Disturbance Decoupling in Descriptor Systems via Output Feedback — A Parametric Eigenstructure Assignment Approach

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Abstract — Based on a recently proposed complete parametric approach for eigenstructure assignment in descriptor linear systems via output feedback, disturbance decoupling using output feedback in descriptor linear systems is investigated. Both the dynamical and static disturbance decoupling problems are tackled. Necessary and sufficient conditions for both problems are proposed in terms of the closed-loop eigenvalues and eigenvectors. By arranging these conditions into constraints on the design parameters provided by eigenstructure assignment, the disturbance decoupling problems are converted into eigenstructure assignment problems with extra parameter constraints. The approach guarantees closed-loop regularity, offers certain flexibility and can provide all the degrees of design freedom. An example is investigated to show the effect of the proposed approach.

1. INTRODUCTION

Disturbance decoupling in descriptor linear systems has been studied by a few researchers. Fletcher and Asaraai (1989) were the first to formulate and to solve the problem of disturbance decoupling with respect to continuous singular systems. However, as the problem has been formulated in Fletcher and Asaraai (1989), disturbance decoupling is achieved if, among other conditions, the output is independent of the input disturbance in the sense that there is a set of admissible initial conditions such that the system's response is zero. But since the disturbance input is naturally unknown, it is not clear how, and if at all, a given initial value can be qualified as an admissible initial condition. Banaszuk *et al.* (1990) investigate the problem of disturbance decoupling in descriptor linear systems by employing the concepts of sliding and coasting subspaces, and solve the problem by means of a set of necessary and sufficient conditions for obtaining disturbance decoupling in implicit discrete systems. Lebret (1991) presents structural equivalent characterizations of the solutions of the disturbance decoupling problems for implicit discrete systems. Ailon (1992) employs Van Dooren's recursive algorithm and Kalman's decomposition theorem in studying disturbance decoupling in descriptor linear systems and shows that the solution to the disturbance decoupling problem is obtained almost directly from Van Dooren's recursive algorithm. Ailon (1993) studies the disturbance decoupling problem via analogy with state space systems. In his problem, the influence of the disturbance is decoupled from the output regardless of the system initial values via analogy with conventional state space systems.

All these reported results, Fletcher and Asaraai (1989), Banaszuk *et al*. (1990), Lebret (1991) and Ailon (1992, 1993), are focused on the state feedback case. In this paper, we consider disturbance decoupling in descriptor linear systems via output feedback. Two problems are formulated, one is the (dynamical) disturbance decoupling problem, the other is the static disturbance decoupling problem. The dynamical disturbance problem is formulated to require the following:

- the output feedback control system is regular and possesses, for simplicity, a set of rank(E) distinct finite closed-loop relative eigenvalues, and
- the effect of the disturbance is decoupled from the system output regardless of the system initial values.

The static disturbance decoupling problem is formulated as to satisfy the above first condition in the dynamical disturbance decoupling problem, but with the second condition replaced by a weaker one:

- the effect of a constant disturbance is decoupled from the output feedback in the steady state of the system after sufficient long time.

For both problems, we establish very neat necessary and sufficient conditions in terms of the closed-loop normalized left and right finite eigenvectors and infinite eigenvectors. Then based on the eigenstructure assignment result proposed in Duan (1999), complete parametric approaches for both the dynamical and static disturbance decoupling problems are presented. Duan (1999) presented a complete parametric approach for eigenstructure assignment in descriptor linear systems via output feedback, which establishes the complete parametric expressions of both the feedback gain and the left and right closed-loop eigenvectors in terms of the closed-loop finite eigenvalues and the design parameters $\{f_{ij}^k\}$, $\{g_{ij}^k\}$ and W_∞ or Z_∞. It is shown in the paper that the corresponding disturbance decoupling conditions can be converted into proper constraints on the system design parameters $\{f_{ij}^k\}$, $\{g_{ij}^k\}$ and W_∞ or Z_∞. As a consequence, the disturbance decoupling problems can then be converted into eigenstructure assignment problems with the designing parameters satisfying these constraints besides the basic ones required by the eigenstructure assignment problem.

