On Stabilization of Rotational Modes of an Inverted Pendulum

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Keywords: passivity, dissipativity, inverted pendulum, V-detectability, back-stepping

Abstract. This paper addresses the problem of stabilization a rotational mode of an inverted pendulum with a prescribed position of the cart. The solution is based on the idea that a desired motion of the inverted pendulum corresponds to some set Γ in the phase space of the system. In fact, the set Γ describes periodic orbit for the closed loop system and for the unforced inverted pendulum this set is not invariant. We constructed a family of no-negative functions V_μ, which are zero on Γ and positive elsewhere, and suggested a globally defined *state feedback* transformation, which makes the inverted pendulum to be passive with V_μ from new input to the output – a speed of the cart. Taking advantage of passivity, we derived stabilizing controller and obtained the qualitative description of behavior of the closed loop system solutions. Moreover, the proposed control scheme is extended for the case, when the inverted pendulum is controlled by an actuator.

1 Introduction

The inverted pendulum is an ubiquitous example of nonlinear control systems analysis and design. It is a popular experiment used for educational purposes, and belongs to a class of under-actuated mechanical systems. The standard problem related to inverted pendulum is stabilization of either downward or upright equilibriums at a prescribed position of the cart. Both problems are widely investigated and solved by different methods, see the papers [5, 2, 12, 6, 1, 3, 11, 7, 9] and others.

A different problem related to this under-actuated system is a stabilization of a given rotational behavior of the pendulum with a prescribed position of the cart. The simplified version of such problem was considered in [8] and others as an example for general procedure of invariant sets stabilization. It was assumed that the mass of the cart is negligible and two-dimensional model of the pendulum could be controlled by force applied to suspension point moving on the horizontal. For this simplified model a desired rotation of pendulum exactly corresponds to some value of total energy of the system. So successful stabilization of this energy level results in solving the original problem.

If the mass of the cart is taken into account, the stabilization of the total energy of the system does not lead to the stabilization of desired rotation of pendulum. Indeed for this case the set of the phase space of the inverted pendulum corresponding to constant value of the total energy is 3-dimensional manifold, which is unbounded, and it could contain the motions of the inverted pendulum which escape from any bounded region of the phase space. Obviously, this is different from the stabilization of the pendulum rotation with a prescribed position of the cart, which corresponds to some compact subset Γ of the phase space.

In this paper we suggest a family of the scalar non-negative functions V_μ, which all are positive outside of Γ and zero on Γ, where the parameter μ is any vector (k_E, k_v, k_x) with positive elements. The functions V_μ by construction provide the way for local stabilization of Γ. Indeed, taking the time derivative of any V_μ along the solution of the system one finds that the inverted pendulum with the output function equal to a speed of the cart can be made *passive system* with V_μ after appropriate *static* state feedback transformation. Unfortunately, this transformation, depending on the choice of the parameters μ, is defined just around Γ.

One of the main contribution of the paper is that we are able to single out the parameters μ, i. e. describe the sub-family of $\{V_\mu\}$, for which the aforementioned transformation can be defined globally. Thus one can re-write the original problem in the form of the well known and widely investigated problem: the stabilization of passive system with proper but just *non-negative* storage function.

The analysis and control of the passive system with non-negative storage functions are more complicated and differs from the case when this function is positive. Taking advantage of the notion V-detectability and the procedure suggested in [8], we identify all connected components for ω-limit set of the closed loop system with the control derived by Speed-Gradient method with the storage (goal) function V_μ. It is showed the closed loop system normally contains 2 equilibriums lying outside of Γ. We derive additional constraints on parameters μ making this equilibrium hyperbolic, i. e. from practical point of view unstable. Furthermore, we add the model of the actuator to the system and using a back-stepping design method develop the form of the regulator which solves the original problem.

The paper is organized as follows. Section 2 contains the description of the problem and some known material. The main results of the paper are collected in Section 3 and Section 4. Conclusions are drawn in Section 5.

