to the transition. In this paper we consider *partially controlled* FCPNs where the external agent, the supervisor, can prevent only some (i.e. not all) transitions from firing. We concern ourselves with the synthesis of supervisory policies that enforce liveness in non-live, partially controlled FCPNs.

In reference [13] it is shown that the existence of a supervisory policy that enforces liveness in par-

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tially controlled FCPNs is monotone with respect to the initial marking. That is, if there is a supervisory policy that enforces liveness in a partially controlled FCPN for a given initial marking $\mathbf{m}^0$, then there exists a supervisory policy that enforces liveness in the same FCPN for any initial marking that is larger than $\mathbf{m}^0$. Consider the situation where the structure of a PN is given, and the initial marking is to be chosen (i.e. the initial marking is not a part of the definition of the PN structure). For any partially controlled FCPN structure N, let $\Delta(N)$ denote the set of initial markings of N for which one could synthesize a supervisory policy that enforces liveness. Using the result in reference [13], we infer that if $\Delta(N) \neq \emptyset$, then $\Delta(N)$ must be be *right-closed*. That is, if this set contains a marking $\mathbf{m} \in \Delta(N)$, then it should contain all markings $\widehat{\mathbf{m}}$ such that $\widehat{\mathbf{m}} \geq \mathbf{m}$, under the standard partial ordering of vectors. Since every right-closed set of vectors contains a finite set of minimal elements, the set $\Delta(N)$ should also contain a finite set of minimal markings $\Gamma(\Delta(N))$. That is, (i) $\forall \mathbf{m} \in \Delta(N), \exists \widetilde{\mathbf{m}} \in \Gamma(\Delta(N))$, such that $\mathbf{m} \geq \widetilde{\mathbf{m}}$, and (ii) if $\exists \mathbf{m} \in \Delta(N)$, and $\exists \widetilde{\mathbf{m}} \in \Gamma(\Delta(N))$, such that $\widetilde{\mathbf{m}} \geq \mathbf{m}$, then $\widetilde{\mathbf{m}} = \mathbf{m}$.

In this paper we show that if a marking $\mathbf{m}$ is reachable under the supervision of a policy that enforces liveness in a partially controlled FCPN structure N for some choice of the initial marking, say $\mathbf{m}^0$, then $\exists \widetilde{\mathbf{m}} \in \Gamma(\Delta(N))$, such that $\mathbf{m} \geq \widetilde{\mathbf{m}}$. Since $\mathbf{m}^0$ is trivially in the set of reachable markings under supervision, it follows that this observation also holds for $\mathbf{m}^0$ as well. It is easy to see that if the set $\Gamma(\Delta(N))$ is known for a given FCPN structure N, the existence of a supervisory policy that enforces liveness, can be tested by checking if $\exists \widetilde{\mathbf{m}} \in \Gamma(\Delta(N))$, such that $\mathbf{m}^0 \geq \widetilde{\mathbf{m}}$. Since the set $\Gamma(\Delta(N))$ is finite, this test is computable. Using the abovementioned relationship between markings that are reachable under the supervision of a policy that enforces liveness and the elements of $\Gamma(\Delta(N))$, we synthesize a minimally restrictive policy that enforces liveness in a given FCPN. We do not address the computation of the set $\Gamma(\Delta(N))$ in this paper, this is suggested as a future research topic.

The following section contains the notations and definitions used in this paper. Section 3 presents a review of the relevant results in the literature. Section 4 contains the main result of this paper and some of its implications. The paper concludes with some future research directions in section 5.

2 Notations and Definitions

A *Petri net structure* (PN structure) $N = (\Pi, T, \Phi)$ is an ordered 3-tuple, where $\Pi = \{p_1, p_2, \ldots, p_n\}$ is a set of n *places*, $T = \{t_1, t_2, \ldots, t_m\}$ is a set of m *transitions*, and $\Phi \subseteq (\Pi \times T) \cup (T \times \Pi)$ is a set of *arcs*. The *initial marking function* (or the *initial marking*) of a PN structure N is a function $\mathbf{m}^0 : \Pi \rightarrow \mathcal{N}$, where $\mathcal{N}$ is the set of nonnegative integers. We will use the term *Petri net* (PN) to denote a PN structure along with its initial marking $\mathbf{m}^0$, and is denoted by the symbol $N(\mathbf{m}^0)$. The *state* of a PN $N(\mathbf{m}^0)$ is the marking $\mathbf{m} : \Pi \rightarrow \mathcal{N}$ that identifies the number of *tokens* in each place. A marking $\mathbf{m} : \Pi \rightarrow \mathcal{N}$ is sometimes represented by an integer-valued vector $\mathbf{m} \in \mathcal{N}^n$, where the i-th component $\mathbf{m}_i$ represents the token load ($\mathbf{m}(p_i)$) of the i-th place. Extending this notation to integer-valued vectors in general, the i-th component of any integer-valued vector $\mathbf{x}$ is denoted by $\mathbf{x}_i$.

