

# Constrained output regulation of linear plants

Jian Han and Ali Saberi  
School of Electrical Engineering  
and Computer Science  
Washington State University  
Pullman, WA 99164-2752  
jhan@eecs.wsu.edu  
saber@eecs.wsu.edu

Anton A. Stoorvogel  
Department of Mathematics  
and Computing Science  
Eindhoven Univ. of Technology  
P.O. Box 513, 5600 MB Eindhoven  
The Netherlands  
A.A.Stoorvogel@tue.nl

Peddapullaiah Sannuti  
Department of Electrical  
and Computer Engineering  
Rutgers University  
94 Brett Road, Piscataway,  
NJ 08854-8058, USA  
sannuti@ece.rutgers.edu

**Abstract.** Output regulation of linear systems with state and/or input magnitude constraints is considered. The structural properties of linear plants are identified under which the so called constrained semi-global and global output regulation problems are solvable. An important aspect of our development is a categorization of constraints to show clearly for what type of constraints what can or cannot be achieved.

## 1 Introduction

One of the foremost reasons a control system is sought for is to achieve *output regulation* under the presence of persistent disturbances. Because of its profound significance, achieving *output regulation* has been studied by several researchers for many years starting with the pioneering work of Davison, Francis, Wonham, Desoer and their coworkers [1–5]. A recent book [9] formulates and studies a number of problems on different facets of *output regulation* of linear systems with or without input constraints. Input constraints arise because of magnitude and rate saturation of actuators. Besides actuator saturation, other constraints such as magnitude and rate constraints on state variables manifest themselves often. The focus of this paper is on *output regulation* of linear systems with state and/or input magnitude constraints. Our emphasis is on fundamental issues of identifying the structural properties of linear plants under which the so called constrained semi-global and global output regulation problems are solvable. Whenever the required structural properties are satisfied, design methodologies for constrained semi-global and global output regulation follow from the constructive methods of proving the obtained results. Whenever the required structural properties are not satisfied, we provide in [7] analysis and design methods to achieve tracking a reference signal (but not disturbance rejection) in a regional sense. These aspects of our work distinguish us from the other works dealing with state and input constraints. No proofs of stated results are given in this conference version of the paper. (see [7] for details).

## 2 Formulation of Problems

Consider a dynamic system,

$$\begin{aligned}\dot{x} &= Ax + Bu + Ew \\ \dot{w} &= Sw \\ y &= C_y x + D_y w \\ e &= C_e x + D_e w \\ z &= C_z x + D_z u\end{aligned}\tag{2.1}$$

where the first equation describes a plant with state  $x \in \mathbb{R}^n$  and input  $u \in \mathbb{R}^m$  subject to the effect of a *disturbance* represented by  $Ew$ . The third equation defines the measurement output  $y \in \mathbb{R}^l$ . The fourth equation defines the error  $e \in \mathbb{R}^r$  between the actual controlled plant output  $z_c = C_e x$  and a *reference* signal  $-D_e w$  which the plant output is required to track. The second equation describes the *exosystem* with state  $w \in \mathbb{R}^s$ . The exosystem models the class of reference signals and/or exogenous disturbances. All the imposed magnitude constraints on the plant manifest themselves in the constrained output  $z \in \mathbb{R}^p$  as shown in the last equation. The constrained output  $z \in \mathbb{R}^p$  is required to remain in a bounded set. Here we assume that the constraints on the input  $u$  are independent of the value of the state  $x$  and conversely. In other words, we have certain constraints involving state variables  $x$  and certain other constraints involving input variables  $u$ . However, we do not have constraints that involve a mixture of state and input variables. This is partially reflected in our assumption that  $D_z^T C_z = 0$ . Such an assumption is reasonable and is made only for ease of presentation. Finally, throughout this paper without loss of generality we assume that  $(C_z \ D_z)$  is surjective. Our objective is to achieve internal stability and output regulation under constraints on the output  $z$ . As usual, internal stability means that, if we disconnect the exosystem and set  $w$  equal to 0, the closed-loop is asymptotically stable. Output regulation means that for a certain class of initial conditions of the closed-loop system, we have  $e(t) \rightarrow 0$  as  $t \rightarrow \infty$ . The output constraints require that for an, a priori given bounded set  $\mathcal{S} \in \mathbb{R}^p$ , the output  $z$  remains in  $\mathcal{S}$  for all  $t \in \mathbb{R}$ .