The proposed approach guarantees the closed-loop regularity. Moreover, since the closed-loop eigenvalues can be set undetermined and treated as a set of design parameters, closed-loop stability can be automatically achieved with the proposed approach.

2. PROBLEM FORMULATION

Consider the following time-invariant linear descriptor system

$$\begin{cases} E\delta x = Ax + Bu + Dd \\ y = Cx \end{cases} \tag{2.1}$$

where δ denotes the differential operator d/dt for continuous-time systems, or the one-step forward operator q (defined by $qx(k) = x(k+1)$) for discrete-time systems; $x \in R^n$, $u \in R^r$, $y \in R^m$ are, respectively, the descriptor-variable vector, the input vector and the output vector; $d \in R^p$ is the disturbance vector; $A, E \in R^{n\times n}$, $B \in R^{n\times r}$, $C \in R^{m\times r}$ and $D \in R^{n\times p}$ are known matrices with $\operatorname{rank}(E) = n_0 \le n$, $\operatorname{rank}(B) = r$, $\operatorname{rank}(C) = m$ and $\operatorname{rank}(D) = p$, and they satisfy the following controllability and observability assumption

Assumption A1:

$$\operatorname{rank}\left([sE - A \;\; B]\right) = \operatorname{rank}\left([sE^T - A^T \;\; C^T]\right) = n \quad \text{for } s \in C$$

When the following output feedback control law

$$u = Ky, \quad K \in R^{r\times m} \tag{2.2}$$

is applied to the system (2.1), the closed-loop system is obtained in the following form

$$E\delta x = A_c x + Dd, \quad A_c = A + BKC \tag{2.3}$$

For the closed-loop system (2.3), the contribution of the disturbance vector d to the system output in the frequency domain is given as follows:

$$y(s) = G_d(s)d(s) \tag{2.4}$$

where

$$G_d(s) = C[sE - (A + BKC)]^{-1}D \tag{2.5}$$

Therefore, the problem of disturbance decoupling in descriptor linear system (2.1) via the output controller (2.2) can be stated as follows.

Problem DDD: *Given system* (2.1), *find* (*if possible*) *an output feedback controller in the form of* (2.2) *such that the following two requirements are met:*

(1) *the closed-loop system* (2.3) *is regular and has* n_0 *number finite distinct eigenvalues* $s_i, i = 1, 2, \cdots, n_0$,

(2) *regardless of the value of* $x(0-) = x_0$, *the disturbance input d has no influence on the system output, that is, the following condition holds:*

$$G_d(s) = C[sE - (A + BKC)]^{-1}D = 0, \forall s \in C \tag{2.6}$$

Problem DDD is referred as dynamical disturbance decoupling since it requires that the effect of an arbitrary disturbance vector d is decoupled from the output for all $t \ge 0$ (the continuous-time system case) or $k \ge 0$ (the discrete-time system case). When the above dynamical disturbance decoupling problem does not have a solution, we can turn to solve the following static decoupling problem, which only requires that the effect of a constant disturbance is decoupled from the output when the system response gets into steady state after sufficient long time.

Problem SDD: *Given system* (2.1), *find* (*if possible*) *an output feedback controller in the form of* (2.2) *such that the first condition in Problem DDD holds together with the following condition*

$$G_d(0) = C(A + BKC)^{-1}D = 0 \quad \text{(continuous-time case)} \tag{2.7}$$

or

$$G_d(1) = C(E - A - BKC)^{-1}D = 0 \quad \text{(discrete-time case)} \tag{2.8}$$

3. AN EIGENSTRUCTURE ASSIGNMENT RESULT

To solve Problem DDD, we will first find the general form for all the controllers in the form of (2.2) that meet the first condition in Problem DDD, and then restrict the degree of freedom in the general form so as to have the disturbance decoupling condition (2.6) or (2.7)/(2.8) satisfied. Using the eigenstructure assignment approach recently proposed by Duan (1999), in this section we will give the general form for all the output feedback controllers which meet the first condition in Problem DDD.