2 Preliminaries

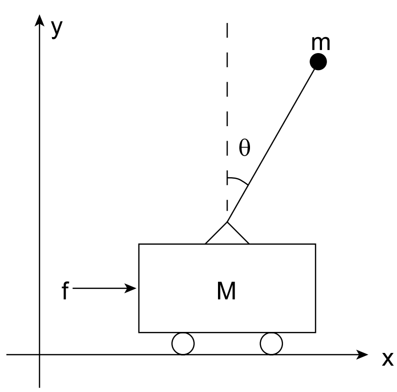

Figure 1: The cart pendulum system

Under the standard assumptions such as a massless rod, point masses, no friction etc., the equations of motion for the inverted pendulum are

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau, \qquad (1)$$

where $q = [x, \theta]^T \in R^1 \times S^1$, x is the horizontal displacement of the cart, θ is the angle between the pendulum rod and the vertical which is zero at the upright position;

$$M(q) = \begin{bmatrix} M+m & ml\cdot\cos\theta \\ ml\cdot\cos\theta & ml^2 \end{bmatrix}, \qquad (2)$$

m, M are the masses of the pendulum and the cart respectively; l is the length of the rod;

$$C(q,\dot{q}) = \begin{bmatrix} 0 & -ml\cdot\sin\theta\cdot\dot{\theta} \\ 0 & 0 \end{bmatrix}, \qquad (3)$$

$$G(q) = \begin{bmatrix} 0 \\ -mgl\cdot\sin\theta \end{bmatrix}, \quad \tau = \begin{bmatrix} f \\ 0 \end{bmatrix}, \qquad (4)$$

where f is the control input to be defined.

The total energy of the inverted pendulum is

$$E(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + mgl\cdot(\cos\theta + 1) \qquad (5)$$

and it could attain any values from the interval $[0, +\infty)$. Let E_0 be any non-negative constant, and let us introduce the function

$$V(q,\dot{q}) = \frac{k_E}{2}\left(E(q,\dot{q}) - E_0\right)^2 + \frac{k_v}{2}\dot{x}^2 + \frac{k_x}{2}x^2, \qquad (6)$$

where k_E, k_v, k_x are some positive constants. Without loss of generality it will be assumed that the desired position x of the cart is 0. The equality of V to zero is obviously equivalent to the equations

$$E(q,\dot{q}) = E_0, \quad \dot{x} = 0, \quad x = 0.$$

The first two equations result in the following

$$E_0 = \frac{1}{2}\dot{\theta}^2 + mgl(\cos\theta + 1).$$

It is worth mentioning that the right-hand side of the last relation is a total energy of the simple pendulum, and it corresponds to the rotational mode to be stabilized.

This simple argument shows, that successful stabilization of the set

$$\Gamma = \{(q,\dot{q}) : V(q,\dot{q}) = 0\}$$

leads to the solution of the problem. Straightforward computations show that the derivative of V along any solution of (1) is

$$\begin{aligned} \dot{V} \;=\; \dot{x}\Bigg[f\left(k_E\left(E(q,\dot{q}) - E_0\right) + \frac{k_v}{1+\sin^2\theta}\right) + \\ + \frac{k_v \sin\theta(\dot{\theta}^2 - g\cos\theta)}{1+\sin^2\theta} + k_x x\Bigg], \end{aligned} \qquad (7)$$

where for simplicity we will assume that $M = m = l = 1$.

To assure the stabilization of the set Γ the function V should decrease (not increase) along the closed loop system solutions. Particularly, this is true if the feedback control law f satisfies the relation

$$f(q,\dot{q})\left(k_{E}\cdot(E(q,\dot{q})-E_0)+\frac{k_v}{1+\sin^2\theta}\right)+ \\ +\frac{k_v\cdot\sin\theta\cdot(\dot{\theta}^2-g\cdot\cos\theta)}{1+\sin^2\theta}+k_x\cdot x=-\phi(q,\dot{q}), \quad (8)$$

where ϕ is some scalar function forming an acute angle with $\dot{x}$, i. e.

$$\dot{x}\phi(q,\dot{q})>0 \quad \forall\,\dot{x}\neq 0,\ \forall\,x,\,\theta,\,\dot{\theta}.$$

In this case $\dot{V}$ is negative along the closed loop system solutions.

It is worth mentioning that the equation (8) means that the closed loop system is passive from ϕ to $\dot{x}$ and the problem of feedback design in this case is equivalent to the problem of state feedback passification. We focus our attention to C^1-smooth functions $\phi(q,\dot{q})$ depending only on $\dot{x}$, i. e. $\phi(q,\dot{q})=\phi(\dot{x})$. Denote the set

$$\begin{aligned}\mathcal{F} \;=\; & \{(k_{E},k_v,k_x,\phi(\cdot)) : k_{E}>0,\, k_v>0,\, k_x>0 \text{ and} \\ & \phi(z) \text{ is a } C^1\text{–smooth function forming} \\ & \text{an acute angle with } z \text{ with } \dot{\phi}(0)>0\}.\end{aligned} \quad (9)$$

Any element in $\mathcal{F}$ corresponds to some feedback controller f.