In general, the constraint set  $\mathcal{S}$  discussed mostly in literature is bounded and convex. In fact typically it is a hypercube. We make the following fundamental assumption on the nature of this constraint set.

**Assumption 2.1** *The following conditions on  $\mathcal{S}$  are satisfied:*

- (i) *The set  $\mathcal{S}$  is bounded, closed, convex, and contains 0 as an interior point.*
- (ii)  $\mathcal{S} = (\mathcal{S} \cap \text{im } C_z) \oplus (\mathcal{S} \cap \text{im } D_z)$ .

**Definition 2.2** *Let the system (2.1) and the constraint set  $\mathcal{S}$  be given. We define*

$$\mathcal{X}(\mathcal{S}) := \{x_0 \in \mathbb{R}^p \mid \exists u_0 \text{ such that } C_z x_0 + D_z u_0 \in \mathcal{S}\}$$

as the **admissible set of initial conditions**.

**Remark.** In view of part (ii) of Assumption 2.1, we observe that  $C_z x_0 + D_z u_0 \in \mathcal{S}$  if and only if  $C_z x_0 \in \mathcal{S}$  and  $D_z u_0 \in \mathcal{S}$ . Since we can always assign  $u$  at  $t = 0$ , the admissible set  $\mathcal{X}(\mathcal{S})$  can be equivalently written as

$$\mathcal{X}(\mathcal{S}) := \{x_0 \in \mathbb{R}^p \mid C_z x_0 \in \mathcal{S}\}.$$

We now define several synthesis problems.

**Problem 2.3** Consider the system (2.1), the constraint set  $\mathcal{S} \subset \mathbb{R}^p$ , and a bounded set  $\mathcal{W}_0 \subset \mathbb{R}^s$ . The **constrained semi-global state feedback output regulation problem** is defined as follows. For any a priori given (arbitrarily large) bounded set  $\mathcal{X}_0$  contained in the interior of  $\mathcal{X}(\mathcal{S})$ , find, if possible, a state feedback law (possibly nonlinear) of the form,

$$\begin{aligned} \dot{v} &= \ell(x, v, w) \\ u &= g(x, v, w) \end{aligned}$$

where  $q$  is the dynamic order of the controller such that for any a priori given bounded set  $\mathcal{V}_0 \in \mathbb{R}^q$ , the following conditions hold:

- (i) The equilibrium point  $(x, v) = (0, 0)$  of the system

$$\begin{aligned} \dot{x} &= Ax(t) + Bg(x, v, 0) \\ \dot{v} &= \ell(x, v, 0) \end{aligned}$$

is locally exponentially stable with  $\mathcal{X}_0 \times \mathcal{V}_0$  contained in its basin of attraction.

- (ii) For any  $(x(0), v(0)) \in \mathcal{X}_0 \times \mathcal{V}_0$ , and  $w(0) \in \mathcal{W}_0$ , we have  $z(t) \in \mathcal{S}$  for all  $t > 0$ .
- (iii) For any  $(x(0), v(0)) \in \mathcal{X}_0 \times \mathcal{V}_0$ , and  $w(0) \in \mathcal{W}_0$ , the solution of the closed-loop system satisfies

$$\lim_{t \rightarrow \infty} e(t) = 0.$$

**Problem 2.4** Consider the system (2.1), the constraint set  $\mathcal{S} \subset \mathbb{R}^p$ , and a set  $\mathcal{W}_0 \subseteq \mathbb{R}^s$ . The **constrained global state feedback output regulation problem** is defined as follows. Find, if possible, a state feedback law (possibly nonlinear) of the form,

$$\begin{aligned} \dot{v} &= \ell(x, v, w) \\ u &= g(x, v, w) \end{aligned}$$

where  $q$  is the dynamic order of the controller such that the following conditions hold:

- (i) The equilibrium point  $(x, v) = (0, 0)$  of the system

$$\begin{aligned} \dot{x} &= Ax(t) + Bg(x, v, 0) \\ \dot{v} &= \ell(x, v, 0) \end{aligned}$$

is locally exponentially stable with  $\mathcal{X}(\mathcal{S}) \times \mathbb{R}^q$  contained in its basin of attraction.

