Global Uniqueness Tests for H^∞ Optima

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Abstract

Optimization of sup norm type performance functions over the space of H^∞ functions is central to the subject of H^∞ design. Problems with a large amount of plant uncertainty are often highly non-convex and therefore may have many solutions. In this article, even for highly non-convex problems, we give a test one can perform, once a local optimum f^* has been computed, to see if it is a global optimum. The uniqueness phenomena we discovered uses H^∞ properties heavily and are considerably stronger than what occurs in other types of general optimization.

One of the least intuitive properties of SISO control is that a (local) optimum for a carefully set up H^∞ problem (cf. [Theorem 9.4.1 HMer98], [HMar90]) even with large amounts of plant uncertainty is unique. Such problems can be quite non -convex so the fact is surprising. While the result is false in general for MIMO control (cf. [HMer94]), in this note we are describing MIMO situations where uniqueness holds.

The setting in this paper is simultaneous (Pareto) optimization of several competing performances $\Gamma_1, \dots, \Gamma_\ell$ and we obtain uniqueness results for its solutions.

Keywords: H^∞ control, Frequency response methods, Uniqueness, Pareto, Optimization, Multiple performances, IQC, QFT.

1 Introduction

Uniqueness results are valuable for assuring an engineer that a local optimum obtained in a computer run is in fact a true global optimum. This can save a practitioner a lot of time and anguish in that it replaces the usual process of initializing an optimization run many times to see if it always goes to the same local optimum; and even after vast numbers of experiments never being sure.

This paper analyzes a problem in which one optimizes performance functions over the space H_N^∞ of bounded analytic vector-valued functions $f = (f_1, \dots, f_N)$ defined on the unit circle, $\mathbf{T}$, where each coordinate function f_j belongs to $L^\infty(\mathbf{T})$ and extends to be analytic on the entire unit disk. Let $C(\mathbf{T})$ be the space of continuous complex-valued functions on the circle and let $C^1(\mathbf{T})$ be those elements in $C(\mathbf{T})$ with continuous first derivatives.

1.1 Definition of Pareto Optimum

The performance criteria we optimize are described in terms of nonnegative continuous functions Γ_j defined on $\mathbf{T} \times \mathbf{C}^N$. We are given positive functions

$$\Gamma_j(e^{i\theta}, z), \quad j = 1, \dots, \ell,$$

for $\ell \leq N$ with $e^{i\theta} \in \mathbf{T}$ and $z \in \mathbf{C}^N$. For function $f \in H_N^\infty$ we define the ℓ performances

$$\gamma_j(f) := \sup_{e^{i\theta} \in \mathbf{T}} \Gamma_j(e^{i\theta}, f(e^{i\theta})), \quad j = 1, \dots, \ell.$$

The goals of this paper are best illustrated by the case of two performance functions Γ_1, Γ_2, even though all results hold for ℓ performance functions.

Definition *A function $f^* \in H_N^\infty$ is called a* Pareto optimum *for Γ_1, Γ_2 if for each $f \in H_N^\infty$ such that $\gamma_1(f) \leq \gamma_1(f^*)$ and $\gamma_2(f) \leq \gamma_2(f^*)$, we must have*

$$\gamma_1(f) = \gamma_1(f^*) \text{ and } \gamma_2(f) = \gamma_2(f^*).$$

By the **MultiOPT** problem we shall mean the problem of finding a Pareto optimum. The definition for more Γ_j's is the obvious analogue.

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1.2 Engineering Motivation

This type of problem is central to frequency domain system design problems where stability is a key constraint. In particular it is important to the area of H^∞-control. The basic physical idea is simple. The following often occurs in a design procedure. We are required to build a system S but part of the system is given (we are stuck with it) and part of the system is designable (denote its frequency response function by f). The performance of the system S at frequency ω is a function $\Gamma(\omega, f(i\omega))$ which depends on ω and on our choice of the designable subsystem f. Let us take the convention that large Γ is bad while small Γ is good. Then in a worst case "broadband" design we consider the worst performance over all frequencies

$$\sup_\omega \Gamma(\omega, f(i\omega))$$

and try to minimize it over all admissible f. If our main constraint is that the designable subsystem f must be stable, then the design problem becomes the MultiOPT problem with only one Γ, after transforming the right half plane to the unit disk. In this paper we deal with the case where f consists of N designable subsystems $f_1, \cdots, f_N$, and where there are ℓ competing performance criteria $\Gamma_1, \Gamma_2, ..., \Gamma_\ell$. Nonconvex performance measures occur in problems with considerable plant uncertainty.

