STABILITY OF CONES OF POLYNOMIALS. AN APPLICATION TO THE DESIGN OF HIGH-GAIN CONTROLLERS FOR SATURATED SYSTEMS.[1]

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Abstract

In [3], necessary and sufficient conditions for a convex conic set of polynomials to be Hurwitz was given. However, that result is not simple to apply. In this paper, an easy-to-check sufficient condition is introduced. The obtained condition is a matrix inequality which is a simple algebraic test for the stability of rays of polynomials. As an application, for stable open-loop systems, a cone of gains c such that the function $u = -kc^T x$ is a stabilizing control feedback for all $k > 0$, is shown to exist. Moreover, for the same cone of gains, it was established that there do not exist any first harmonic periodic orbits despite saturation.

Key words: Convex cones of Hurwitz polynomials, high-gain controllers, bounded inputs, first harmonic approach.

1 Introduction

The aim of this paper is to derive simple algebraic conditions for the stability of rays of polynomials. Since the Routh-Hurwitz criterion and related results become usually quite complicated when applied to theoretical matters, we have develop here an *ad hoc* approach to the problem of obtaining characterizations of Hurwitz polynomials in terms of the corresponding coefficients. More specifically, we obtain a sufficient condition for a conic set $p_0 + K$ to consist only of stable polynomials. Here p_0 is an $n-$order stable polynomial and K is a convex cone of $(n-1)-$order polynomials. In the framework of [3] this correspond to have infinite robustness of the polynomial p_0 with respect to perturbations in the directions contained in K. In that work the authors present necessary and sufficient conditions for a convex conic set of polynomials to be Hurwitz. However, that result is not simple to apply. Here, in Theorem 1, an easy-to-check sufficient condition is introduced. That condition is the matrix inequality (3) which is a very simple algebraic test.

As an application of the above mentioned result, we address the problem of design high-gain controllers. Define a feedback control u by

$$u = -kc^T x \tag{1}$$

where $c \in \mathrm{R}^n$, and $k > 0$. If $k \gg 1$ the control feedback (1) is known as *high-gain feedback*, since high control gains kc^T are induced. In practical applications, high-gain feedback is commonly used to reduce the effects of bounded disturbance and nonlinearities on system outputs and internal stability. It is well known (see [9, 13]) that when $k \to \infty$ a closed-loop eigenvalue, say λ_1, has the property that $\frac{\lambda_1}{k} \to -c_1$ and, the other ones eigenvalues converge to the roots of the polynomial $c_1 t^{n-1} + c_2 t^{n-2} + ... + c_n$. Consequently, the closed-loop system is asymptotically stable at the origin when k is sufficiently large. However, the origin is not necessarily asymptotically stable for all $k > 0$. In this paper, we will use the matrix inequality (3), in order to find a convex set of gains c such that the control $u = -kc^T x$ is a stabilizing control for all values of $k > 0$. Only open-loop stable systems are considered since the result is not true for unstable ones.

We shall also be concerned with the following control problem. In the last thirty years, the high-gain control design and analysis problems have been addressed using a variety of approaches (see for instance [8, 9, 12, 13]). However, as far as we know, the constrains in the magnitude of the control used to regulate the system, $|u| \leq u_{\max}$, which is an important restriction in a great number of applications, has not been taken into account by the reported investigations. A drawback of high-gain feedback is that it leads to

[1]This research was supported in part by CONACYT Grant 2115-31014.

very large control actions and, the boundedness in the magnitude of the control, limits the achievable stability of the controlled system. In particular, even for open-loop stable systems, when a high-gain feedback is considered, the origin could not be a global attractor (see [10]) and, periodic orbits can be created by the saturation of the controller. We conjecture that under the same conditions on the vector c, matrix inequality (3), there do not exist any periodic orbits despite saturation. However, the proof of the existence or not of periodic orbits is a rather complex problem (there are not systematic tools for the analysis in dimensions greater than two). Thus, we will apply an approximated method, the *first harmonic balance **(FHB)** method*, to analyze the non existence of periodic orbits. Although the **FHB** approach is approximate, it is usually applied for detecting periodic orbits of nonlinear systems [6]. The justification of such approximate methods is that they can give reasonable accurate predictions of the existence or not of periodic orbits, but at a fraction of the effort needed by rigorous methods (e.g. Poincaré maps). In fact, we have shown recently [2] that the **FHB** method predicts reasonably the asymptotic behavior of linear control systems subjected to certain high-gain saturated feedback. In summary, our results can be used, assisted by simulations, as a guide for the design of high-gain control systems with bounded inputs.