- (ii) For any  $(x(0), v(0)) \in \mathcal{X}(\mathcal{S}) \times \mathbb{R}^q$ , and  $w(0) \in \mathcal{W}_0$ , we have  $z(t) \in \mathcal{S}$  for all  $t > 0$ .
- (iii) For any  $(x(0), v(0)) \in \mathcal{X}(\mathcal{S}) \times \mathbb{R}^q$ , and  $w(0) \in \mathcal{W}_0$ , the solution of the closed-loop system satisfies

$$\lim_{t \rightarrow \infty} e(t) = 0.$$

**Problem 2.5** Consider the system (2.1), the constraint set  $\mathcal{S} \subset \mathbb{R}^p$ , and a bounded set  $\mathcal{W}_0 \subset \mathbb{R}^s$ . The **constrained semi-global measurement feedback output regulation problem** is defined as follows. For any a priori given (arbitrarily large) bounded set  $\mathcal{X}_0$  contained in the interior of  $\mathcal{X}(\mathcal{S})$ , find, if possible, a measurement feedback law (possibly nonlinear) of the form,

$$\begin{aligned} \dot{v} &= \ell(y, v, w) \\ u &= g(y, v, w) \end{aligned}$$

where  $q$  is the dynamic order of the controller such that for any a priori given bounded set  $\mathcal{V}_0 \in \mathbb{R}^q$ , the following conditions hold:

- (i) The equilibrium point  $(x, v) = (0, 0)$  of the system,

$$\begin{aligned} \dot{x} &= Ax(t) + Bg(C_y x, v, 0) \\ \dot{v} &= \ell(C_y x, v, 0) \end{aligned}$$

is locally exponentially stable with  $\mathcal{X}_0 \times \mathcal{V}_0$  contained in its basin of attraction.

- (ii) For any  $(x(0), v(0)) \in \mathcal{X}_0 \times \mathcal{V}_0$  and  $w(0) \in \mathcal{W}_0$ , we have  $z(t) \in \mathcal{S}$  for all  $t > 0$ .
- (iii) For any  $(x(0), v(0)) \in \mathcal{X}_0 \times \mathcal{V}_0$ , and  $w(0) \in \mathcal{W}_0$ , the solution of the closed-loop system satisfies

$$\lim_{t \rightarrow \infty} e(t) = 0.$$

We can obviously also formulate the above problem in the global setting. However, if we start on the boundary of the set of initial conditions then no matter how fast our observer will be, there will be a short period of time that we have little information about the state. During this period of time the state of the system must remain inside this admissible set of initial conditions with the help of only a static output feedback. In other words this set must be controlled-invariant with respect to static output feedback and this is a very restrictive condition. For that reason we will not study this case although our techniques are perfectly suitable for this case as well.

Throughout this paper we let  $\mathbb{C}$ ,  $\mathbb{C}^+$ ,  $\mathbb{C}^-$  and  $\mathbb{C}^0$  denote respectively the entire complex plane, the open right-half complex plane, the open left-half complex plane, and the imaginary axis.

### 3 Categorization of Constraints

It turns out that the structural properties of the subsystem  $\Sigma_z$  characterized by the quadruple  $(A, B, C_z, D_z)$  play dominant roles in the study of constrained semi-global and global output regulation. These structural properties need to be delineated or categorized based on their impact for control engineers. Such a categorization can be done in different directions. Two directions, one in terms of right invertibility and the other in terms of invariant zeros of the plant, play profound roles in dictating what can or cannot be achieved. Our intention in this section is to present such a categorization. We first focus on categorizing the constraints based on whether the subsystem  $\Sigma_z$  is right invertible or not.

- The constraints are said to be *right invertible constraints* if the subsystem  $\Sigma_z$  associated with the constrained output  $z$  and characterized by the quadruple  $(A, B, C_z, D_z)$  is right invertible.
- The constraints are said to be *non-right invertible constraints* if the subsystem  $\Sigma_z$  associated with the constrained output  $z$  and characterized by the quadruple  $(A, B, C_z, D_z)$  is non-right invertible.