Under Assumption A1, there exist a pair of right coprime polynomial matrices $N(s) \in R^{n\times r}$ and $D(s) \in R^{r\times r}$ and a pair of right coprime polynomial matrices $H(s) \in R^{n\times m}$ and $L(s) \in R^{m\times m}$ satisfying

$$(A - sE)N(s) + BD(s) = 0 \tag{3.1}$$

and

$$(A - sE)^T H(s) + C^T L(s) = 0 \tag{3.2}$$

Let the infinite eigenvalue of the closed-loop system be denoted by s_∞. Then s_∞ is a multiple eigenvalue with both geometric and algebraic multiplicities being equal to $n - n_0$. Therefore, there are $n - n_0$ left and right eigenvectors associated with s_∞. Denote the left and the right eigenvector matrices of the closed-loop system (2.3) by $T_\infty \in C^{(n-n_0)\times n}$ and $V_\infty \in C^{n\times(n-n_0)}$, then by definition,

$$EV_\infty = 0, \quad \operatorname{rank}(V_\infty) = n - n_0 \tag{3.3}$$

and

$$T_\infty^T E = 0, \qquad \mathrm{rank}(T_\infty) = n - n_0 \tag{3.4}$$

For convenience, in the following we will call the left and the right eigenvector matrices T_∞ and V_∞ associated with the infinite closed-loop eigenvalue respectively the *left and right infinite closed-loop eigenvector matrices*. Let T and V be respectively a pair of left and right finite closed-loop eigenvector matrices. Then

$$\tilde{T} = [T \quad T_\infty] \quad \text{and} \quad \tilde{V} = [V \quad V_\infty] \tag{3.5}$$

are respectively the left and the right entire closed-loop eigenvector matrices for the closed-loop system.

Following the main result in Duan (1999) we can obtain the following result which gives the general form for all the output feedback controllers which meet the first condition in Problem DDD.

Lemma 3.1: *Let Assumption A1 be satisfied, and* T_∞ *and* V_∞ *be the infinite left and right closed-loop eigenvector matrices given in* (3.3) *and* (3.4), *respectively. Then all the output feedback controllers in the form of* (2.2) *for the descriptor linear system* (2.1), *which satisfy the first condition in Problem DDD can be parameterized by*

$$K = \left(WV^T + W_\infty V_\infty^T\right)C^T\left[C(VV^T + V_\infty V_\infty^T)C^T\right]^{-1} \tag{3.6}$$

or

$$K = \left[B^T(TT^T + T_\infty T_\infty^T)B)\right]^{-1}B^T\left(TZ^T + T_\infty Z_\infty^T\right) \tag{3.7}$$

with the matrices T, V, W and Z given by

$$V = [v_1 \quad v_2 \quad \cdots \quad v_{n_0}], \quad v_i = N(s_i)f_i \tag{3.8a}$$

$$W = [w_1 \quad w_2 \quad \cdots \quad w_{n_0}], \quad w_i = D(s_i)f_i \tag{3.8b}$$

$$T = [t_1 \quad t_2 \quad \cdots \quad t_{n_0}], \quad t_i = H(s_i)g_i \tag{3.9a}$$

$$Z = [z_1 \quad z_2 \quad \cdots \quad z_{n_0}], \quad t_i = L(s_i)g_i \tag{3.9b}$$

where W_∞, Z_∞, f_i *and* g_i, $i = 1,2,\cdots,n_0$, *are the design parameters satisfying the following constraints:*

C1: $f_i = \bar{f}_l \,, g_i = \bar{g}_l$ if $s_i = \bar{s}_l$

C2: $g_i^T H^T(s_i)EN(s_j)f_j = \delta_{ij}, \quad i,j = 1,2,\cdots,n_0$

C3: $\begin{cases} g_i^T H^T(s_i)Bw_j^\infty = g_i^T L^T(s_i)Cv_j^\infty \\ (t_j^\infty)^T BD(s_i)f_i = (z_j^\infty)^T CN(s_i)f_i \\ \quad i = 1,2,\cdots,n_0, \;\; j = 1,2,\cdots,n-n_0 \end{cases}$

C4: $(t_k^\infty)^T Bw_j^\infty = (z_k^\infty)^T Cv_j^\infty, \quad j,k = 1,2,\cdots,n-n_0$

C5: $\det(T_\infty^T AV_\infty + T_\infty^T BW_\infty) \neq 0$

or $\det(T_\infty^T AV_\infty + Z_\infty^T CV_\infty) \neq 0$

where δ_{ij} *represents the Kronecker function, and* t_i^∞ *and* v_i^∞ *are the columns of matrices* T_∞ *and* V_∞, *respectively. Moreover, the matrices* T *and* V *given above are a pair of normalized left and right finite eigenvector matrices for the closed-loop system.*

The following lemma further gives the response of the closed-loop system (2.3) resulted in by the output feedback control law given in Lemma 3.1 (proof omitted).