2 Main results

The aim of this section is to obtain conditions for the stability of rays of polynomials. The main result is based in the following lemma where sufficient conditions for a real polynomial to be Hurwitz are given.

Lemma 1. Let $F(t)$ and $f(t)$ be real polynomials of degrees n and $n-1$, respectively, such that $f(0) \neq 0$ and the roots of $F(t)$ are contained in C^+. Consider the degree $2n-1$ polynomial given by $F(t)f(t)$. If $F(i\omega)f(i\omega)$ does not intersect the real axis for all $\omega > 0$, then, all the roots of $f(t)$ are in C^-.

Proof: Let l and r be the number of roots of $F(t)f(t)$ contained in C^- and C^+, respectively. Since $F(i\omega)f(i\omega)$ does not intersect the real axis for $\omega > 0$, then $F(t)f(t)$ does not have roots on the imaginary axis.

Let $\theta(\omega)$ be the argument of $F(i\omega)f(i\omega)$. Denote by $\Delta_0^\infty\theta(\omega) = \theta(\infty) - \theta(0)$ to the net change in the argument. It is known that $\Delta_0^\infty\theta(\omega) = \frac{\pi}{2}(l-r)$ [7]. The fact that $F(i\omega)f(i\omega)$ does not intersect the real axis for $\omega > 0$ implies $|\Delta_0^\infty\theta(\omega)| \leq \pi$. On the other hand, we know that at least n roots of $F(t)f(t)$ are in C^+, then it follows that $r \geq n$ and $l \leq n-1$. Hence, $l-r < 0$. Additionally, $l-r$ is an odd number since $l+r = 2n-1$. Thus, the equality $\Delta_0^\infty\theta(\omega) = \frac{\pi}{2}(l-r)$ implies that $\Delta_0^\infty\theta(\omega) = -\frac{\pi}{2}$. Consequently, $l-r=-1$, from where it follows that $r = n$ and $l = n-1$. Finally, the $n-1$ roots of $f(t)$ are contained in C^-, as we wanted to prove. ∎

Given a real polynomial $h(t) = t^n + a_1t^{n-1} + ... + a_n$ define the matrix

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ -a_2 & a_1 & -1 & 0 & \ldots & 0 & 0 \\ a_4 & -a_3 & a_2 & -a_1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & a_{n-2} & -a_{n-3} \\ 0 & 0 & 0 & 0 & \ldots & -a_n & a_{n-1} \end{pmatrix} \quad (2)$$

and let D^i denote the i-th row of matrix D.

Theorem 1. Let $h(t) = t^n + a_1t^{n-1} + ... + a_n$ be a Hurwitz polynomial. Let D be the corresponding matrix defined by (2). If the vector $c = (c_1, c_2, ..., c_n)^T > 0$ is a solution to the system of linear inequalities

$$\boxed{D^i c > 0, i = 1, ..., n,} \quad (3)$$

then, the polynomial $f(t) = \sum_{i=1}^n c_i t^{n-i}$ is Hurwitz.

Proof: We present the proof for n even ($n = 2m$). Let $F(t) = h(-t)$. Then, $F(t)$ is a real polynomial of degree n with all its roots in C^+. Consider the polynomial $F(t)f(t)$, which has degree $2n-1$. Notice that $h(i\omega)$ and $f(i\omega)$ can be written as $h(i\omega) = P(\omega^2) + i\omega Q(\omega^2)$ and $f(i\omega) = p(\omega^2) + i\omega q(\omega^2)$, where $P, Q, p,$ and q are real polynomials. We have $F(i\omega)f(i\omega) = [Pp + \omega^2 Qq] + i\omega(Pq - Qp)$. After some calculations we get

$$(Pq - Qp)(\omega^2) = -\sum_{i=1}^n (D^i c)\omega^{2(n-i)}$$

Since $D^i c > 0$, $i = 1, ..., n$, it follows that $F(i\omega)f(i\omega)$ does not intersect the real axis for all $\omega > 0$. The above lemma implies that all the roots of $f(t)$ are in C^- as we wanted to prove. ∎