Next, we focus on categorizing the constraints based on the invariant zeros of the subsystem  $\Sigma_z$ .

**Definition 3.1** *The invariant zeros of the subsystem  $\Sigma_z$  characterized by the quadruple  $(A, B, C_z, D_z)$  are called as **constraint invariant zeros** of the plant associated with the constrained output  $z$ .*

The following categorization of constraints emerges based on the nature of constraint invariant zeros.

- The constraints are said to be *minimum phase constraints* if all the constraint invariant zeros of the plant associated with the constrained output  $z$  are in  $\mathbb{C}^-$ .

- The constraints are said to be *weakly minimum phase constraints* if all the constraint invariant zeros of the plant associated with the constrained output  $z$  are in  $\mathbb{C}^- \cup \mathbb{C}^0$  with the restriction that at least one such constraint invariant zero is in  $\mathbb{C}^0$  and any such constraint invariant zero in  $\mathbb{C}^0$  is simple.
- The constraints are said to be *weakly non-minimum phase constraints* if all the constraint invariant zeros of the plant associated with the constrained output  $z$  are in  $\mathbb{C}^- \cup \mathbb{C}^0$  with at least one non-simple constraint invariant zero in  $\mathbb{C}^0$ .
- The constraints are said to be *at most weakly non-minimum phase constraints* if all the constraint invariant zeros of the plant associated with the constrained output  $z$  are in  $\mathbb{C}^- \cup \mathbb{C}^0$ .
- The constraints are said to be *strongly non-minimum phase constraints* if any one or more of the constraint invariant zeros of the plant associated with the constrained output  $z$  are in  $\mathbb{C}^+$ .

## 4 Main Results

Our aim is to study in detail both the constrained semi-global and global output regulation problems utilizing state as well as measurement feedback. Our study is divided into two subsections, one for right invertible constraints and the other for non-right invertible constraints. The rationale for such a division lies in the fundamental way the study of either semi-global or global output regulation problem differs for these two categories of constraints. In fact, there exists a definite perceptible demarkation line between the right and non-right invertible constraints. For instance, *the solvability conditions for the constrained semi-global and global output regulation problems do not depend on the shape of the constraint set for right invertible constraints where as for non-right invertible constraints they indeed do so*. Because of this reason our development is divided into two main subsections one considering the right invertible constraints and the other non-right invertible constraints. However, owing to lack of space, we will not discuss here the case of non-right invertible constraints. For details of proofs, discussion on non-right invertible constraints, and for other aspects such as tracking a reference signal but not disturbance rejection in a regional sense, we refer the reader to [7].

### 4.1 Right Invertible Constraints

Our development throughout this subsection focuses on right invertible constraints. Such a focus on right invertible constraints is not as restrictive as it might seem. In fact, it is very broad. Let us observe that all the stabilization and output regulation problems studied so far in literature (see e.g. [9]) for linear systems subject to only control amplitude constraints but not magnitude constraints on states

(i.e.  $C_z = 0$ ) fall under the umbrella of right invertible constraints. This is because any system characterized by the quadruple  $(A, B, 0, D_z)$  is right invertible.

Before we proceed with our development in this section, we need to recall one important aspect of the work done in [8], namely the semi-global or global stabilization of a system with state and control magnitude constraints can be done only if the constraints are at most weakly non-minimum phase. Since output regulation requires stabilization, this implies that both constrained semi-global and global output regulation cannot be achieved if there exist strongly non-minimum phase constraints. Throughout this subsection we consider only systems with at most weakly non-minimum phase constraints.

From the study of classical output regulation even when there are no control and/or state magnitude constraints, it is well known that certain standard assumptions are necessary for stabilization as well as output regulation. In order to avoid repetition, we label and state the following standard assumptions.