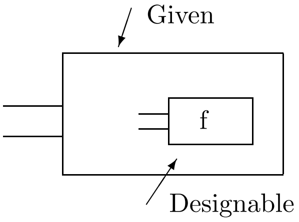

Figure 1: For a given plant we want to find the best designable part, represented by f

A number of authors, Mayne-Polak-et al, Fan-Tits-et al, Streit, Boyd-Barratt, Daleh, Pearson, Balas-Doyle-Glover-Packard-Smith, Helton-Merino and Sideris, have theory and computer programs on searching for an optimal f^* with certain kinds of Γ. The main H^∞ optimization problem of Quantitative Feedback Theory (QFT) is essentially the MultiOPT problem. Also Integral Quadratic Constraints (IQC's; see [MR97]) address such problems but in a different set of co-ordinates (behavioral coordinates).

1.3 Geometric version of the problem

The MultiOPT problem can be stated geometrically in a way which is physically appealing. The sublevel sets

$$\mathcal{S}^j(\gamma_j) := \{(e^{i\theta}, z) \in \mathbf{T} \times \mathbf{C}^N : \Gamma_j(e^{i\theta}, z) \leq \gamma_j\}$$
$$\mathcal{S}^j_\theta(\gamma_j) := \{z \in \mathbf{C}^N : \Gamma_j(e^{i\theta}, z) \leq \gamma_j\}$$

of the performance functions Γ_j correspond to values of the frequency response function where the j^{th} performance measure is better (less) than $\gamma_j \in \mathbf{R}^+$. For fixed $\overrightarrow{\gamma} := (\gamma_1, \cdots, \gamma_\ell)$,

$$\mathcal{S}_\theta(\overrightarrow{\gamma}) := \mathcal{S}^1_\theta(\gamma_1) \cap \ldots \cap \mathcal{S}^\ell_\theta(\gamma_\ell) \qquad \forall e^{i\theta} \in \mathbf{T} \quad (1)$$

is the set of values simultaneously yielding peformance level $(\gamma_1, \cdots, \gamma_\ell)$.

Given target sets $\mathcal{S}_\theta(\overrightarrow{\gamma})$ in $\mathbf{C}^N$, the suboptimal MultiOPT problem is to find a stable system f whose values $f(e^{i\theta})$ lie in the target sets:

$$f(e^{i\theta}) \in \mathcal{S}_\theta(\overrightarrow{\gamma}).$$

Standard Assumption: Assume that each Γ_j is three times differentiable. Assume that sets $\mathcal{S}_\theta(\overrightarrow{\gamma})$ have non-empty interior for each $\overrightarrow{\gamma}$ and are uniformly bounded. We also assume throughout that each of the domains $\mathcal{S}_\theta(\overrightarrow{\gamma})$ is arcwise connected (i.e. none have isolated components) and none of them contain holes.

Clearly the class of $\mathcal{S}(\overrightarrow{\gamma})$ which satisfy the Standard Assumption contains the class of $\mathcal{S}(\overrightarrow{\gamma})$ whose $\mathcal{S}_\theta(\overrightarrow{\gamma})$ are all convex, a class of $\mathcal{S}(\overrightarrow{\gamma})$ upon which most uniqueness theory is based. Our assumption is weak and without it most theory and algorithms of any existing type appear impossible (unless one is in a situation where the holes do not matter and only one component matters.)

1.4 The Gist of the Main Results

It is common for computer optimization algorithms at the k^{th} step to keep track of both the "primal variables" (in our case f^k) and "dual variables". These are called primal-dual algorithms. We shall see in Sections 2 and 3.4 that such an algorithm for MultiOPT which stops in a local optimum f^* with performance levels $\overrightarrow{\gamma}^* := (\gamma_1(f^*), \cdots, \gamma_\ell(f^*))$ produces information with the simple geometric interpretation that we know the primal local optimum f^* and a [1] tangent plane $T^{*\theta}$ to the boundary $\partial\mathcal{S}_\theta(\overrightarrow{\gamma}^*)$ at $f^*(e^{i\theta})$. The tangent plane is a good way to visualize optimal "dual information".