3 Main results

Definition of controller $f(q,\dot{q})$ made in (8) obviously leads to a local result. One can easily check that for any positive parameters k_{E}, k_v, k_x the equation (8) could be solved with respect to f in some neighborhood of Γ. To establish the global properties of the controller $f(q,\dot{q})$ implicitly defined in (8), one should provide the criterion of solvability of the equation (8) with respect to the control variable f for any vector $(q,\dot{q})$. The next statement contains such a condition.

Proposition 1 *Let k_{E}, k_v be any positive constants. Given $E_0\geq 0$, the inequality*

$$k_{E}\cdot(E(x_1,x_2,\theta_1,\theta_2)-E_0)+\frac{k_v}{1+\sin^2\theta_1}\neq 0, \quad (10)$$

holds for any x_1, x_2, θ_1, θ_2 if and only if the constants k_{E}, k_v satisfy the inequality

$$k_v>\rho\cdot k_{E}, \quad (11)$$

where

$$\rho=\begin{cases} E_0, & E_0\leq\frac{1}{2}g \\ \frac{g}{27}\left[2\frac{E_0-g}{g}+\sqrt{\left(\frac{E_0-g}{g}\right)^2+6}\right]\times \\ \left[18-\left(\frac{E_0-g}{g}-\sqrt{\left(\frac{E_0-g}{g}\right)^2+6}\right)^2\right], & E_0>\frac{1}{2}g\end{cases} \quad (12)$$

Moreover, the inequality (11) guarantees that the left hand side of (10) is a positive function.

Thus, due to Proposition 1, if inequality (11) holds, equation (8) is globally solvable and $f(q(t),\dot{q}(t))$ is bounded provided that $(q(t),\dot{q}(t))$ is bounded.

Denote $\mathcal{F}_\rho$ as a subset of elements of $\mathcal{F}$ for which the inequality (11) holds for given nonnegative value of E_0. We will write $f\in\mathcal{F}_\rho$ having in mind that f is uniquely defined by the controller given by equation (8), corresponding to some point in $\mathcal{F}_\rho$. The following simple statement contains some qualitative result on the system behavior even for the case when equation (8) cannot be solved.

Proposition 2 *Let k_{E}, k_v be any positive constants. Given $E_0\geq 0$, suppose that the equality*

$$k_{E}\cdot(E(q(t),\dot{q}(t))-E_0)+\frac{k_v}{1+\sin^2\theta(t)}=0, \quad \forall\,t\in\mathcal{T} \quad (13)$$

is valid for some time interval $\mathcal{T}$. Then the functions $\dot{x}(t)$, $\dot{\theta}(t)$ are uniformly bounded on $\mathcal{T}$

Proof. Indeed, equality (13) is equivalent to

$$g\cdot(\cos\theta+1)-E_0+\frac{k_v}{k_{E}\cdot(1+\sin^2\theta)}= \\ -\left((\dot{x}+\tfrac{1}{2}\dot{\theta}\cos\theta)^2+\tfrac{1}{2}\dot{\theta}^2(1-\tfrac{1}{2}\cos^2\theta)\right).$$

This implies that the sum of $(\dot{x}+\frac{1}{2}\dot{\theta}\cos\theta)^2$ and $\frac{1}{2}\dot{\theta}^2(1-\frac{1}{2}\cos^2\theta)$ is uniformly bounded from below and above. But both items are positive so the value of $\dot{\theta}^2(1-\frac{1}{2}\cos^2\theta)$ is bounded from below and above and, in particular, $|\dot{\theta}|$ is uniformly bounded. Therefore, $|\dot{x}|$ is also uniformly bounded. ■

One can easily verify that, under the assumption $M=m=l=1$, the system (1) can be rewritten in the equivalent form

$$\ddot{x}=\frac{1}{1+\sin^2\theta}\left[\sin\theta\left(\dot{\theta}^2-g\cos\theta\right)+f\right] \quad (14)$$

$$\ddot{\theta}=\frac{1}{1+\sin^2\theta}\left[-\dot{\theta}^2\sin\theta\cos\theta+2g\sin\theta-f\cos\theta\right] \quad (15)$$

One of the main results of this paper is

Theorem 3 *Consider the controlled inverted pendulum (1). Take any state feedback controller $f \in \mathcal{F}_\rho$ defined by equation (8) with appropriate parameters $\{k_E, k_v, k_x, \phi(\cdot)\}$. Then*