Lemma 3.2: *Let the gain matrix* K *be given by Lemma 3.1, and denote*

$$\Lambda = \mathrm{diag}[\, s_1 \quad s_2 \quad \cdots \quad s_{n_0}\,] \tag{3.10}$$

Then, under the relation $x = \tilde{V}\tilde{x}$ *or* $\tilde{x} = \tilde{V}^{-1}x$, *the closed-loop system* (2.3), *with* $u(t) = 0$, *can be equivalently transformed into the following canonical form*

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\delta\tilde{x} = \begin{bmatrix} \Lambda & 0 \\ 0 & \Sigma_\infty \end{bmatrix}\tilde{x} + \begin{bmatrix} T^T \\ T_\infty^T \end{bmatrix}Dd \tag{3.11}$$

which can be divided, by letting

$$\tilde{x} = \begin{bmatrix} \tilde{x}_d \\ \tilde{x}_\infty \end{bmatrix}, \quad \tilde{x}_d \in R^{n_0}, \quad \tilde{x}_\infty \in R^{n-n_0} \tag{3.12}$$

into the normal dynamical part

$$\delta\tilde{x}_d = \Lambda\tilde{x}_d + T^T Dd \tag{3.13}$$

and the non-dynamical part

$$\tilde{x}_\infty = -\Sigma_\infty^{-1}T_\infty^T Dd \tag{3.14}$$

where in (3.11) *and* (3.14), *the matrix* Σ_∞ *is given by*

$$\Sigma_\infty = T_\infty^T(A + BKC)V_\infty \tag{3.15}$$

which also has the expression

$$\Sigma_\infty = T_\infty^T AV_\infty + T_\infty^T BW_\infty \text{ or } \Sigma_\infty = T_\infty^T AV_\infty + Z_\infty^T CV_\infty \tag{3.16}$$

Remark 3.1: For solutions of the right coprime polynomial matrices $N(s)$ and $D(s)$ satisfying (3.1) (or $H(s)$ and $L(s)$ satisfying (3.2)), several ways have been given in Duan (1996) under the controllability condition of the open-loop system (2.1). General computational methods for such right coprime polynomial matrices can also be found in Beelen and Veltkamp (1987), Bongers and Heuberger (1990), Datta and Gangopadhyay (1991) and (Duan and Nichols 1998).

4. SOLUTION TO PROBLEMS DDD AND SDD

This section presents the solutions to Problems DDD and SDD based on Lemmas 3.1 and 3.2.

4.1. Dynamical disturbance decoupling

The following theorem gives the conditions, in terms of the closed-loop eigenvector matrices, for the second requirement in Problem DDD (proof omitted)

Theorem 4.1: *Let Assumption A1 be satisfied, and K be given as in Lemma 3.1. Then*

(1) *The transfer function* $G_d(s)$ *has the following form*

$$G_d(s) = CV(sI - \Lambda)^{-1}T^T D - CV_\infty\Sigma_\infty^{-1}T_\infty^T D \tag{4.1}$$

(2) $G_d(s) \equiv 0, \; \forall s \in C$, *if and only if*

$$R_i = Cv_i t_i^T D = 0, \quad i = 1,2,\cdots,n_0 \tag{4.2}$$

and

$$R_\infty = CV_\infty\Sigma_\infty^{-1}T_\infty^T D = 0 \tag{4.3}$$

Using the parametric forms in (3.8) and (3.9) for the closed-loop eigenvectors, condition (4.2) can be converted into the following constraint on the parameters $\{f_i\}$ and $\{g_i\}$:

C6: $CN(s_i)f_ig_i^TH^T(s_i)D=0,\ \ i=1,2,\cdots,n_0$

Using the two parametric forms in (3.16) for the matrix Σ_∞, condition (4.3) can be converted into the following constraint on the parameter matrix W_∞ or Z_∞:

$$\textbf{C7:}\ \begin{cases} CV_\infty(T_\infty^TAV_\infty+T_\infty^TAW_\infty)^{-1}T_\infty^TD=0 \\ \text{or} \\ CV_\infty(T_\infty^TAV_\infty+Z_\infty^TAV_\infty)^{-1}T_\infty^TD=0 \end{cases}$$

Based on the above reasoning, and using Lemmas 3.1 and 4.1, we now have the following Theorem for solution to Problem DDD.