Figure 2 illustrates this situation as well as our uniqueness test when the set $\mathcal{S}_\theta(\overrightarrow{\gamma}^*)$ has smooth boundary.

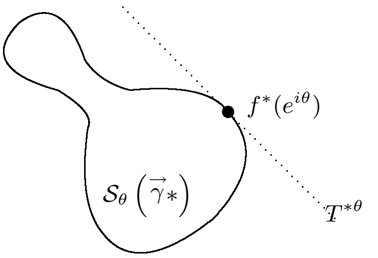

Figure 2: Non-convex but f^* is a unique global optimum.

For each θ, if the tangent plane $T^{*\theta}$ at $f^*(e^{i\theta})$ intersects $\mathcal{S}_\theta(\overrightarrow{\gamma}^*)$ at only the point $f^*(e^{i\theta})$, then f^* is the unique global optimum for MultiOPT.

We emphasize that $\mathcal{S}_\theta(\overrightarrow{\gamma}^*)$ need not be convex; our condition is much less stringent. For comparison recall that a closed strictly convex set C with smooth boundary ∂C has the defining property that at each point z_0 on ∂C the tangent plane T_{z_0} to ∂C at z_0 intersects C only at z_0. Thus to check strict convexity one must check this property at **every point** z_0 on ∂C. Our test for uniqueness requires checking this condition at **only one point**, for each θ. This is a surprising property of optimization over spaces of analytic functions. Uniqueness theorems for analytic function optimization which assume some sort of convexity at all points may be found in [V,W1,W2],[HH86].

The test works in more generality than described here. The sets $\mathcal{S}_\theta(\overrightarrow{\gamma}^*)$ can have corners as would be the case with multi-performance problems. Also, a smaller set than the tangent plane, the "complex tangent plane" suffices in our test.

2 Optimality Conditions and Computation

We begin the detailed description of our global uniqueness test by saying precisely what is meant by primal and dual variables.

2.1 Primal-Dual Optimality Conditions

Recall the optimality conditions for MultiOPT.

Theorem 2.1 *(Theorem 17.7.1 [HMer98], [HV97])*

ASSUME Γ_j *for* $j = 1, \cdots, \ell$ *satisfies the Standard Assumption. SUPPOSE that a continuous local optimum* f^* *does exist with performance values denoted*

$$\gamma_1^*, \cdots, \gamma_\ell^*$$

which makes

$$\frac{\partial\Gamma}{\partial z}(e^{i\theta}, f^*(e^{i\theta}))\frac{\partial\Gamma^T}{\partial z}(e^{i\theta}, f^*(e^{i\theta}))$$

invertible[2] *on all* $e^{i\theta}$. *(This implies* $\ell \leq N$*).*

THEN there exist functions ψ_j *in* $L^1(\mathbf{T})$, *for* $j = 1, \cdots, \ell$, *which satisfy*

Flatness: $\psi_j(e^{i\theta})(\gamma_j^* - \Gamma_j(e^{i\theta}, f^*(e^{i\theta}))) = 0$ *for* $j = 1, \cdots, \ell$.

Gradient Alignment:

$$\begin{aligned}\psi_1(e^{i\theta})\tfrac{\partial\Gamma_1}{\partial\bar{z}}(e^{i\theta}, f^*(e^{i\theta})) + \ldots + \psi_\ell(e^{i\theta})\tfrac{\partial\Gamma_\ell}{\partial\bar{z}}(e^{i\theta}, f^*(e^{i\theta}))\\ = e^{-i\theta}\bar{F}(e^{i\theta}), \qquad F \in H_N^2.\end{aligned}$$

Normalization and Positivity: $\sum_{j=1}^{\ell}\int_0^{2\pi}\psi_j d\theta = 2\pi$, $\gamma_j^* - \Gamma_j(e^{i\theta}, f^*(e^{i\theta})) \geq 0$ *and* $\psi_j(e^{i\theta}) \geq 0$ *for* $j = 1, \cdots, \ell$.

Here A^T denotes the conjugate transpose of$A$ and H_N^2 is vector valued H^2 of the circle.