1) any solution of the closed loop system is globally well defined and bounded;

2) for any solution $[q(t), \dot{q}(t)]$ of the closed loop system, its ω-limit set Ω_* consists of either the upright equilibrium

$$[x, \theta, \dot{x}, \dot{\theta}] = [0, 0, 0, 0], \tag{16}$$

the downward equilibrium

$$[x, \theta, \dot{x}, \dot{\theta}] = [0, \pi, 0, 0], \tag{17}$$

or the set

$$\Gamma = \left\{ [x, \theta, \dot{x}, \dot{\theta}] : \frac{\dot{\theta}^2}{2} + g(1 + \cos\theta) = E_0,\, \dot{x} = x = 0 \right\} \tag{18}$$

Proof. Let $[q_0, \dot{q}_0]$ be any point in the state space. Consider the solution $[q(t), \dot{q}(t)]$ of the closed loop system with $(q(0), \dot{q}(0)) = (q_0, \dot{q}_0)$. From the assumption $f \in \mathcal{F}_\rho$ and Proposition 1, equation (8) is solvable with respect to f. Then the derivative of V along the solution $q(t)$ takes the form

$$\dot{V}(q(t), \dot{q}(t)) = -\dot{x}(t)\phi(\dot{x}(t)). \tag{19}$$

Therefore, one can conclude that the solution $q(t) \in R^1 \times S^1$ is bounded, has a non-empty ω-limit set Ω_*, and that $V(q(t), \dot{q}(t))$ tends to some constant value as $t \to +\infty$. Indeed, this follows from the facts that V is proper with respect to $R^3 \times S^1$, nonnegative and nonincreasing along the solution $q(t)$. Moreover, the value of $\dot{x}(t)$ tends to zero as $t \to +\infty$. Indeed, due to (19) the value of the integral

$$\int_0^{+\infty} \dot{x}(\tau)\phi(\dot{x}(\tau))d\tau$$

is bounded. Obviously the function $[q(t), \dot{q}(t)]$ is differentiable and its derivative is uniformly bounded. Thus by Barbalat lemma the value of

$$\dot{x}(t)\phi(\dot{x}(t))$$

tends to zero as $t \to +\infty$. This implies that $\dot{x}(t) \to 0$ as $t \to +\infty$.

Thus one can conclude that on the set Ω_*, which is non-empty and consists of the whole trajectories of the closed loop system, the function V is constant, $\dot{x} = 0$ and

$$\begin{gathered} f(q, \dot{q}) \left(k_E \left(E(q, \dot{q}) - E_0 \right) + \frac{k_v}{1 + \sin^2\theta} \right) + \\ \frac{k_v \sin\theta(\dot{\theta}^2 - g\cos\theta)}{1 + \sin^2\theta} + k_x x = -\phi(0) = 0. \end{gathered} \tag{20}$$

The relation $\dot{x} = 0$ immediately implies that x is some constant and $\ddot{x} = 0$ on Ω_*. The last, due to equation (14), is equivalent to

$$f(q, \dot{q}) = \sin\theta \left(g \cdot \cos\theta - \dot{\theta}^2 \right) \tag{21}$$

on Ω_*.

Substituting (21) into (15) one obtains that any motion of the closed loop system subjected to the above mentioned constraints is: x equals to a constant, θ is some trajectory of the unforced pendulum. Indeed, due to (15) one has

$$\begin{aligned} \ddot{\theta} &= \frac{1}{1 + \sin^2\theta} \left[-\dot{\theta}^2 \sin\theta\cos\theta + 2g\sin\theta - f\cos\theta \right] \\ &= \frac{1}{1 + \sin^2\theta} \Big[-\dot{\theta}^2 \cdot \sin\theta \cdot \cos\theta + 2g \cdot \sin\theta \\ &\qquad - \sin\theta \left(g \cdot \cos\theta - \dot{\theta}^2 \right) \cdot \cos\theta \Big] \\ &= g\sin\theta. \end{aligned} \tag{22}$$

Substituting the value (21) of f into (20), one has

$$0 = f(q, \dot{q}) \cdot k_E \cdot (E(q, \dot{q}) - E_0) + k_x \cdot x. \tag{23}$$

Here x is a constant. Moreover, the value of the function $V(q, \dot{q})$ on the set Ω_* is constant. Therefore due to relation (6) and $\dot{x} = 0$, the value of $E(q, \dot{q})$ on the set Ω_* is also constant. The last arguments and relation (23) result in two possible cases.