We will use this result to find a convex set of gains c such that the control $u = -kc^T x$ is a stabilizing control for all values of $k > 0$. Let us consider the following system

$$\dot{x} = Ax + bu \quad (4)$$

where the pair (A, b) is controllable, $x, b \in \mathrm{R}^n$ and u is a control function. Without loosing generality, we suppose that the pair (A, b) is in the canonical form

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

It is well known that one of the eigenvalues of the closed-loop system $\dot{x} = Ax - kb\bar{c}^T x$, say λ_1, has the property that $\frac{\lambda_1}{k} \to -c_1$ when $k \to \infty$ and, the other ones converge to the roots of the polynomial $f(t) = c_1 t^{n-1} + c_2 t^{n-2} + ... + c_n$ [13]. On the other hand, if $c > 0$ is a solution to (3), it follows from Theorem 1 that the polynomial $f(t)$ is Hurwitz. Consequently, the closed-loop system is asymptotically stable at the origin for k sufficiently large. This shows that the control $u(t) = -kc^T x$ is a high-gain feedback. However, we have a stronger result which is useful for the design of high-gain stabilizing feedbacks.

Theorem 2. Consider the linear system (4) written in the canonical form (5). Suppose A is Hurwitz, that is, the open-loop polynomial $h(t) = t^n + a_1 t^{n-1} + ... + a_n$ is Hurwitz. If $c > 0$ is a solution to (3), then, for all $k > 0$, the control $u(t) = -kc^T x$ is a stabilizing control feedback.

Proof: Suppose n is even (the case when n is odd is analogous). Let $n = 2m$ and $k > 0$. To prove this item, it is enough to see that the closed-loop polynomial is Hurwitz. Let $P_c(t)$ and $P_o(t)$ denote the closed-loop and the open-loop polynomials, respectively. Let p, q, P, Q denote the next polynomials

$$\begin{aligned} p(L) &= c_{2m} - c_{2(m-1)} L + \ldots + (-1)^{m-1} c_2 L^{m-1}, \\ q(L) &= c_{2m-1} - c_{2m-3} L + \ldots + (-1)^{m-1} c_1 L^{m-1}, \\ P(L) &= a_{2m} - a_{2(m-1)} L + \ldots + (-1)^m L^m, \\ Q(L) &= a_{2m-1} - a_{2m-3} L + \ldots + (-1)^{m-1} a_1 L^{m-1}. \end{aligned} \tag{6}$$

Then, it holds that

$$\begin{aligned} P_c(i\omega) &= (P + kp)(\omega^2) + i\omega(Q + kq)(\omega^2), \\ P_o(i\omega) &= P(\omega^2) + i\omega Q(\omega^2). \end{aligned}$$

Consider the polynomial $P_c(t)P_o(-t)$. Thus we get

$$P_c(i\omega)P_o(-i\omega) = P(\omega^2)\left[P(\omega^2) + kp(\omega^2)\right] + \omega^2 Q(\omega^2)$$

$$\left[Q(\omega^2) + kq(\omega^2)\right] + i\omega k\left[P(\omega^2)q(\omega^2) - Q(\omega^2)p(\omega^2)\right].$$

Since $P(\omega^2)q(\omega^2) - Q(\omega^2)p(\omega^2) = -\sum_{i=1}^n (D^i c)\omega^{2(n-i)}$ and the vector $c > 0$ is a solution to the system of the linear inequalities (3), the polynomial $Pq - Qp$ does not have positive roots. That is, $(Pq - Qp)(\omega^2) \neq 0$ for all $\omega > 0$. Consequently, for all $\omega > 0$, $P_c(i\omega)P_o(-i\omega)$ does not intersect the real axis.