Given two integer-valued vectors $\mathbf{x}, \mathbf{y} \in \mathcal{N}^n$, we use the notation $\mathbf{x} \geq \mathbf{y}$ if some component of $\mathbf{x}$ is greater than the corresponding component of $\mathbf{y}$ and no component of $\mathbf{x}$ is less than the corresponding component of $\mathbf{y}$. A set of integer-valued vectors $\Delta \subseteq \mathcal{N}^n$ is said to be *right-closed*, if $((\mathbf{x} \in \Delta) \& (\mathbf{y} \geq \mathbf{x})) \Rightarrow \mathbf{y} \in \Delta$. Every right-closed set of vectors $\Delta \subseteq \mathcal{N}^n$ contains a finite set of minimal-elements $\Gamma(\Delta) \subseteq \Delta$, such that (i) $\forall \mathbf{x} \in \Delta, \exists \mathbf{y} \in \Gamma(\Delta)$, such that $\mathbf{x} \geq \mathbf{y}$, and (ii) if $\exists \mathbf{x} \in \Delta, \exists \mathbf{y} \in \Gamma(\Delta)$, such that $\mathbf{y} \geq \mathbf{x}$, then $\mathbf{x} = \mathbf{y}$.

For a given marking $\mathbf{m}$ a transition $t \in T$ is said to be *enabled* if $\forall p \in^\bullet t$, $\mathbf{m}(p) \geq 1$, where $^\bullet x = \{y \mid (y, x) \in \Phi\}$. The set of enabled transitions in the PN $N(\mathbf{m}^0)$ at the marking $\mathbf{m}$ is denoted by the symbol $T_e(\mathbf{m}, N)$. An enabled transition $t \in T_e(\mathbf{m}^1, N)$ can *fire*, which changes the marking $\mathbf{m}^1$ to $\mathbf{m}^2$ according to the equation $\mathbf{m}^2(p) = \mathbf{m}^1(p) - card(p^\bullet \cap \{t\}) + card(^\bullet p \cap \{t\})$, where $x^\bullet = \{y \mid (x, y) \in \Phi\}$, and the symbol $card(\bullet)$ is used to denote the cardinality of the set argument. This notation is also used to denote the predecessor or successor set of a set of places or transitions.

A string of transitions $\sigma = t_{j_1} t_{j_2} \cdots t_{j_k}$, where $t_{j_i} \in T$ ($i \in \{1, 2, \ldots, k\}$) is said to be a *valid firing string* starting from the marking $\mathbf{m}$, if, (i) the transition t_{j_1} is enabled at the marking $\mathbf{m}$, and (ii) for $i \in \{1, 2, \ldots, k-1\}$ the firing of the transition t_{j_i} produces a marking at which the transition

$t_{j_{i+1}}$ is enabled. The set of *reachable markings* from $\mathbf{m}^0$, denoted by $\Re(N, \mathbf{m}^0)$, is the set of markings generated by all valid firing strings starting with marking $\mathbf{m}^0$ in the PN structure N. At a marking $\mathbf{m}^1$, if the firing of a valid firing string σ results in a marking $\mathbf{m}^2$, we represent it as $\mathbf{m}^1 \rightarrow \sigma \rightarrow \mathbf{m}^2$. A transition $t \in T$ is *live* if $\forall \mathbf{m}^1 \in \Re(N, \mathbf{m}^0), \exists \mathbf{m}^2 \in \Re(N, \mathbf{m}^1)$ such that $t \in T_e(\mathbf{m}^2, N)$.

A PN structure $N = (\Pi, T, \Phi)$ is a *Free-Choice* PN structure (FCPN structure) if $\forall p \in \Pi, card(p^\bullet) > 1 \Rightarrow^\bullet (p^\bullet) = \{p\}$. In words, a PN structure is Free-Choice if and only if an arc from a place to a transition is either the unique output arc from that place, or, is the unique input arc to the transition.