First, for any closed loop system trajectory $[q_*(t), \dot{q}_*(t)]$ in Ω_*, the constant value of $E(q_*(t), \dot{q}_*(t))$ is equal to E_0. Moreover, by the equality (23) the constant value of x is also zero. Thus, the set Ω_* is the set (18).

Second, if for any trajectory $[q_*(t), \dot{q}_*(t)]$ of the closed loop system in Ω_*, the value of $E(q_*(t), \dot{q}_*(t))$ is not E_0, then due to (23) the value of

$$f(q_*(t), \dot{q}_*(t)) = \sin\theta_*(t) \left(g\cos\theta_*(t) - \dot{\theta}_*^2(t) \right) \tag{24}$$

is constant for all $t \geq 0$. Now it is worth to point out that for any trajectory of system (22), except the downward and upright positions, there exists a moment of time $T_* \geq 0$ such that $\theta_*(T_*) = \pi$. In particular, this means that the constant in (24) is always zero, i. e. for $t \geq 0$

$$f(q_*(t), \dot{q}_*(t)) = \sin\theta_*(t) \left(g\cos\theta_*(t) - \dot{\theta}_*^2(t) \right) = 0. \tag{25}$$

In turn, due to relation (23) and because of the positiveness of k_x, it immediately implies that $x = 0$.

To complete the proof of the part 2) one should determine all the motions of the system (22), which satisfy the constraint (25). The system (22) describes the motions of a simple pendulum and has the conserved quantity

$$H_0(\theta,\dot\theta) = \frac{1}{2}\,\dot\theta^2 + g\cdot(\cos\theta + 1)\,.$$

Suppose that there exists a trajectory $[\theta_*(t), \dot\theta_*(t)]$ of the system (22), which differs from the upright and downward equilibria, $[0,0]$ and $[\pi,0]$, and such that relation (25) is valid for all $t \geq 0$. Thus there exist $\varepsilon > 0$ and a time interval $\mathcal{T} = (T_* - \varepsilon, T_* + \varepsilon)$ such that $\sin\theta_*(t) \neq 0 \;\forall\, t \in \mathcal{T}$. Due to (25) this implies that

$$g\cdot\cos\theta_*(t) - \dot\theta_*^2(t) = 0, \quad \forall\, t \in \mathcal{T}. \tag{26}$$

Then for this special trajectory one has that for $\forall\, t \in \mathcal{T}$

$$\begin{aligned} H_0(\theta_*(0), \dot\theta_*(0)) &= \frac{1}{2}\,\dot\theta_*^2(t) + g\cdot\cos\theta_*(t) + g \\ &= \frac{1}{2}\,\dot\theta_*^2(t) + \dot\theta_*^2(t) + g. \end{aligned} \tag{27}$$

In particular, this implies that $\ddot\theta_*(t) \equiv 0$ for all $t \in \mathcal{T}$. Coming back to equation (22) one concludes that $\sin\theta_*(t) \equiv 0$ on the interval $\mathcal{T}$. This obviously corresponds to only one of the equilibriums of the pendulum (22): $[0,0]$ and $[\pi,0]$. Thus, it is shown that the assumption $E(q_*(t), \dot q_*(t)) \neq E_0$, for any closed loop system trajectory lying in Ω_*, implies that $[q_*(t), \dot q_*(t)]$ is either the equilibrium (16) or (17). ∎

Remark 4 Essentially the proof of Theorem 3 can be divided into two parts. The first part contains standard arguments based on the properties of some appropriate storage function and Barbalat's lemma. To complete the second part of the proof one should analyze *motions of the closed loop system* subjected to some constraints (equalities) with respect to state variables. Such an analysis corresponds to the verification of V-detectability of the system with some suitable output, see definition 2.2 [8]. In our case such an output is $y(q,\dot q) = \dot x$. ∎

Theorem 3 singles out three sets in the state space of the inverted pendulum, which serve as ω-limit sets for the closed loop system solutions. But Theorem 3 does not reflect the global properties of the closed loop system in a sense that it does not say which of these three sets are 'generically' attractive sets, i. e. which of these three sets attracts almost all trajectories of the closed loop system. This problem is discussed in the next statements.