Let l_1 be the number of roots of $P_c(t)P_o(-t)$ in C^- and r_1 the number of roots of $P_c(t)P_o(-t)$ in C^+. If $\theta_1(\omega)$ is the argument of $P_c(i\omega)P_o(-i\omega)$, then $\Delta_0^\infty \theta_1(\omega) = \theta_1(\infty) - \theta_1(0)$ denotes the net change in the argument of $P_c(i\omega)P_o(-i\omega)$. Since $P_c(i\omega)P_o(-i\omega)$ does not intersect the real axis if $\omega > 0$, it follows that $P_c(t)P_o(-t)$ does not have roots on the imaginary axis. Therefore $\Delta_0^\infty \theta_1(\omega) = \frac{\pi}{2}(l_1 - r_1)$.

The fact that $P_c(i\omega)P_o(-i\omega)$ does not intersect the real axis if $\omega > 0$ also implies $|\Delta_0^\infty \theta_1(\omega)| \leq \pi$. On the other hand, $P_c(0)P_o(0) = a_{2m}(a_{2m} + kc_{2m})$, which is a positive real number. Hence $\theta_1(0) = 0$.

Now we will analyze $\theta_1(\omega)$ when ω is large. First, we have for large ω that $P_c(i\omega)P_o(-i\omega) \approx \omega^{4m} - ic_1\delta\omega^{4m-1}$. Therefore, $P_c(i\omega)P_o(-i\omega)$ always lies in the 4th. quadrant and $\dfrac{\mathrm{Im}[P_c(i\omega)P_o(-i\omega)]}{\mathrm{Re}[P_c(i\omega)P_o(-i\omega)]} \to 0$ when $\omega \to \infty$. Hence, $\theta_1(\infty) = 2s\pi$, where s is an integer. Then, since $\Delta_0^\infty \theta_1(\omega) = \theta_1(\infty) - \theta_1(0) = 2s\pi$ and $|\Delta_0^\infty \theta_1(\omega)| \leq \pi$, we get that $\Delta_0^\infty \theta_1(\omega) = 0$.

Consequently, the polynomial $P_c(t)P_o(-t)$ has as many roots in C^- as in C^+. Since such polynomial has degree $2n$, then there are n roots in C^+. In fact the roots in C^+ correspond to the roots of $P_o(-t)$ because the open-loop polynomial $P_o(t)$ is a Hurwitz polynomial. Finally, it follows that the n roots in C^- correspond to the roots of $P_c(t)$, which means that $P_c(t)$ is Hurwitz. ∎

Corollary. Given a Hurwitz polynomial $h(t)$ let G be the family of polynomials $f(t) = c_1 t^{n-1} + c_2 t^{n-2} + ... + c_n$ such that $c^T = (c_1, c_2, ..., c_n)^T$ satisfies the inequality (3). We have that for each $f \in G$ and $k \geq 0$, the polynomial $h(t) + kf(t)$ is Hurwitz.

3 The set of solutions of the linear inequalities

In this section we explicitly characterize the set of solutions to the system of inequalities (3) and prove that it is not empty. Given a Hurwitz real polynomial $h(t) = t^n + a_1 t^{n-1} + ... + a_n$, consider the matrix D as defined in (2). Let H be the set of solutions to (3), that is,

$$H = \{c \in \mathrm{R}^n : c > 0 \text{ and } D^i c > 0, \quad i = 1, ..., n\}$$

Since the matrix A is Hurwitz and the matrix D is defined in terms of the elements of A, we will be able to get the whole set of solutions of the linear inequalities using matrices of monotone kind (R is a matrix of monotone kind if $Rz \geq 0$ implies $z \geq 0$.) It is known that a square real matrix R is of monotone kind if and only if there exists R^{-1} and $R^{-1} \geq 0$, where $R^{-1} \geq 0$ means that all its entries are nonnegative (see [5]). It is immediate that the first row of D^{-1} is $\begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}$. We will prove that the elements of D^{-1} (see (2)) are nonnegative.

Lemma 2. Let $E = (e_{ij})$ be the inverse matrix of D. Then $e_{ij} > 0$ for all $i = 2, 3, ..., n$, $j = 1, 2, 3, ..., n$.

Before proving Lemma 2 we need the next result.