Theorem 4.2: *Let Assumption A1 be satisfied. Then*

(1) *Problem DDD has a solution if and only if there exist the design parameter vectors* f_i *and* g_i, $i=1,2,\cdots,n_0$, *and the parameter matrices* W_∞ *or* Z_∞, *such that Constraints C1-C7 are met.*

(2) *When the above condition is met, the solution to Problem DDD is given by* (3.6) *or* (3.7), *with the matrices V, W and T, Z given by* (3.8) *and* (3.9), *and the design parameters* f_i, g_i, $i=1,2,\cdots,n_0$, W_∞ *and* Z_∞ *satisfying Constraints C1-C7.*

Based on the above Theorem 4.2, an algorithm for solution to Problem DDD can be given as follows.

Algorithm DD:

1. solve the two pairs of matrix polynomials $N(s)$ and $D(s)$, $H(s)$ and $L(s)$ satisfying the two equations (3.1) and (3.2);
2. solve the common open-loop and closed-loop left and right eigenvector matrices T_∞ and V_∞ associated with the infinite eigenvalues of the system defined by the equations in (2.13) and (2.14);
3. find parameters $\{f_i\},\{g_i\}, W_\infty$ and Z_∞ satisfying Constraints C1-C7. If such parameters do not exist, the disturbance decoupling problem for this system has no solution;
4. calculate matrices *T* and *V*, or *W* and *Z*, according to formulae (3.8) or (3.9) based on the parameter vectors $\{f_i\}$ or $\{g_i\}$ obtained in the step 3.
5. calculate the feedback gain matrix K by formula (3.6) or (3.7) based on the matrices *T* and *V*, or *W* and *Z*, obtained in step 4 and the parameter matrix W_∞ or Z_∞ obtained in step 3.

In the case of $n_0=\operatorname{rank}(E)=n$, condition (4.3) vanishes, and the parameter matrices W_∞ and Z_∞ do not exist. It follows from the above Theorem 4.2 and the Corollary 1 in Duan (1996) that the following corollary holds.

Corollary 4.1: *Let Assumption A1 be satisfied, and* $n_0=\operatorname{rank}(E)=n$. *Then*

(1) *Problem DDD has a solution if and only if there exist the design parameter vectors* f_i *and* g_i, $i=1,2,\cdots,n_0$, *such that Constraints C1-C2 and C6 are met.*

(2) *When the above condition is met, the solution to Problem DDD is given by*

$$K=W(CV)^T[(CV)(CV)^T]^{-1} \tag{4.7}$$

or

$$K=[(T^TB)^T(T^TB)]^{-1}(T^TB)^TZ^T \tag{4.8}$$

with the matrices V, W and T, Z given by (3.8) *and* (3.9), *and the design parameters* f_i *and* g_i, $i=1,2,\cdots,n_0$, *satisfying Constraints C1-C2 and C6.*

4.2. *Static disturbance decoupling*

4.2.1 *The Continuous-time case*

It follows from (4.6) that (2.7) holds if and only if

$$\sum_{i=1}^{n_0}\frac{Cv_it_i^TD}{s_i}+CV_\infty\Sigma_\infty^{-1}T_\infty^TD=0 \tag{4.9}$$

Further, with the help of the parametric forms for the closed-loop eigenvectors given in (3.8) and (3.9) and the parametric forms in (3.16) for the matrix Σ_∞, condition (4.9) can be converted into the following constraint