For the purposes of supervisory control the set of transitions $T = \{t_1, t_2, \ldots, t_m\}$ is partitioned into two sets $T_u = \{t_1, t_2, \ldots, t_p\}$ and $T_c = \{t_{p+1}, t_{p+2}, \ldots, t_m\}$. The set of transitions T_c (T_u) is referred to as the set of *controllable* (*uncontrollable*) transitions. A *supervisory policy* $\mathcal{P} : \mathcal{N}^n \rightarrow \{0, 1\}^m$, is a total map that returns an m-dimensional binary vector for each reachable marking. For reasons that should be obvious in the following paragraph, the supervisory policy must also satisfy the requirement that $\mathcal{P}(\mathbf{m})_i = 1, 1 \leq i \leq p, \forall \mathbf{m} \in \mathcal{N}^n$.

The supervisory policy $\mathcal{P}$ permits the firing of transition t_i at marking $\mathbf{m}$, only if $\mathcal{P}(\mathbf{m})_i = 1$. If at a marking $\mathbf{m}$ all input places to a transition t_i contain at least one token, we say the transition t_i is *state-enabled* at $\mathbf{m}$. If $\mathcal{P}(\mathbf{m})_i = 1$, we say the transition t_i is *control-enabled* at $\mathbf{m}$. A transition has to be state-enabled and control-enabled before it can fire. Since uncontrollable transitions can never be prevented from firing, they remain control-enabled under all markings. Hence the requirement that $\mathcal{P}(\mathbf{m})_i = 1, 1 \leq i \leq p, \forall \mathbf{m} \in \mathcal{N}^n$.

A string of transitions $\sigma = t_{j_1} t_{j_2} \cdots t_{j_k}$, where $t_{j_i} \in T$ ($i \in \{1, 2, \ldots, k\}$) is said to be a *valid firing string* under the supervision of $\mathcal{P}$, starting from the marking $\mathbf{m}$, if, (i) the transition t_{j_1} is enabled at the marking $\mathbf{m}$, $\mathcal{P}(\mathbf{m})_{j_1} = 1$, and (ii) for $i \in \{1, 2, \ldots, k-1\}$ the firing of the transition t_{j_i} produces a marking $\mathbf{m}^i$ at which the transition $t_{j_{i+1}}$ is enabled and $\mathcal{P}(\mathbf{m}^i)_{j_{i+1}} = 1$. The set of reachable markings under the supervision of $\mathcal{P}$ in $N(\mathbf{m}^0)$ from the initial marking $\mathbf{m}^0$ is denoted by $\Re(N, \mathbf{m}^0, \mathcal{P})$. A transition t_{j_i} is *live* under the supervision of $\mathcal{P}$ if $\forall \mathbf{m}^1 \in \Re(N, \mathbf{m}^0, \mathcal{P}), \exists \mathbf{m}^2 \in \Re(N, \mathbf{m}^1, \mathcal{P})$ such that $t_{j_i} \in T_e(\mathbf{m}^2, N)$ and $\mathcal{P}(\mathbf{m}^2)_{j_i} = 1$. A supervisory policy $\mathcal{P}$ enforces liveness if all transitions in N are live under $\mathcal{P}$.

Given two supervisory policies $\mathcal{P}_1$ and $\mathcal{P}_2$ that enforce liveness in a PN $N(\mathbf{m}^0)$, we say $\mathcal{P}_1$ is *less restrictive* than $\mathcal{P}_2$ if, $\forall \mathbf{m} \in \Re(N, \mathbf{m}^0, \mathcal{P}_1) \cap \Re(N, \mathbf{m}^0, \mathcal{P}_2)$, (i) $\mathcal{P}_1(\mathbf{m}) \geq \mathcal{P}_2(\mathbf{m})$, componentwise, and (ii) $\exists i \in \{1, 2, \ldots, m\}$ such that $\mathcal{P}_1(\mathbf{m})_i > \mathcal{P}_2(\mathbf{m})_i$. A supervisory policy $\mathcal{P}$ that enforces liveness is said to be *minimally restrictive*, if there does not exist a supervisory policy that is less restrictive than $\mathcal{P}$.

In the next section we present a brief overview of some results in the literature on the synthesis of supervisory policies that enforce liveness in PNs.