Proposition 5 *Consider the the controlled inverted pendulum (1). For any controller $f \in \mathcal{F}$ the upright equilibrium (17) is hyperbolic. Moreover, the dimension of its stable manifold is 3 and the dimension of its unstable manifold is 1.*

Proposition 6 *Consider the controlled inverted pendulum (1). For any controller $f \in \mathcal{F}_\rho$ the downward equilibrium (16) has at least a 2-dimensional stable manifold. If the parameters of the controller f satisfy the inequality*

$$2g\cdot E_0\cdot k_E > g\cdot k_v + k_x, \tag{28}$$

then the downward equilibrium (16) is a hyperbolic stationary point of the closed loop system. Moreover, the dimension of its stable manifold is 2 and the dimension of its unstable manifold is 2.

These statements make it possible to formulate the next theorem.

Theorem 7 *Consider the controlled inverted pendulum (1). Take any state feedback controller $f \in \mathcal{F}_\rho$ defined by (8). Suppose that the parameters of the controller f satisfy inequality (28). Then the set Γ, defined by (18), is the 'generic' attractive set, i. e. for all initial conditions $[q_0, \dot q_0]$ (except a stable manifolds of the downward and upright equilibria with zero displacement of the cart) the solution $[q(t), \dot q(t)]$ of the closed loop system, starting at this point, tends to the set Γ and satisfies the limit relation*

$$\lim_{t\to+\infty} V(q(t), \dot q(t)) = 0.$$

4 Stabilization of Rotational Modes of an Inverted Pendulum with an Actuator

Here we are going to consider the model of the inverted pendulum, see the equations (1) or (14)–(15), incorporating an actuator dynamics to it

$$\dot f = -f + u. \tag{29}$$

The problem under consideration is to stabilize a rotational behavior of the pendulum, corresponding to the given value E_0 of the total energy, with a prescribed position of the cart. The solution of this problem is based on the combination of the results obtained in the previous section and the back-stepping design methodology, see also [4, Lemma 2.8, p.34], [10].

Given $E_0 \geq 0$, let $\gamma \in \mathcal{F}_\rho$, where ρ is given from (12) and $\mathcal{F}$ is defined by (9). As we agreed before, the last inclusion simply means that γ is uniquely defined by the equation (8), corresponding to some point in $\mathcal{F}_\rho$. One could easily verify that the time derivative of $\gamma(q,\dot q)$ does not depend explicitly on u and the function of $[x, \dot x, \theta, \dot\theta, f]$. Therefore the control law

$$\begin{aligned} u \;=\;& (1-c)f + c\gamma + \dot\gamma - \\ & -\dot x\left(k_E\,(E(q,\dot q) - E_0) + \frac{k_v}{1+\sin^2\theta}\right) \end{aligned} \tag{30}$$

is correctly defined. Here c is any positive constant.

Theorem 8 *Consider the controlled inverted pendulum (14), (15), (29). Take state feedback controller defined by equation (30). Then*

1) any solution of the closed loop system is globally well defined and bounded;

2) for any solution $[q(t), \dot{q}(t)]$ of the closed loop system, its ω-limit set Ω_* consists of either the upright equilibrium

$$[x, \theta, \dot{x}, \dot{\theta}] = [0, 0, 0, 0], \tag{31}$$

the downward equilibrium

$$[x, \theta, \dot{x}, \dot{\theta}] = [0, \pi, 0, 0], \tag{32}$$

or the set

$$\Gamma = \left\{ [x, \theta, \dot{x}, \dot{\theta}] : \frac{1}{2}\dot{\theta}^2 + g(1 + \cos\theta) = E_0, \, \dot{x} = x = 0 \right\}$$

3) If the inequality (28) is valid then equilibrium points (31), (32) are hyperbolic and Γ is a 'generic' attractive set of the closed loop system.

5 Conclusions

This paper is devoted to the solution of relatively new problem for the ubiquitous example, the cart-pendulum system: how to stabilize a given rotational mode of the simple (2-dimensional) unforced pendulum provided the cart tends to a prescribed position. The main difficulty in the problem arises from the fact that desired motion is not a motion of the unforced cart-pendulum system. If the mass of the cart is taken into account then one should somehow compensate the internal dynamics of the cart.

The idea of the solution is based on possibility, *firstly*, to describe a desired rotational mode as a compact subset Γ of the phase space of the inverted pendulum, and, *secondly*, to construct a family of non-negative functions V_μ, which are positive outside Γ and possess a key property: the inverted pendulum with the speed of the cart as the output, could be made passive with any V_μ after some globally defined state feedback transformation. Taking advantage of this property, we suggested the set of stabilizing controllers and performed the qualitative analysis of the behavior of the closed loop system solutions.

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