2.2 Computer optimization

It is common for computer optimization algorithms at the k^{th} step to keep track of both the "primal variables" (in our case f^k) and "dual variables" ψ_j^k and consequently F^k. We start with bad guesses f^0, ψ_j^0 and update them

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[1] When the set $\mathcal{S}_\theta(\overrightarrow{\gamma}^*)$ has smooth boundary (as in problems with a single performance measure Γ^1) the tangent plane $T^{*\theta}$ is unique.

[2] Here

$$\frac{\partial\Gamma}{\partial z} := \begin{pmatrix} \frac{\partial}{\partial z_1}\Gamma_1 & \cdots & \frac{\partial}{\partial z_N}\Gamma_1 \\ \cdots & \cdots & \cdots \\ \frac{\partial}{\partial z_1}\Gamma_\ell & \cdots & \frac{\partial}{\partial z_N}\Gamma_\ell \end{pmatrix} \tag{2}$$

in various ways ultimately to approach a solution to the flatness and gradient alignment equations. At optimum a key property is that F is analytic. These are called primal-dual algorithms and are popular. (See [AHO98], [VB96].) As we shall see in our H^∞ case the dual variable F^k has the interpretation that $e^{-i\theta}\bar{F}^k(e^{i\theta})$ is pointed "normally" to the sets $\mathcal{S}(e^{i\theta})$ at the point $f^k(e^{i\theta})$. Exactly what this means geometrically requires discussion (see Section 3.4) but it motivates calling the optimal dual function $e^{-i\theta}\bar{F}(e^{i\theta})$ the **conjugate analytic normal** at f^*. The point is that many H^∞ optimization algorithms produce both a primal optimum f^* and a conjugate analytic normal $e^{-i\theta}\bar{F}(e^{i\theta})$ to $\mathcal{S}_\theta(e^{i\theta})$ at $f^*(e^{i\theta})$.

3 Checking Global Uniqueness

We now describe our main result in a (high level) algorithmic format. Although a bit redundant it gives a casual reader a description of the method which (except for the most technical hypotheses) is self-contained. The subsequent sections give theorems supplying technical hypotheses and verify that the algorithm works.

Suppose you have run your favorite numerical algorithm for solving the optimization problem MultiOPT in Section 1.1 and that you have obtained $f^*, \vec{\gamma}^*$ and the corresponding dual function F.

Before we describe our test, we need a few definitions. For any integer $N > 0$, the N dimensional complex vector space $\mathbf{C}^N$ has the usual inner product $\langle z, w\rangle_C := \sum_j z_j\bar{w}_j =: z\cdot\bar{w}$, but $\mathbf{C}^N$ can be viewed as a $2N$ dimensional real vector space with the inner product $\langle z, w\rangle_R := \mathrm{Re}\sum_j z_j\bar{w}_j =: \mathrm{Re}\,[z\cdot\bar{w}]$. Define the complex plane which is **complex orthogonal** to the vector $\mathbf{N}$ at location b by

$$\mathbf{N}^{c\perp} := \{z \in \mathbf{C}^N : \bar{\mathbf{N}}\cdot(z-b) = 0\}. \qquad (3)$$

This complex orthogonal is a subset of the ordinary **real orthogonal** complement

$$\mathbf{N}^{r\perp} := \{z \in \mathbf{C}^N : \mathrm{Re}\,[\bar{\mathbf{N}}\cdot(z-b)] = 0\}. \qquad (4)$$

to $\mathbf{N}$ at b.

Now we describe our test for global optimality of f^*.

3.1 The Algorithm

1. Define Γ by

$$\Gamma(e^{i\theta}, z) := \max\{\frac{\Gamma_1(e^{i\theta}, z)}{\gamma_1^*}, \cdots, \frac{\Gamma_\ell(e^{i\theta}, z)}{\gamma_\ell^*}\}. \qquad (5)$$

2. For each θ, compute a linearly independent set of $N-1$ vectors $\phi^1(e^{i\theta}), \phi^2(e^{i\theta}), ..., \phi^{N-1}(e^{i\theta})$ satisfying the equation $\phi^k(e^{i\theta})\cdot F(e^{i\theta}) = 0$ for $k = 1$ to $N-1$. (These $N-1$ vectors form a basis for $\mathbf{N}^{c\perp}$.)