3 Review of Relevant Results

In the interest of space a short, annotated-list of results that are pertinent to the synthesis of supervisory policies that enforce liveness in PNs are presented below.

1. There is a necessary and sufficient condition for the existence of a supervisory policy that enforces liveness in an arbitrary PN (cf. theorem 5.1, [8]).
2. If all transitions in a PN are controllable (i.e. ($card(T_u) = 0$), then it is possible to test the existence, and synthesize a (least restrictive) policy that enforces liveness (cf. theorem 5.2, [8]).
3. If some transitions in a PN are not controllable (i.e. ($card(T_u) = p) \neq 0$), and the plant PN is unbounded, then in general, it is impossible to test the existence of a supervisory policy that enforces liveness (cf. theorem 5.3, [8]; Hilbert's Tenth Problem [4]).
4. If the plant PN is bounded, then we can test the existence of a supervisory policy that enforces liveness (cf. theorem 5.4, [8]; figure 10, [8]).
5. The existence of a policy that enforces liveness in a plant PN is necessary and sufficient for the existence of a minimally-restrictive policy that enforces liveness for that plant PN (cf. theorem 6.1, [8]).
6. Testing the existence of a supervisory policy that enforces liveness in bounded plant PNs is at least $PSPACE-complete$ (cf. page 943, [8]).
7. It is possible to characterize policies that enforce liveness in completely controlled FCPNs (cf. theorem 1, [7]). This result also applies, *mutatis*

mutandis, to partially controlled FCPNs.
8. The existence of a supervisory policy that enforces liveness in partially controlled FCPN structures is monotonic with respect to the initial marking [13].

The following results are motivated by mitigating the computational complexity of testing the existence and synthesis of supervisory policies that enforce liveness in arbitrary/specific plant PNs.
1. If the PN can be represented hierarchically, then the complexity of testing the existence and synthesis of policies that enforce liveness can be significantly improved (cf. [12]).
2. There is a ready-made policy that enforces liveness in PNs that belong to a class of completely controlled FCPNs called *Independent, Increasing, Free-Choice PNs* (II-FCPNs) (cf. theorem 2, [7]). The exact definition of an II-FCPN is not important, but it is important to note that for this class of FCPNs the test for the existence and synthesis of policies that enforce liveness is entirely unnecessary.
3. Any arbitrary, completely controlled PN can be converted into a bisimulation-equivalent, completely controlled FCPN by the addition of a few extra places and transitions (cf. Hack's procedure in [1]). Reference [6] shows that there is a supervisory policy that enforces liveness in an arbitrary, completely controlled PN if and only if there is a similar policy for its free-choice equivalent. For instance if the arbitrary, completely controlled PN is converted into its equivalent FCPN as per the construction in reference [1], and this equivalent FCPN turns out to be an II-FCPN (cf. section 4.1, [7]), then following the result in reference [6], we know there is a supervisory policy that enforces liveness in the original PN. This eliminates the computationally expensive step of deciding the existence of a policy that enforces liveness entirely.
4. If an arbitrary, completely controllable PN is such that its equivalent FCPN is an II-FCPN, then the ready-made policy that enforces liveness in the II-FCPN can be easily converted into a policy that enforces liveness in the original PN (that is not necessarily a FCPN) [9].
5. If an arbitrary, completely controllable PN belongs to a new, hitherto unknown class of PNs, (the definition of this class of PNs can be found in reference [10], we refrain from presenting it here in the interest of space) then there is a readily available supervisory policy that enforces liveness.
6. If an arbitrary, completely controllable PN has specific structural features known as *directed cut-places*, or, *cut-transitions*, it is possible to test the existence and synthesize policies that enforce liveness in the plant using a "divide-and-conquer" approach with a significantly reduced computational burden [11].

4 Main Results

We now present observations that are used in the proof of the two main results in this paper. The first observation is about the right-closed nature of the set $\Delta(N)$ for a partially controlled FCPN structure N.

Observation 4.1 *Given a partially controlled FCPN structure* $N = (\Pi, T, \Phi)$*, the set* $\Delta(N) = \{\mathbf{m} \in \mathcal{N}^n \mid$ *there is a supervisory policy* $\mathcal{P}$ *that enforces liveness in* $N(\mathbf{m})\}$ *is right-closed.*

This observation follows directly from the results in reference [13] where it is shown that if there is a supervisory policy that enforces liveness in the FCPN $N(\mathbf{m}^0)$ then there is a supervisory policy that enforces liveness in the FCPN $N(\overline{\mathbf{m}}^0)$, if $\overline{\mathbf{m}}^0 \geq \mathbf{m}^0$. The following observation is about the existence of a finite set of minimal elements $\Gamma(\Delta(N)) \subseteq \Delta(N)$. This result on right-closed sets is well-known and we refrain from presenting a detailed proof in the interest of space.