**Assumption 4.1** *There exist matrices  $\Pi$  and  $\Gamma$  satisfying the so-called regulator equation,*

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + E \\ 0 &= C_e\Pi + D_e. \end{aligned} \quad (4.1)$$

**Assumption 4.2** *The matrix  $S$  has all its eigenvalues in the closed right half plane.*

**Assumption 4.3** *The pair  $(A, B)$  is stabilizable.*

**Assumption 4.4** *The pair*

$$\left[ \begin{array}{cc} (C_y & D_y), \begin{pmatrix} A & E \\ 0 & S \end{pmatrix} \end{array} \right]$$

*is observable.*

It is known that Assumption 4.1 is basically necessary for the solvability of the output regulation problems for systems without state and control magnitude constraints both for state feedback and for measurement feedback. Assumption 4.2 does not involve a loss of generality, because asymptotically stable modes in the exosystem do not affect the regulation of the output. Assumption 4.3 is necessary for stabilization of any system both for state feedback and for measurement feedback. For the measurement feedback case, Assumption 4.4 is not necessary (we only need detectability). However, the unobservable dynamics must satisfy stringent technical conditions that are needed to guarantee that the unobservable dynamics will not violate the state constraints. We have the following results.

**Theorem 4.5** *Consider the system  $\Sigma$  as given by (2.1), a constraint set  $\mathcal{S}$  that satisfies Assumption 2.1, and a bounded set  $\mathcal{W}_0 \subset \mathbb{R}^s$ . Moreover, let Assumptions 4.1, 4.2, and*

*4.3 hold. Assume also that the constraints are right invertible, i.e. the subsystem  $\Sigma_z$  characterized by the quadruple  $(A, B, C_z, D_z)$  is right invertible. Then the constrained semi-global state feedback output regulation problem (i.e. Problem 2.3) is solvable if the following conditions hold:*

(i) *The constraints are at most weakly non-minimum phase, i.e. the constraint invariant zeros of the plant associated with the constrained output  $z$  are in  $\mathbb{C}^- \cup \mathbb{C}^0$ .*

(ii) *There exists a  $\rho \in (0, 1)$  such that*

$$(C_z\Pi + D_z\Gamma)w(t) \in (1 - \rho)\mathcal{S}$$

*for all  $w(t)$  with  $w(0) \in \mathcal{W}_0$ , and for all  $t \geq 0$ .*

*Moreover, condition (i) is necessary.*

The statement of Theorem 4.5 implies that condition (i) is necessary, but condition (ii) is only sufficient and not necessary in order to solve the constrained semi-global state feedback output regulation problem. However, under an additional mild assumption, the theorem below proves that condition (ii) in Theorem 4.5 is very close to being necessary.

**Theorem 4.6** *Consider the system  $\Sigma$  as given by (2.1), a constraint set  $\mathcal{S}$  that satisfies Assumption 2.1, and a bounded set  $\mathcal{W}_0 \subset \mathbb{R}^s$ . Moreover, let Assumptions 4.1, 4.2, and 4.3 hold. Assume that the constraints are right invertible. Under the additional condition that the system from  $u$  to  $e$  is left-invertible, the constrained semi-global state feedback output regulation problem (i.e. Problem 2.3) for the given set  $\mathcal{W}_0$  is not solvable if there exists  $w_0 \in \mathcal{W}_0$  such that*

$$(C_z\Pi + D_z\Gamma)w(t) \notin \mathcal{S}$$

*for some  $t > 0$ .*

Next, we consider the global case. For this case to be solvable, we note that we only need an extra condition which is known to be necessary for globally stabilizing a system with state and control magnitude constraints (see [8]).

**Theorem 4.7** *Consider the system  $\Sigma$  as given by (2.1), a constraint set  $\mathcal{S}$  that satisfies Assumption 2.1, and a bounded set  $\mathcal{W}_0 \subset \mathbb{R}^s$ . Moreover, let Assumptions 4.1, 4.2, and 4.3 hold. Assume also that the constraints are right invertible, i.e. the subsystem  $\Sigma_z$  characterized by the quadruple  $(A, B, C_z, D_z)$  is right invertible. Then the constrained global state feedback output regulation problem (i.e. Problem 2.4) is solvable if the following conditions hold:*

(i) *The constraints are at most weakly non-minimum phase, i.e. the constraint invariant zeros of the plant associated with the constrained output  $z$  are in  $\mathbb{C}^- \cup \mathbb{C}^0$ .*

(ii) *The subsystem  $\Sigma_z$  characterized by  $(A, B, C_z, D_z)$  has no infinite zeros of order greater than one.*

(iii) There exists a  $\rho \in (0, 1)$  such that

$$(C_z \Pi + D_z \Gamma)w(t) \in (1 - \rho)\mathcal{S}$$

for all  $w(t)$  with  $w(0) \in \mathcal{W}_0$ , and for all  $t \geq 0$ .