C8$_\mathbf{c}$:

$$\begin{cases} \displaystyle\sum_{i=1}^{n_0}\frac{CN(s_i)f_ig_i^TH^T(s_i)D}{s_i}+CV_\infty(T_\infty^TAV_\infty+T_\infty^TAW_\infty)^{-1}T_\infty^TD=0 \\ \text{or} \\ \displaystyle\sum_{i=1}^{n_0}\frac{CN(s_i)f_ig_i^TH^T(s_i)D}{s_i}+CV_\infty(T_\infty^TAV_\infty+Z_\infty^TAV_\infty)^{-1}T_\infty^TD=0 \end{cases}$$

Therefore, to solve the static disturbance decoupling problem for a continuous-time system, we need only to replace Constraints C5 and C6 required in the solution to Problem DDD by Constraint C8$_\text{c}$. In the special case of $n_0=\operatorname{rank}(E)=n$, this constraint becomes

$$\textbf{C'8}_\mathbf{c}\textbf{:}\ \sum_{i=1}^{n}\frac{CN(s_i)f_ig_i^TH^T(s_i)D}{s_i}=0$$

4.2. *The discrete-time case*

It follows from (4.6) that (2.8) holds if and only if

$$\sum_{i=1}^{n_0}\frac{Cv_it_i^TD}{1-s_i}-CV_\infty\Sigma_\infty^{-1}T_\infty^TD=0 \tag{4.10}$$

Further, with the help of the parametric forms for the closed-loop eigenvectors given in (3.8) and (3.9) and the parametric forms in (3.16) for the matrix Σ_∞, condition (4.10) can be converted into the following constraint

C8$_\mathbf{d}$:

$$\begin{cases} \displaystyle\sum_{i=1}^{n_0}\frac{CN(s_i)f_ig_i^TH^T(s_i)D}{1-s_i}-CV_\infty(T_\infty^TAV_\infty+T_\infty^TAW_\infty)^{-1}T_\infty^TD=0 \\ \text{or} \\ \displaystyle\sum_{i=1}^{n_0}\frac{CN(s_i)f_ig_i^TH^T(s_i)D}{1-s_i}-CV_\infty(T_\infty^TAV_\infty+Z_\infty^TAV_\infty)^{-1}T_\infty^TD=0 \end{cases}$$

Therefore, to solve the static disturbance decoupling problem for a discrete-time linear system, we need only to replace Constraints C5 and C6 required in the solution to Problem DDD by Constraint C8$_d$. In the special case of $n_0 = \mathrm{rank}(E) = n$, this constraint becomes

C'8$_d$: $\displaystyle\sum_{i=1}^{n} \frac{CN(s_i) f_i g_i^T H^T(s_i) D}{1 - s_i} = 0$

5. AN ILLUSTRATIVE EXAMPLE

Consider a system in the form of (2.1) with the following coefficient matrices (Fletcher 1988, Duan 1995 and Duan 1999):

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is easy to verify that with this system Assumption A1 holds. Moreover, note that the left and right infinite eigenvector matrices T_∞ and V_∞ defined by (3.3) and (3.4) are

$$V_\infty = T_\infty = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$$

it can be verified that condition (3.19) holds. Therefore, this system is both impulse controllable and impulse observable.

5.1. *Eigenstructure assignment*

Restrict the closed-loop eigenvalues s_i, $i = 1,2,3$ to be distinct and real, and denote

$$f_i = f_{i1}^1 = \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \\ \alpha_{i3} \end{bmatrix}, \ g_i = g_{i1}^1 = \begin{bmatrix} \beta_{i1} \\ \beta_{i2} \end{bmatrix}, \quad i = 1,2,3$$

and

$$W_\infty = w_1^\infty = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \ Z_\infty = z_1^\infty = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{5.1}$$

then the matrices *T*, *V*, *W* and *Z* are given as follows

$$T = \begin{bmatrix} \beta_{11} & \beta_{21} & \beta_{31} \\ s_1\beta_{11} & s_2\beta_{21} & s_3\beta_{31} \\ \beta_{12} & \beta_{22} & \beta_{32} \\ s_1\beta_{12} - \beta_{11} & s_2\beta_{22} - \beta_{21} & s_3\beta_{32} - \beta_{31} \end{bmatrix} \tag{5.2}$$