Observation 4.2 *Given a partially controlled FCPN structure* $N = (\Pi, T, \Phi)$*, and the set* $\Delta(N) = \{\mathbf{m} \in \mathcal{N}^n \mid$ *there is a supervisory policy* $\mathcal{P}$ *that enforces liveness in* $N(\mathbf{m})\}$*, if* $\Delta(N) \neq \emptyset$*, then* $\exists \Gamma(\Delta(N)) \subseteq \mathcal{N}^n$ *such that (i)* $\Gamma(\Delta(N))$ *is finite, (ii)* $\forall \mathbf{m} \in \Delta(N), \exists \widetilde{\mathbf{m}} \in \Gamma(\Delta(N))$ *such that* $\mathbf{m} \geq \widetilde{\mathbf{m}}$*, and (iii) if* $\exists \mathbf{m} \in \Delta(N), \exists \widetilde{\mathbf{m}} \in \Gamma(\Delta(N))$ *such that* $\widetilde{\mathbf{m}} \geq \mathbf{m}$*, then* $\widetilde{\mathbf{m}} = \mathbf{m}$*.*

Theorem 4.1 *Given a partially controlled FCPN structure* $N = (\Pi, T, \Phi)$*, and an initial marking* $\mathbf{m}^0$*, there is a supervisory policy* $\mathcal{P}$ *that enforces liveness in* $N(\mathbf{m}^0)$*, if and only if* $\exists \widetilde{\mathbf{m}} \in \Gamma(\Delta(N))$ *such that* $\mathbf{m}^0 \geq \widetilde{\mathbf{m}}$*.*

There is a supervisory policy that enforces liveness in $N(\mathbf{m}^0)$ if and only if $\mathbf{m}^0 \in \Delta(N)$. Since $\Gamma(\Delta(N))$ is the set of minimal elements of $\Delta(N)$, there is a supervisory policy that enforces liveness in $N(\mathbf{m}^0)$ if and only if $\exists \widetilde{\mathbf{m}} \in \Gamma(\Delta(N))$ such that $\mathbf{m}^0 \geq \widetilde{\mathbf{m}}$. Hence the result.

Even though the set $\Delta(N)$ is right-closed, for any $\mathbf{m}^0 \in \Delta(N)$, and any supervisory policy $\mathcal{P}$ that enforces liveness in $N(\mathbf{m}^0)$, the set of reachable markings under supervision in $N(\mathbf{m}^0)$ is not necessarily right-closed. This follows directly from the fact that the reachable set of any bounded PN under any supervisory policy (not necessarily one that enforces liveness) is not right-closed. This is formally stated in the following observation .

Observation 4.3 *Given a partially controlled FCPN structure $N = (\Pi, T, \Phi)$, and an initial marking $\mathbf{m}^0$ such that there is a supervisory policy $\mathcal{P}$ that enforces liveness in $N(\mathbf{m}^0)$, then the set $\Re(N, \mathbf{m}^0, \mathcal{P})$ is not necessarily right-closed.*

Observation 4.4 *Given a partially controlled FCPN structure $N = (\Pi, T, \Phi)$, and an initial marking $\mathbf{m}^0$ such that there is a supervisory policy $\mathcal{P}$ that enforces liveness in $N(\mathbf{m}^0)$, let $\mathbf{m}^1 \in \Re(N, \mathbf{m}^0, \mathcal{P})$, then $\mathbf{m}^1 \in \Delta(N)$.*

Since $\mathcal{P}$ enforces liveness in $N(\mathbf{m}^0)$, it follows that $\forall t_i \in T, \forall \mathbf{m}^1 \in \Re(N, \mathbf{m}^0, \mathcal{P}), \exists \mathbf{m}^2 \in \Re(N, \mathbf{m}^1, \mathcal{P})$ such that $t_i \in T_e(\mathbf{m}^2, N)$ and $\mathcal{P}(\mathbf{m}^2)_i = 1$. Since $\Re(N, \mathbf{m}^1, \mathcal{P}) \subseteq \Re(N, \mathbf{m}^0, \mathcal{P})$, $\mathcal{P}$ also enforces liveness in $N(\mathbf{m}^1)$. Hence $\mathbf{m}^1 \in \Delta(N)$.