Moreover, conditions (i) and (ii) are necessary.

We observe that, compared to the conditions for the semi-global case as given in Theorem 4.5, condition (ii) is an added necessary condition for solving the constrained global state feedback output regulation problem. In fact, condition (ii) is even necessary to globally stabilize a system with state and control magnitude constraints (see [8]). Also, as in the semi-global case, condition (iii) is almost necessary with an arbitrarily small gap between it and the necessary condition presented in Theorem 4.6.

**Theorem 4.8** Consider the system  $\Sigma$  as given by (2.1), a constraint set  $\mathcal{S}$  that satisfies Assumption 2.1, and a bounded set  $\mathcal{W}_0 \subset \mathbb{R}^s$ . Moreover, let Assumptions 4.1, 4.2, 4.3, and 4.4 hold. Assume also that the constraints are right invertible, i.e. the subsystem  $\Sigma_z$  characterized by the quadruple  $(A, B, C_z, D_z)$  is right invertible. Then the constrained semi-global measurement feedback output regulation problem (i.e. Problem 2.5) is solvable if the following conditions hold:

(i) The constraints are at most weakly non-minimum phase, i.e. the constraint invariant zeros of the plant associated with the constrained output  $z$  are in  $\mathbb{C}^- \cup \mathbb{C}^0$ .

(ii) There exists a  $\rho \in (0, 1)$  such that

$$(C_z \Pi + D_z \Gamma)w(t) \in (1 - \rho)\mathcal{S}$$

for all  $w(t)$  with  $w(0) \in \mathcal{W}_0$ , and for all  $t \geq 0$ .

Moreover, condition (i) is necessary.

**Remark.** We observe that the results obtained above are generalizations of the results reported in [6, 9] which deal with output regulation of linear systems with input constraints only, i.e. the case when  $C_z = 0$  and  $D_z = I$ . In fact, whenever  $C_z = 0$  and  $D_z = I$ , the constraint invariant zeros of the plant simply coincide with the poles of the given system and thus for this special case, the stabilizability of the pair  $(A, B)$  and the constraints being at most weakly non-minimum phase turn out to be the two basic assumptions used in [6, 9]. Consider next the case when  $C_z = 0$  and  $D_z$  has full row-rank, i.e. only a subset of the input channels is subject to constraints. It is then straightforward to show that the constraint invariant zeros of  $\Sigma$ , i.e. the invariant zeros of subsystem  $(A, B, 0, D_z)$  belong to a subset of the eigenvalues of  $A$ . Then that subset of the eigenvalues of  $A$  must be in closed left-half plane. On the other hand, if the system  $\Sigma$  is controllable via the unconstrained input channels, it is straightforward to see that the subsystem characterized by

$(A, B, 0, D_z)$  does not have any invariant zeros, i.e.  $\Sigma$  does not have any constraint invariant zeros. Thus, for this special case there will not be any constraints on the eigenvalues of  $A$  as obviously can be expected.

**Remark.** In [6, 9], the condition similar to

$$(C_z \Pi + D_z \Gamma)w(t) \in (1 - \rho)\mathcal{S} \quad (4.2)$$

is not imposed for all  $t \geq 0$  but only for  $t \geq T$  for some  $T \geq 0$ . However, note that due to Assumption 4.2, the condition that  $(C_z \Pi + D_z \Gamma)w(t)$  is bounded implies that this signal is almost periodic and therefore if (4.2) is satisfied for  $t > T$  for some  $T \geq 0$  then it must also be satisfied for all  $t > 0$ .