$$Z = \begin{bmatrix} s_1^2\beta_{11} - \beta_{12} & s_2^2\beta_{21} - \beta_{22} & s_3^2\beta_{31} - \beta_{32} \\ \beta_{12} & -\beta_{22} & -\beta_{32} \end{bmatrix} \tag{5.3}$$

$$V = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \\ \varphi_1 & \varphi_2 & \varphi_3 \end{bmatrix} \tag{5.4}$$

$$W = \begin{bmatrix} s_1\alpha_{11} - \alpha_{13} & s_2\alpha_{21} - \alpha_{23} & s_3\alpha_{31} - \alpha_{33} \\ \psi_1 & \psi_2 & \psi_3 \\ -\alpha_{13} & -\alpha_{23} & -\alpha_{33} \end{bmatrix} \tag{5.5}$$

with

$$\varphi_i = (-\alpha_{i1} + \alpha_{i2} + \alpha_{i3})s_i - \alpha_{i1} - \alpha_{i2} + 3\alpha_{i3} \tag{5.6}$$

$$\psi_i = (s_i + 1)\alpha_{i1} - s_i\alpha_{i2} - 3\alpha_{i3} \tag{5.7}$$

Since all the finite closed-loop eigenvalues are real, we can also restrict the parameters α_{ij}'s and β_{ij}'s to be real. Therefore, Constraint C1 holds automatically, while Constraints C2~C5 give the following set of equations:

$$\xi_3 = \eta_2 \tag{5.8}$$

$$\xi_3 = \eta_2 \neq 0 \tag{5.9}$$

$$\alpha_{i2}\eta_1 + \varphi_i\eta_2 = -\alpha_{i3}, i = 1,2,3 \tag{5.10}$$

$$(\alpha_{j1} + s_i\alpha_{j2})\beta_{i1} + \alpha_{j3}\beta_{i2} = \delta_{ij} \quad i, j = 1,2,3 \tag{5.11}$$

$$\beta_{i1}(s_i + 1)\xi_1 + (\beta_{i2} - s_i\beta_{i1})\xi_2 = \rho_i, i = 1,2,3 \tag{5.12}$$

where

$$\rho_i = -[(2s_i - 1)\beta_{i1} + s_i\beta_{i2}]\eta_2 - \beta_{i2}, i = 1,2,3 \tag{5.13}$$

For a thorough treatment of this set of constraints, refer to the subsections 4.1 and 4.2 in Duan (1999). Particularly, it is shown in Duan (1999) that (5.11) is satisfied if and only if

$$\Delta_1 = \det \begin{bmatrix} \beta_{11} & s_1\beta_{11} & \beta_{12} \\ \beta_{21} & s_2\beta_{21} & \beta_{22} \\ \beta_{31} & s_3\beta_{31} & \beta_{32} \end{bmatrix} \neq 0 \tag{5.14}$$

and in this case, there holds

$$\begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix} = \begin{bmatrix} \beta_{11} & s_1\beta_{11} & \beta_{12} \\ \beta_{21} & s_2\beta_{21} & \beta_{22} \\ \beta_{31} & s_3\beta_{31} & \beta_{32} \end{bmatrix}^{-1} \tag{5.15}$$

When the closed-loop finite eigenvalues are chosen as $s_1 = -1$, $s_2 = -2$, $s_3 = -3$, under the restriction of $\beta_{21} = 0$, a special class of solutions are obtained. In this case the gain matrix is given by

$$K = \frac{1}{\beta_{12} - 1} \begin{bmatrix} 5 - 6\beta_{12} - \beta_{32} & 1 - \frac{3}{2}\beta_{12} - \frac{1}{2}\beta_{32}(1 + 2\beta_{12}) \\ 5 - \beta_{12} & 3 - \beta_{12} \\ 2 & 1 \end{bmatrix}, \ \beta_{12} \neq 1 \tag{5.16}$$

where β_{32} is an arbitrary real scalar.

5.2. *Disturbance decoupling*

This subsection considers the disturbance decoupling in this example system with

$$D = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

5.2.1. *Dynamical disturbance decoupling*

To solve the dynamical disturbance decoupling problem in this example system, let us first investigate the disturbance decoupling condition represented by Constraints C6 and C7.