Lemma 3. Let $E_j = (e_{ij})_{i=1,...,n}$ be the $j-th$ column of E. Define $g(t) = \sum_{i=1}^{n} e_{ij}t^{n-i}$. Since the matrix E is not singular, there is some e_{ij} different from zero. Choose s such that $e_{n-i,j} = 0$ for all $i = 1, ..., s-1$ and $e_{n-s,j} \neq 0$ (s is the least exponent so that the coefficient of t^s is different from zero). Define $h(t) = \sum_{i=1}^{n-s} e_{ij}t^{n-s-i}$. Then the polynomial $P_o(-t)h(t)$ does not have roots on the imaginary axis.

Proof. It holds that $P_o(-t)g(t) = t^s P_o(-t)h(t)$. Let us consider two cases:

1) s **is even**. Evaluating the polynomial $P_o(-t)g(t)$ in $t = i\omega$ we get $P_o(-i\omega)g(i\omega) = (-1)^{\frac{s}{2}}\omega^s P_o(-i\omega)h(i\omega)$. Equaling the imaginary parts we have

$$\sum_{i=1}^{n} D^i E_j \omega^{2(n-i)} = (-1)^{\frac{s}{2}+1-n}\omega^{s-1}\,\mathrm{Im}[P_o(-i\omega)h(i\omega)].$$

In virtue that $\mathrm{Im}[P_o(-i\omega)h(i\omega)] = i\omega G(\omega^2)$, where G is a polynomial, we have $D^i E_j = 0$ for all $i > n - \frac{s}{2}$. On the other hand, $D^i E_j = 0$ for $i \neq j$ and $D^j E_j = 1$. Consequently, $1 \leq j \leq n - \frac{s}{2}$ and

$$\mathrm{Im}[P_o(-i\omega)h(i\omega)] = (-1)^{n-\frac{s}{2}-1}\omega \sum_{i=1}^{n-\frac{s}{2}} D^i E_j \omega^{2(n-i)-s} = (-1)^{n-\frac{s}{2}-1}\omega\omega^{2(n-j)-s}.$$

Then, if ω is such that $P_o(-i\omega)h(i\omega) = 0$ it follows that $\omega = 0$. But this is a contradiction to the fact $P_o(0)h(0) = a_n e_{n-s,j} \neq 0$. Thus, $[P_o(-i\omega)h(i\omega)]$ does not have roots in the imaginary axis.

2) s **is odd**. We have that $P_o(-i\omega)g(i\omega) = (-1)^{\frac{s-1}{2}} i\omega^s P_o(-i\omega)h(i\omega)$. Equaling the imaginary parts we get $(-1)^{n-1} i\omega \sum_{i=1}^{n} D^i E_j \omega^{2(n-i)} = (-1)^{\frac{s-1}{2}} i\omega^s\,\mathrm{Re}[P_o(-i\omega)h(i\omega)]$. Since $\mathrm{Re}[P_o(-i\omega)h(i\omega)]$ is a polynomial in the variable ω^2, we have $D^i E_j = 0$ for all $i > n - \frac{s-1}{2}$. Besides, we know that $D^i E_j = 0$ for $i \neq j$ and $D^j E_j = 1$. Hence, $1 \leq j \leq n - \frac{s-1}{2}$ and

$$\mathrm{Re}[P_o(-i\omega)h(i\omega)] = (-1)^{n-\frac{s+1}{2}} \sum_{i=1}^{n-\frac{s-1}{2}} D^i E_j \omega^{2(n-i)-s+1}$$

$$= (-1)^{n-[\frac{s+1}{2}]}\omega^{2(n-j)-s+1}.$$

On the other hand, we know that $P_o(0)h(0) = a_n e_{n-s,j} \neq 0$. This implies $2(n-j) - s + 1 = 0$ or, $s = 2(n-j) + 1$. So then, $\mathrm{Re}[P_o(-i\omega)h(i\omega)] = (-1)^{n-\frac{s+1}{2}} = (-1)^{n-[n-j+1]} = (-1)^{j-1} \neq 0$. Therefore, $P_o(-i\omega)h(i\omega) \neq 0$ for all ω. In other words, $P_o(-t)h(t)$ does not have roots in the imaginary axis, as we wanted to prove. ■

Proof to Lemma 2. Let E_j be the $j-th$ column of E and D^i the $i-th$ row of D. Since E is the inverse of D, it follows that

$$\left(D^1E_j, D^2E_j, ..., D^{j-1}E_j, D^jE_j, D^{j+1}E_j, ..., D^nE_j\right) = (0, 0, ..., 0, 1, 0, ...0)$$