3. For each θ, define $v_\theta(w)$ by

$$v_\theta(w) := \Gamma(e^{i\theta}, f^*(e^{i\theta}) + w_1\phi^1(e^{i\theta}) + \ldots + w_{N-1}\phi^{N-1}(e^{i\theta})),$$

where $w = (w_1, w_2, ..., w_{N-1}) \in \mathbf{C}^{N-1}$.

4. For each θ, compute the minimum of $v_\theta(w)$ for $w \in \mathbf{C}^{N-1}$. If $w = 0$ is the unique "nondegenerate" minimum of $v_\theta(w)$ for every θ, then f^* is the unique optimum (provided the $\mathcal{S}_\theta(\vec{\gamma}^*)$ are "uniformly contractible").

Section 3.3 discusses the technical assumptions "nondegenerate" and "uniformly contractive" mentioned in **Step 4**.

3.2 Remarks on Implementation

Step 2 (this is easy). Given $F(e^{i\theta}) = (F_1(e^{i\theta}), F_2(e^{i\theta}), ..., F_N(e^{i\theta})) \in \mathbf{C}^N$, there are many ways to compute $\phi^k(e^{i\theta})$ for each θ. One way is to select $\phi^1(e^{i\theta}) = (-F_2(e^{i\theta}), F_1(e^{i\theta}), 0, 0, 0, ..., 0)$, $\phi^2(e^{i\theta}) = (-F_3(e^{i\theta}), 0, F_1(e^{i\theta}), 0, 0, 0, ..., 0)$, etc., provided that for that θ, $F_1(e^{i\theta}) \neq 0$. The Gram-Schmidt process may be used to obtain an orthogonal set of $\{\phi^k(e^{i\theta})\}_{k=1}^{N-1}$; this may help with numerics.

Step 4 (this might be costly) The problem is that we must show that $v_\theta(\cdot)$ **a function on** $\mathbf{C}^{N-1}$ has no minimum besides $w = 0$. While this is a great improvement over the infinite dimensional MultiOPT problem, it still may require special structure of Γ and cleverness.

3.3 Technicalities in the Algorithm

The sets $\mathcal{S}_\theta$ are **uniformly contractible**; means there exist mappings $I_t(e^{i\theta}, z)$ from $\mathcal{S}(\vec{\gamma})$ to $\mathcal{S}(\vec{\gamma})$ continuous in t, θ, z such that for each θ, $(e^{i\theta}, z) \mapsto I_t(e^{i\theta}, z)$ is the identity for $t = 0$, and for each θ, $z \mapsto I_t(e^{i\theta}, z)$ is constant when

$t=1$. This is just a "uniform version" of the statement (found in the Standard Assumption) that the sets $\mathcal{S}_\theta$ consist of one component and have no holes. For such a set to contain holes or to be disconnected, one must be working with a highly nonlinear situation. Under these circumstances, even convergence of your computer runs to functions f^*, F can be problematic; possibly it would be worthwhile to reconsider the setup of the original problem (especially your specs).

Nondegenerate minimum is a very technical second order contact assumption. It involves so fine a distinction that we do not think one would worry about it in practice.

The proof of the algorithm is too long for the CDC and can found in a preprint at

math.ucsd.edu/ ∼ helton or
math.ucsd.edu/ ∼ mwhittle.

Section 4 gives one of the ideas of the proof which any engineer might apply to other situations.

3.4 Uniqueness Theorem Expressed Geometrically

All of the results of this paper can be stated geometrically. This way of looking at these optimization problems strongly enhances intuition, and also geometry plays a role in our proofs (see Section 4.) Critical to a geometric understanding are the sublevel sets

$$\mathcal{S}_\theta^j(\gamma_j) := \{z \in \mathbf{C}^N : \Gamma_j(e^{i\theta}, z) \leq \gamma_j\}$$

in $\mathbf{C}^N$ of the performance functions Γ_j. Fix $\overset{\rightarrow}{\gamma}^* := (\gamma_1^*, \cdots, \gamma_\ell^*)$. Let $\partial\mathcal{S}_\theta$ denote the topological boundary of $\mathcal{S}_\theta$. Let $d\partial\mathcal{S}_\theta$ denote

$$d\partial\mathcal{S}_\theta(\gamma^*) := \partial\mathcal{S}_\theta^1(\gamma_1^*) \cap \ldots \cap \partial\mathcal{S}_\theta^\ell(\gamma_\ell^*) \qquad \forall e^{i\theta} \in \mathbf{T} \tag{6}$$

Of course $d\partial\mathcal{S}_\theta \subset \partial\mathcal{S}_\theta$.