Observation 4.5 *Given a partially controlled FCPN structure $N = (\Pi, T, \Phi)$, and an initial marking $\mathbf{m}^0 \in \Delta(N)$, let $\mathbf{m}^0 \rightarrow \sigma \rightarrow \mathbf{m}^1$ in $N(\mathbf{m}^0)$, for some $\sigma \in T_u^*$, then $\mathbf{m}^1 \in \Delta(N)$.*

This observation follows from the fact that (i) if $\mathbf{m}^0 \in \Delta(N)$, there is a supervisory policy that enforces liveness in $N(\mathbf{m}^0)$, (ii) $\mathbf{m}^0 \rightarrow \sigma \rightarrow \mathbf{m}^1$ in $N(\mathbf{m}^0)$ under the supervision of any supervisory policy in $N(\mathbf{m}^0)$ as $\sigma \in T_u^*$, and (iii) observation 4.4.

Given a partially controlled FCPN structure $N = (\Pi, T, \Phi)$ and an initial marking $\mathbf{m}^0 \in \Delta(N)$, we define a supervisory policy $\widehat{\mathcal{P}}$ for $N(\mathbf{m}^0)$ as follows – $\forall \mathbf{m} \in \Re(N, \mathbf{m}^0, \widehat{\mathcal{P}})$, (i) if $t_i \in T_u$, then $\widehat{\mathcal{P}}(\mathbf{m})_i = 1$, and (ii) if $t_i \in T_c$, then $\widehat{\mathcal{P}}(\mathbf{m})_i = 1$ only if either (i) $t_i \notin T_e(\mathbf{m}, N)$, or (ii) $\mathbf{m} \rightarrow t_i \rightarrow \overline{\mathbf{m}}$ in $N(\mathbf{m})$, and $\exists \widetilde{\mathbf{m}} \in \Gamma(\Delta(N))$, such that $\overline{\mathbf{m}} \geq \widetilde{\mathbf{m}}$ (i.e. $\overline{\mathbf{m}} \in \Delta(N)$). Since the set $\Gamma(\Delta(N))$ is finite, we know that the test that is required for the case when $t_i \in T_c$ is computable. In theorem 4.2 we show that $\widehat{\mathcal{P}}$ is a minimally restrictive policy that enforces liveness in $N(\mathbf{m}^0)$. Before we state and prove theorem 4.2, we first make an observation about the markings that are reachable under the supervision of $\widehat{\mathcal{P}}$ in $N(\mathbf{m}^0)$.

Observation 4.6 *Given a partially controlled FCPN structure $N = (\Pi, T, \Phi)$, an initial marking $\mathbf{m}^0 \in \Delta(N)$, and the supervisory policy $\widehat{\mathcal{P}}$ as defined above, then if $\mathbf{m}^0 \rightarrow \sigma \rightarrow \mathbf{m}^1$ under the supervision of $\widehat{\mathcal{P}}$ in $N(\mathbf{m}^0)$, then $\mathbf{m}^1 \in \Delta(N)$.*

This observation can be established by an induction argument over the length of σ. The base-case is trivially established when σ is the empty string. The induction hypothesis supposes the observation is true for all σ of length $k \in \mathcal{N}$. For the induction step we suppose $\mathbf{m}^1 \rightarrow t_i \rightarrow \mathbf{m}^2$ under the supervision of $\widehat{\mathcal{P}}$ in $N(\mathbf{m}^0)$. If $t_i \in T_c$ the observation is established by the definition of $\widehat{\mathcal{P}}$ and the fact that $t_i \in T_e(\mathbf{m}^1, N)$. If $t_i \in T_u$, from the induction hypothesis we infer $\mathbf{m}^1 \in \Delta(N)$. Using observation 4.5, we infer $\mathbf{m}^2 \in \Delta(N)$ also.