On the other hand, let $g(t)$, s and $h(t)$ be as in Lemma A1. Consider the polynomial $P_o(-t)h(t)$. Let l_2 and r_1 be the number of roots of $P_o(-t)h(t)$ in C^- and C^+, respectively. If we evaluate $P_o(-t)h(t)$ in $i\omega$ we can write the complex number $P_o(-i\omega)h(i\omega)$ as $P_o(-i\omega)h(i\omega) = \alpha(\omega^2) + i\omega\beta(\omega^2)$.

Suppose s is even (the case when s is odd is analogous). As in Lemma 3, $\mathrm{Im}[P_o(-i\omega)h(i\omega)] = (-1)^{n-\frac{s}{2}-1}\omega\omega^{2(n-j)-s}$. Consequently, $P_o(-i\omega)h(i\omega)$ does not intersect the x axis for all $\omega > 0$. Denote by $\theta_2(\omega)$ the argument of $P_o(-i\omega)h(i\omega)$. Let $\Delta_0^\infty\theta_2(\omega) = \theta_2(\infty) - \theta_2(0)$, the net change in the argument of $P_o(-i\omega)h(i\omega)$. From Lemma 3 $P_o(-t)h(t)$ does not have roots in the imaginary axis. This implies that $\Delta_0^\infty\theta_2(\omega) = \frac{\pi}{2}(l_2 - r_2)$.

In view that $P_o(-i\omega)h(i\omega)$ does not intersect the x axis for $\omega > 0$ it hods that $|\Delta_0^\infty\theta_2(\omega)| \leq \pi$. On the other hand, $n \leq$ degree of $P_o(-t)h(t) \leq 2n-1$ and since $P_o(-t)$ has n roots in C^+, then $r_2 \geq n$. Hence, $l_2 - r_2 \leq -1$. Thus, $\Delta_0^\infty\theta_2(\omega) \leq -\frac{\pi}{2}$. This, together with the fact that $|\Delta_0^\infty\theta_2(\omega)| \leq \pi$, gives that $\Delta_0^\infty\theta_2(\omega) = -\frac{\pi}{2}$ or $-\pi$. Hence $l_2 - r_2 = -1$ or $l_2 - r_2 = -2$, that is $l_2 = r_2 - 1$ or $l_2 = r_2 - 2$. Since all roots with negative real part of $P_o(-t)h(t)$ are roots of $h(t)$, which is a polynomial of degree $\leq n-1$. Then, $n > l_2$. In virtue that $r_2 \geq n$, it follows $n > l_2 \geq n-1$ or $n > l_2 \geq n-2$. Therefore $l_2 = n-1$ or $l_2 = n-2$. That is, the polynomial $h(t)$ has $n-1$ or $n-2$ roots with negative real part. Consequently, all its coefficients are not zero and have the same sign. Moreover, we get that the polynomial $h(t)$ has degree $n-1$ (which corresponds to the study of the column E_1) or has degree $n-2$ (which corresponds to the study of the columns E_j with $j \geq 2$). Hence the elements of the first column are positive because $e_{11} = 1$ and $e_{ij} \neq 0$ for $j \geq 2$; besides sign $ee_{i_1,j}$ =sign$e_{i_2,j}$ for all $2 \leq i_1, i_2 \leq n,\ j = 2, ..., n$. It also follows that $s = 0$.

Now we will prove that $e_{ij} > 0\ \forall j = 2, ..., n$. Suppose that $n = 2m$ [the case for n odd is analogous]. We have that $P_o(-i\omega)h(i\omega) = \mathrm{Re}[P_o(-i\omega)h(i\omega] + (-1)^{n-1} i\omega\omega^{2(n-j)}$. . Then, we get $\mathrm{Im}[P_o(-i\omega)h(i\omega)] = (-1)^{2m-1}\omega\omega^{2(n-j)+1}$. Consequently, $P_o(-i\omega)h(i\omega)$ is contained in the third or fourth quadrant. On the other hand, $P_o(0)h(0) = a_{2m}e_{2m,j}$. We will suppose that $e_{2m,j} < 0$ and obtain a contradiction.