It is clear that Constraint C7 holds if only $d_4 = 0$. Therefore, in the following we take $d_4 = 0$. Note

$$CN(s_i)f_ig_i^TH^T(s_i)D=\begin{bmatrix}[(d_1+s_id_2)\beta_{i1}+d_3\beta_{i2}]\alpha_{i2}\\ [(d_1+s_id_2)\beta_{i1}+d_3\beta_{i2}]\varphi_i\end{bmatrix}$$

where $\varphi_i, i=1,2,3$ are given by (5.6), Constraint C6 is

$$\begin{cases}[(d_1+s_id_2)\beta_{i1}+d_3\beta_{i2}]\alpha_{i2}=0\\ [(d_1+s_id_2)\beta_{i1}+d_3\beta_{i2}]\varphi_i=0\end{cases},\quad i=1,2,3 \qquad (5.17)$$

For solution of the above equations in (5.17), we have the following fact (proof ommited).

Fact 5.1: *For this example system, there does not simultaneously exist*

1. *distinct real numbers* $s_i,\ i=1,2,3$ *in* $(-\infty 1)$*;*
2. *real parameters* $\alpha_{ij},\ i,j=1,2,3$*;* $\beta_{ij},\ i=1,2,3,\ j=1,2$*,*
3. *real parameters* $\xi_i,\ i=1,2,3$ *and* $\eta_i,\ i=1,2$ *, and*
4. *real parameters* $d_i,\ i=1,2,3$ *which are not all zero,*

such that Constraints C2 (*equation* (*5.2*)), *C3* (*equation* (*5.10*) *and* (*5.12*)) *and C6* (*equation* (5.17)) *are simultaneously satisfied.*

The above fact tells us that for this system, there does not exist a nonzero matrix D of proper dimensions, distinct eigenvalues $s_i, i=1,2,3$ in $(-\infty 1)$, parameters $\alpha_{ij},\ i,j=1,2,3$; $\beta_{ij},\ i=1,2,3,\ j=1,2$; $\xi_i, i=1,2,3$ and $\eta_i, i=1,2$, satisfying Constraints C1~C7. Therefore, according to Theorem 4.2, we have the conclusion that, for arbitrary distinct eigenvalues $s_i,\ i=1,2,3$ in $(-\infty\ 1)$, and arbitrary nonzero matrix D of proper dimensions, the disturbance decoupling problem for this example system does not have a solution.

5.2.2. *Static disturbance decoupling*

In the following, let us treat this system as a continuous-time one, and turn to consider the static disturbance decoupling in this system with the closed-loop finite eigenvalues $s_i,\ i=1,2,3$ assigned to $-1,-2$ and -3, and the disturbance distribution matrix given by

$$D=\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}^T \qquad (5.20)$$

In this case, the disturbance decoupling condition, that is, Constraint C8$_c$ becomes

$$\sum_{i=1}^{3}\beta_{i1}\alpha_{i2}=0 \quad \text{and} \quad \sum_{i=1}^{3}\beta_{i1}\varphi_i=0 \qquad (5.21)$$

It can be observed that the above first condition holds automatically following the relation in (5.15). As for the second one, we point out that it is a very week condition and there indeed exist parameters which satisfy this condition together with the other necessary ones. In fact, it may be observed that this condition is satisfied with the whole class of special solutions given in (5.16),.which corresponds to the following choices of parameters

$$\beta_{11}=\beta_{22}=\beta_{31}=1,\ \beta_{21}=0$$

$$\varphi_1=-1,\ \varphi_2=1-\beta_{32},\ \varphi_3=1$$

Therefore, a group of solutions to this static disturbance decoupling problem are characterized by (5.16) with $\beta_{12}\neq 1$ and β_{32} arbitrary. Specially choosing $\beta_{12}=-1,\ \beta_{32}=1$ and $\beta_{12}=\beta_{32}=0$ gives the following two specific solutions:

$$K_1=\begin{bmatrix}-5 & -1.\\ -3. & -2.\\ -1. & -0.5\end{bmatrix} \quad \text{and} \quad K_2=\begin{bmatrix}-5 & -1\\ -5 & -3\\ -2 & -1\end{bmatrix}$$

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