Theorem 4.2 *Given a partially controlled FCPN structure $N = (\Pi, T, \Phi)$, an initial marking $\mathbf{m}^0 \in \Delta(N)$, and the finite set of minimal-elements $\Gamma(\Delta(N))$, the supervisory policy $\widehat{\mathcal{P}}$ defined above enforces liveness in $N(\mathbf{m}^0)$, and it is minimally restrictive.*

Proof: Let $\mathbf{m}^0 \rightarrow \sigma_1 \rightarrow \mathbf{m}^1$ under the supervision of $\widehat{\mathcal{P}}$ in $N(\mathbf{m}^0)$. From observation 4.6 we know that $\mathbf{m}^1 \in \Delta(N)$. So, there is a supervisory policy $\mathcal{P}$ that enforces liveness in $N(\mathbf{m}^1)$.

$\forall t_i \in T, \exists \sigma_2 \in T^*$, such that $\mathbf{m}^1 \rightarrow \sigma_2 \rightarrow \mathbf{m}^2$ under the supervision of $\mathcal{P}$ in $N(\mathbf{m}^1)$, and $t_i \in T_e(N, \mathbf{m}^2)$ and $\mathcal{P}(\mathbf{m}^2)_i = 1$. Let us suppose $\mathbf{m}^2 \rightarrow t_i \rightarrow \mathbf{m}^3$ under the supervision of $\mathcal{P}$ in $N(\mathbf{m}^1)$. Additionally, $\forall \sigma_3 \in pr(\sigma_2)$ (the term $pr(\bullet)$ is used to denote the prefix set of the string argument), if $\mathbf{m}^1 \rightarrow \sigma_3 \rightarrow \mathbf{m}^4$ under the supervision of $\mathcal{P}$ in $N(\mathbf{m}^1)$, from observation 4.4 we know that $\mathbf{m}^4 \in \Delta(N)$.

By the definition of $\widehat{\mathcal{P}}$ we infer $\mathbf{m}^1 \rightarrow \sigma_3 \rightarrow \mathbf{m}^4$ under the supervision of $\widehat{\mathcal{P}}$ in $N(\mathbf{m}^0)$ also. Therefore $\mathbf{m}^1 \rightarrow \sigma_2 \rightarrow \mathbf{m}^2$ under the supervision of $\widehat{\mathcal{P}}$ in $N(\mathbf{m}^0)$. If $t_i \in T_u$, then $\widehat{\mathcal{P}}(\mathbf{m}^2) = 1$. Since $\mathbf{m}^3 \in \Delta(N)$, we infer that if $t_i \in T_c$, $\widehat{\mathcal{P}}(\mathbf{m}^2)_i = 1$. Hence $\widehat{\mathcal{P}}$ enforces liveness in $N(\mathbf{m}^0)$.

The minimally restrictive nature of $\widehat{\mathcal{P}}$ follows from the fact that if for some marking $\mathbf{m} \in \Re(N, \mathbf{m}^0, \widehat{\mathcal{P}}), \widehat{\mathcal{P}}(\mathbf{m})_i = 0$, then $t_i \in T_c \cap T_e(N, \mathbf{m}), \mathbf{m} \rightarrow t_i \rightarrow \overline{\mathbf{m}}$ in $N(\mathbf{m})$, and $\overline{\mathbf{m}} \notin \Delta(N)$. From observation 4.4 it follows that $\overline{\mathbf{m}}$ cannot be reached under any policy that enforces liveness in $N(\mathbf{m})$. ∎

Reference [3] considers a *forbidden-marking problem* (cf. [2] for a formal definition) where the forbidden-marking set of a partially controllable PN is right-closed. A minimally restrictive supervisory policy that avoids this right-closed, forbidden-marking set is synthesized in this reference. The results of this paper suggest that the problem of enforcing liveness in partially controllable FCPNs can be looked upon as a forbidden-marking problem, where the set of legal markings is $\Delta(N)$, which is right closed, which is different from that of reference [3], where the forbidden-marking set is assumed to be right-closed.

5 Conclusions

Starting from the observation that the set of initial markings $\Delta(N)$ for which there is a supervisory policy that enforces liveness in a partially controlled *Free Choice Petri net* (FCPN) $N(\mathbf{m}^0)$ is right-closed, following a series of observations, we (i) developed a test for the existence of a supervisory policy that enforces liveness in $N(\mathbf{m}^0)$, and (ii) synthesized a minimally restrictive supervisory policy that enforces liveness in $N(\mathbf{m}^0)$. This synthesis procedure supposes the ready availability of the finite set of minimal-elements of $\Delta(N)$. The computation of this set is suggested as a future research topic.

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