The fact that $e_{2m,j} < 0$ implies that $\theta_2(0) = \pi$. When $\omega >> 1$, $P_o(-i\omega)h(i\omega) \approx -e_{2,j}\omega^{2(2m-1)} - i\omega^{2(n-j)+1}$. Since $e_{2,j}$ has the same sign to $e_{2m,j}$, then $P_o(-i\omega)h(i\omega)$ is contained in the fourth quadrant for sufficiently large ω. Since $j \geq 2$, then it is satisfied that $2(2m-1) > 2(n-j)+1$ and thus $\dfrac{\mathrm{Im}[P_o(-i\omega)h(i\omega)]}{\mathrm{Re}[P_o(-i\omega)h(i\omega)]} \to 0$ when $\omega \to \infty$. Hence $\theta_2(\infty) = 2\pi$, from where $\Delta_0^\infty \theta_2(\omega) = 2\pi - \pi = \pi$, which is a contradiction since $\Delta_0^\infty \theta_2(\omega) = \frac{\pi}{2}(l_2 - r_2) = \frac{\pi}{2}(n - 2 - n) = -\pi$. Then, it follows that $e_{2m,j} > 0$ and $e_{ij} > 0\ \forall i = 2, 3, ..., n$. ■

We have as an immediately consequence to Lemma 2.

Proposition 1. The matrix D is of monotone kind.

Now, denote by $U = \{z \in R^n \mid z > 0\}$ to the positive orthant. Then the set of solutions to (3) is characterized in the next theorem.

Theorem 3. The set of solutions to the system of linear inequalities (3) can be written as $H = EU$.

Proof. $H \subseteq V$) Let $v \in H$, then $v > 0$ and $Dv > 0$. Consequently, $v = EDv$ with $Dv > 0$. That is $v \in EU$.

$H \supseteq V$) Let $v \in EU$, then $v = Eu$ with $u > 0$. Hence, $Dv = u > 0$. Since the matrix D is a matrix of monotone kind, we finally get that $v > 0$. Therefore, $v \in H$. ■

Remark. From Lemma 2 and the Kuhn-Fourier theorem [4] it follows the existence of a solution to (3), however it is not clear how to obtain the set H by this way.

4 Systems with bounded inputs

Let us consider the following system

$$\dot{x} = Ax + b\mathcal{S}(u) \qquad (7)$$

where the pair (A, b) is controllable, $x, b \in \mathrm{R}^n$, u is a control function and $\mathcal{S}$ is the saturation function

$$\mathcal{S}(v) = \begin{cases} -1 & \text{if} \quad v < -1 \\ v & \text{if} \quad -1 \leq v \leq 1 \\ 1 & \text{if} \quad v > 1 \end{cases}$$

Even for open-loop stable systems (*i.e.* the open-loop matrix A is Hurwitz), there are stabilizing control feedbacks $c^T x$, for which the closed-loop system $\dot{x} = Ax + b\mathcal{S}(c^T x)$ is not globally asymptotically stable [10]. One of the consequences of the saturation is precisely the undesirable existence of equilibrium points or periodic orbits of the closed-loop system. Since the only equilibrium point of system(7)-(1) is the origin (see [10]), the next problem to address is the nonexistence of periodic orbits. In this section we will prove that for any $\overline{c} \in H$ and $k > 0$, the closed-loop saturated system $\dot{x} = Ax + b\mathcal{S}(-k\overline{c}^T x)$ does not have first harmonic periodic orbits.

The first harmonic balance method is based in the following ideas: First, one assume the existence of a periodic orbit of the form $\alpha_0 + a \sin \omega t$, then, substitute it into the equation, and after some approximation (discarding higher-order terms of the Fourier series), determine the coefficients in the solution (see [11]). The obtained solutions are called *first harmonic periodic orbits* (**FHPO's**). For each **FHPO,** it is suspected that the system has a periodic solution near it. In this paper we study the conditions under which there do not exist any **FHPO's** for the saturated system (7)-(1).