**Remark.** Note that the above theorems do not discuss any solution to the constrained global measurement feedback output regulation problem. This is due to the fact that this problem is only solvable under very restrictive conditions. Basically we need to assume that there exists a static feedback  $u = n(y)$  and  $\varepsilon > 0$  such that if  $\xi(0) \in \mathcal{X}(\mathcal{S})$ , then  $\xi(t) \in \mathcal{X}(\mathcal{S})$  for all  $t \in [0, \varepsilon]$  where  $\dot{\xi}(t) = A\xi(t) + Bn(C_y \xi(t))$ . In other words, the system must satisfy the state constraints on the interval  $[0, \varepsilon]$  with a static output feedback. This is clearly very restrictive and therefore a result for this case is not of much interest.

**Remark.** It is crucial to observe that, for constrained semi-global output regulation, it is necessary and natural to have the steady state of the state of the closed loop system remain in the domain of attraction. This is evidently so because if the reference signal switches to zero then one would indeed need the state of the closed loop system to be inside the domain of attraction. We did not include such a requirement in our problem formulation as it essentially boils down to a proper choice of  $\mathcal{X}_0$  and  $\mathcal{V}_0$ . Let us explain this in the case of constrained semi-global state feedback output regulation problem, i.e. Problem 2.3. In this case, in order to guarantee that the steady state of the state of the closed loop system remain in the domain of attraction, we need that  $\Pi w(t)$  be in  $\mathcal{X}_0$  for all  $t > 0$ . This condition can simply be achieved by choosing  $\mathcal{X}_0$  sufficiently large. Similar comments and arguments apply for constrained semi-global measurement feedback output regulation as well.

**Remark.** A fundamental aspect of solvability conditions as given by Theorems 4.5, 4.7 and 4.8 is that they are independent of any specific features of the given constraint set. That is, for right invertible constraints, if the constrained semi-global or global output regulation problems are solvable for some given constraint set satisfying Assumption 2.1, then these problems are also solvable for all constraint sets satisfying Assumption 2.1.

## 5 Conclusions

Linear systems with state and/or control magnitude constraints are considered. For such systems, semi-global and global output regulation problems are formulated and studied in depth when state as well as measurement feedback controllers are used. To start with the imposed state and control magnitude constraints are categorized in two main directions that have a profound impact on what can or cannot be achieved. One direction of categorization is based on what is labeled as constraint invariant zeros of the plant while the other is based on the right invertibility or lack of it of the mapping from the input to the constrained output vector that contains as its components all the state and control magnitude constraints. That is, in one direction the constraints are categorized into two main mutually exclusive categories, (1) right invertible constraints and (2) non-right invertible constraints. In the other direction the constraints are delineated into two other mutually exclusive categories, (1) at most weakly non-minimum phase constraints and (2) strongly non-minimum phase constraints.

Our development in this conference version of paper is focused on right invertible constraints. Sufficient conditions (which are almost necessary) are developed for the solvability of both the semi-global as well as global output regulation problems while using both state and measurement feedback controllers. The type of systems as well as the constraints dealt with here encompass all those for which output regulation has been studied so far in the literature for linear systems subject to only control amplitude constraints. Moreover, the solvability conditions developed here for the semi-global as well as global output regulation are natural extensions of those that are formulated in the literature for linear systems subject to only control amplitude constraints. One of the basic and important solvability conditions is that the constraints be at most weakly non-minimum phase. Whenever there are constraints only on control variables but not on state variables, this solvability condition simplifies to the one already formulated in the literature, namely that the poles of the given open-loop system be in the closed left half complex plane.

As discussed in [7], the study of semi-global and global output regulation problems for non-right invertible constraints is highly complex. The simplicity and the elegance of the solvability conditions developed for right invertible constraints will not prevail for non-right invertible constraints.

The solvability conditions depend on the shape of the constraint set for non-right invertible constraints where as for right invertible constraints they do not do so. In this sense there exists a definite perceptible demarkation line between the right and non-right invertible constraints.

Since either semi-global or global output regulation is not possible for strongly non-minimum phase constraints, an important issue arises as to what can be achieved for strongly non-minimum phase constraints. In this regard, it is shown in [7] that whenever there exists a controller which achieves stabilization with a certain domain of attraction, it is also possible to achieve tracking (but not disturbance rejection) with the same domain of attraction.

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