Flatness corresponds to the geometric statement $f^*(e^{i\theta}) \in \partial\mathcal{S}_\theta^j$ whenever $\psi_j(e^{i\theta}) \neq 0$.

Step 4 of The Algorithm corresponds to the geometric statement

$$\mathcal{S}_\theta \text{ intersects } (e^{-i\theta}\bar{F}(e^{i\theta}))^{c\perp},$$
a "complex tangent plane" to $\partial\mathcal{S}_\theta$ at $f^*(e^{i\theta})$, only at $f^*(e^{i\theta})$ [3] .

To make these last two statements comprehensible we need some definitions and also we need to justify the statements. Tangent planes to a smooth surface can be defined as the set of points orthogonal to a normal to the surface; there are two notions of orthogonal, real and complex, which lead to two notions of tangent plane, the ordinary tangent plane and the complex tangent plane. We are dealing with surfaces which possibly have corners. Then at a corner there are many normal directions and as a consequence many tangent planes. To get formulas for tangent planes to $\partial\mathcal{S}_\theta$ we need some background.

Consider a once continuously differentiable function ρ from $\mathbf{C}^N$ to $\mathbf{R}^+$. Let $S = \{z : \rho(z) \leq 1\}$ and $\partial S = \{z : \rho(z) = 1\}$ denote its boundary; ρ is called a defining function for ∂S. The surface ∂S is a hypersurface, that is, it has real codimension 1. The gradient $\nabla\rho$ is directed normally to ∂S at z_0 which in complex notation is

$$\nabla\rho(z_0) = \frac{\partial\rho(z_0)}{\partial\bar{z}}$$

Thus if $z_0 \in \partial\mathcal{S}_\theta^j$, then $\frac{\partial\Gamma_j}{\partial\bar{z}}(e^{i\theta}, z_0)$ is directed normally to $\partial\mathcal{S}_\theta^j$. At a corner of $\mathcal{S}_\theta$ there is a family of normals $\mathcal{N}_\theta$ pointing "out" of $\mathcal{S}_\theta$ which we define to be all vectors of the form

$$\{\psi_1(e^{i\theta})\tfrac{\partial\Gamma_1}{\partial\bar{z}}(e^{i\theta}, z_0) + \cdots + \psi_\ell(e^{i\theta})\tfrac{\partial\Gamma_\ell}{\partial\bar{z}}(e^{i\theta}, z_0) : \text{for some } \psi_j \geq 0\} \tag{7}$$

This leads to a formal definition of the **tangent plane** to $\partial\mathcal{S}_\theta$ at $f^*(e^{i\theta})$ as $\mathbf{N}_\theta^{r\perp}$ for some $\mathbf{N}_\theta \in \mathcal{N}_\theta$, and the **complex tangent plane** as $\mathbf{N}_\theta^{c\perp}$ for some $\mathbf{N}_\theta \in \mathcal{N}_\theta$. So far we have used only the complex tangent plane. The **Gradient Alignment** condition says precisely that

$$e^{-i\theta}\bar{F}(e^{i\theta}) \in \mathcal{N}_\theta.$$

Thus the **Gradient Alignment** condition amounts to selecting a normal and corresponding tangent planes to $\mathcal{S}_\theta$ at $f^*(e^{i\theta})$. Note the nature of the corner of $\mathcal{S}_\theta$ is determined by which ψ_j are not 0 (called the active ψ_j).

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[3] Nondegeneracy says that it has second order contact there.