To use the **FHB** method for the analysis of the existence of periodic orbits for the system $\dot{x} = Ax + b\mathcal{S}(u)$, first observe that the following differential equation for the command input signal $v(t) = K^T x(t)$ is satisfied:

$$v(t) = -W(p)\mathcal{S}(v(t)), \quad \text{where } p = d/dt, \qquad (8)$$

and $W(p) = K^T(A - pI)^{-1}b$ is the transfer function from the signal $\mathcal{S}(v(t))$ to the signal $v(t)$. Suppose that the differential equation (8) has a periodic solution which can be approximated as a first harmonic solution: $v_0(t) = \alpha_0 + \alpha_1 \sin(\omega t)$, $a, \omega > 0$. Assume that $\mathcal{S}(v_0(t))$ admits a Fourier series. By discarding the higher-order than one terms, and defining $F(\alpha_0, \alpha_1) = \frac{1}{2\pi} \displaystyle\int_0^{2\pi} \mathcal{S}(\alpha_0 + \alpha_1 \sin\theta) d\theta$, $G(\alpha_0, \alpha_1) = \frac{1}{\alpha_1 \pi} \displaystyle\int_0^{2\pi} \mathcal{S}(\alpha_0 + \alpha_1 \sin\theta) \sin\theta d\theta$, the following equations for the parameters α_0, α_1 and the frequency ω are obtained:

$$\alpha_0 + W(0)F(\alpha_0, \alpha_1) = 0; 1 + W(i\omega)G(\alpha_0, \alpha_1) = 0 \quad (9)$$

Theorem 4. If the matrix A is Hurwitz and $c > 0$ is a solution to (3), then the system $\dot{x} = Ax + b\mathcal{S}(kc^T x)$ does not have first harmonic periodic orbits for all $k > 0$.

Proof. Suppose that the system (4) has dimension n even (the case when n is odd is similar), then $n = 2m$ for some m. The function $G(\alpha_0, \alpha_1)$ satisfies $0 \leq G(\alpha_0, \alpha_1) \leq 1$ (see [1]). On the other hand, the second equation of (9) can be written as $1 + W(i\omega) = 1 - \frac{1}{G(\alpha_0,\alpha_1)} \in (-\infty, 0]$. Then, the existence of first harmonic periodic orbits implies that $1 + W(i\omega)$ is a nonpositive real number.

Now consider the polynomials $p(L), q(L), P(L), Q(L)$ defined by (6). Observe that

$$P_c^*(i\omega) = p(\omega^2) + i\omega q(\omega^2),$$

$$P_c(i\omega) = (P + kp)(\omega^2) + i\omega(Q + kq)(\omega^2),$$

$$P_o(i\omega) = P(\omega^2) + i\omega Q(\omega^2).$$

Since $(Pq - Qp)(\omega^2) = (-1)^{n-1} \sum_{i=1}^{n} (D^i c)\omega^{2(n-i)}$ with $D^i c > 0$ for all $i = 1, ..., n$ and taking into account that $1 + W(p) = \frac{P_c(p)}{P_0(p)}$, it follows after some algebraic manipulations that the imaginary part of $1 + W(i\omega)$ is different from zero for all $k > 0$. This implies that the system does not have first harmonic periodic orbits. ■

Example. Consider the following system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathcal{S}\left(kc^T x\right). \quad (10)$$

where $c^T = (-6.5, -11, -5)$. The matrix D and its inverse E are given by

$$D = \begin{pmatrix} 1 & 0 & 0 \\ -11 & 6 & -1 \\ 0 & -6 & 11 \end{pmatrix},$$

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 2.0167 & 0.1833 & 0.0167 \\ 1.1 & 0.1 & 0.1 \end{pmatrix}.$$

Then, the set of solutions to (3) is the following conic set

$H = \{c \in R^3 \mid c_1 = z_1, c_2 = 2.0167z_1 + 0.1833z_2 + 0.0167z_3, c_3 = 1.1z_1 + 0.1z_2 + 0.1z_3$ for $z_1 > 0,\ z_2 > 0,\ z_3 > 0.\}$

Observe that the vector $c = (5, 11, 6.5)^T$ is an element of H. The non existence of periodic orbits of (10) was checked by means of numerical simulations. We found that the trajectories converge to the origin even for large values of k and initial conditions. The simulations evidence permit us to conjecturer that the saturated closed-loop system is globally asymptotically stable for all positive values of k.

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