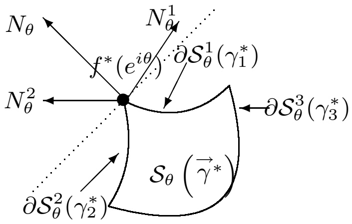

Figure 3: $\mathbf{N}_\theta^j = \frac{\partial\Gamma_j}{\partial\bar{z}}(e^{i\theta}, f^*(e^{i\theta}))$ and $\mathbf{N}_\theta = \psi_1(e^{i\theta})\mathbf{N}_\theta^1 + \psi_2(e^{i\theta})\mathbf{N}_\theta^2 + \psi_3(e^{i\theta})\mathbf{N}_\theta^3$

To prove the geometric interpretation of **Step 4** observe that $\mathcal{S}_\theta(\vec{\gamma}) = \{z : \Gamma(e^{i\theta}, z) \leq 1\}$. Thus the **Step 4** condition

$$\Gamma(e^{i\theta}, z) > \min_z \Gamma(e^{i\theta}, z) = \Gamma(e^{i\theta}, z_0) = 1$$

for $z \neq z_0$ in a set T says that T touches $\mathcal{S}_\theta(\vec{\gamma})$ only at z_0 where $\Gamma(e^{i\theta}, z_0) = 1$. Thus $z_0 = f^*(e^{i\theta})$ being the location of a unique minimum is equivalent to T touching $\mathcal{S}_\theta(\vec{\gamma})$ only at $f^*(e^{i\theta})$.

4 A Taste of the Proof

We prove validity of a less powerful test than that in the Algorithm. It replaces the complex tangent plane $\mathbf{N}_\theta^{c\perp}$, with the larger conventional tangent plane $\mathbf{N}_\theta^{r\perp}$ (see **Step 2**).

Start with primal and dual optima $f^* \in H^\infty$, F and performance levels $\gamma_1^*, \cdots \gamma_\ell^*$. Consider the transformation

$$\pi_\theta(z) := e^{i\theta} F(e^{i\theta}) \cdot (z - f^*(e^{i\theta})).$$

If $z \in \mathcal{S}_\theta$, then the geometric interpretation of the $\mathbf{N}_\theta^{r\perp}$ analog of **Step 4**:

$\mathcal{S}_\theta$ intersects $(e^{-i\theta}\bar{F}(e^{i\theta}))^{r\perp}$,
a "tangent plane" to $\partial\mathcal{S}_\theta$ at $f^*(e^{i\theta})$,
only at $f^*(e^{i\theta})$

says exactly that

$$\operatorname{Re} \pi_\theta(z) \leq 0.$$

Also we transform the sublevel sets $\mathcal{S}_\theta^j(\gamma_j^*)$ using π_θ to obtain sets

$$\tilde{\mathcal{S}}_\theta^j(\gamma_j^*) := \pi_\theta(\mathcal{S}_\theta^j(\gamma_j^*)) \subset \mathbf{C}$$

The $\mathcal{S}_\theta^j(\gamma_j^*)$ have been collapsed to $\mathbf{C}$ in a way which makes $\tilde{\mathcal{S}}_\theta^j(\gamma_j^*)$ contain zero .

Suppose f^{**} is an optimizer in $C(\mathbf{T})$ different from f^*. The mapping

$$\begin{gathered} P : \mathbf{T} \to \mathbf{C} \\ e^{i\theta} \to \pi_\theta(f^{**}(e^{i\theta})) \end{gathered}$$

extends to the analytic function

$$P(s) = sF(s) \cdot (f^{**}(s) - f^*(s))$$

for s on the closed disk and has a zero at the origin. P is not identically 0, since the only point $v \in \mathcal{S}_\theta$ such that $\pi_\theta(v) = 0$ is $f^*(e^{i\theta})$, and for some θ, $f^*(e^{i\theta}) \neq f^{**}(e^{i\theta})$. Thus there is a $\tau > 0$ such that

(P vs. τ) $[0, \tau]$ is in the image of P applied to the unit disk

From $\operatorname{Re} \pi_\theta(z) \leq 0$ for $z \in \mathcal{S}_\theta$, we get $\operatorname{Re} P(e^{i\theta}) \leq 0$. Moreover, $P(s)$ is analytic and bounded for s in the unit disk, since f^*, f^{**}, F are. Thus $\operatorname{Re} P(s) \leq 0$ on the unit disk, and in particular $P(s) \neq \tau$ at any $|s| \leq 1$. This contradicts (P vs τ). ☐

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