

# Global Stability Analysis of On/Off Systems

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## Abstract

This paper considers quadratic surface Lyapunov functions in the study of global stability analysis of on/off systems (OFS), including those OFS with unstable nonlinearity sectors. In previous work, quadratic surface Lyapunov functions were successfully applied to prove global asymptotic stability of limit cycles of relay feedback systems. In this work, we show that these ideas can be used to prove global asymptotic stability of equilibrium points of piecewise linear systems (PLS). We present conditions in the form of LMIs that, when satisfied, guarantee *global* asymptotic stability of an equilibrium point. A large number of examples was successfully proven globally stable. These include systems with an unstable affine linear subsystem, systems of relative degree larger than one and of high dimension, and systems with unstable nonlinearity sectors, for which all classical fail to analyze. In fact, existence of an example with a globally stable equilibrium point that could not be successfully analyzed with this new methodology is still an open problem. This work opens the door to the possibility that more general PLS can be systematically globally analyzed using quadratic surface Lyapunov functions.

## 1 Introduction

Although widely used and intuitively simple, *piecewise linear systems* (PLS) are computationally hard and very few results are available to *globally* analyze most PLS. In the analysis of equilibrium points of PLS, recent research has been concentrating on developing LMI based tools to construct piecewise quadratic Lyapunov functions. These ideas have been proposed and developed by [9], [13], and [8]. Partitioning of the state-space is the key in this approach. For most PLS, construction of piecewise quadratic Lyapunov functions is only possible after a more refined partition of the state space, in addition to the already existent natural state space

partition of the PLS. As a consequence, the analysis method is efficient only when the number of partitions required to prove stability is small. As illustrated in an example in [3], however, even for second order systems, the method can become computationally intractable. Also, for high-order systems, it is extremely hard to obtain a refinement of partitions in the state-space to efficiently analyze the PLS. Another disadvantage of finding Lyapunov functions in the state space is that it does not allow to study stability of limit cycles.

In [5, 6], global asymptotic stability of limit cycles of relay feedback systems was efficiently proven for a large number of examples analyzed, including minimum-phase systems, systems of relative degree larger than one, and of high dimension. The idea was to construct quadratic Lyapunov functions on switching surfaces that were used to show that *impact maps*, i.e., maps from one switching surface to the next switching surface, were contracting in some sense. Here, there was an obvious choice on how the stability problem needed to be setup since the fixed point, consisting of the intersection of the limit cycle with a switching surface, belonged to the switching surface. Therefore, all we needed was to show that consecutive switches were getting closer in some norm to the fixed point. This could even be done by just analyzing a single impact map due to the symmetry of RFS.

In this paper, we show that the approach introduced in [5, 6] can also be used to efficiently prove global asymptotic stability of equilibrium points, even when these equilibrium points do not belong to the switching surface. In case an equilibrium point belongs to the switching surface, the problem is similar to the one in [5, 6], with the added difficulty that now we have to simultaneously analyze two impact maps, instead of just one. If an equilibrium points does *not* belong to the switching surface, setting up the stability problem on the switching surface is not so straightforward. We show, however, that even in this case, analysis using quadratic surface Lyapunov functions can still be applied.

To demonstrate these ideas, we chose a class of PLS known as on/off systems (OFS). An OFS can be

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thought of as an LTI system that switches between open and closed loop. The switches are determined by the values of the output of the LTI system. OFS can be found in many engineering applications. In electronic circuits, diodes can be approximated by on/off controllers. Transient behavior of logical circuits that involve latches/flip-flops performing very fast on/off switching can be modeled using on/off circuits and saturations. In general, on/off circuits have many applications in electronics and circuit design. Another area of application of OFS is aircraft control. For instance, in [1], a *max* controller is designed to achieve good tracking of the pilot’s input without violating safety margins.

We are interested in checking if a unique locally stable equilibrium point of an OFS is also globally stable. The idea is to construct quadratic Lyapunov functions on the switching surface of an OFS to show contraction in some sense of impact maps. Under certain easily verifiable conditions, quadratic stability of impact maps implies globally asymptotically stability of the system. The search for quadratic surface Lyapunov functions is efficiently done by solving a set of LMIs.

As in relay feedback systems, a large number of examples was successfully proven globally stable. These include systems with an unstable affine linear subsystem, systems of relative degree larger than one and of high dimension, and systems with unstable nonlinearity sectors, for which classical methods like small gain theorem, Popov criterion, Zames-Falb criterion [14], and integral quadratic constraints [10, 2, 12, 11], fail to analyze. In fact, existence of an example with a globally stable equilibrium point that could not be successfully analyzed with this new methodology is still an open problem.

This paper is organized as follows. Section 2 starts by formulating the problem. Section 3 presents the main results of this paper followed by some illustrative examples in section 4. Section 5 presents a special case of OFS where the switching surface includes the origin. Finally, conclusions and future work are discussed in section 6.

## 2 Problem formulation

An OFS is defined as follows. Consider a SISO LTI system satisfying the following linear dynamic equations

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases} \quad (1)$$

where  $x \in \mathbb{R}^n$ , in feedback with a on/off controller (see figure 1) given by

$$u(t) = \max\{0, y(t) - d\} \quad (2)$$

where  $d \in \mathbb{R}$ . By a solution of (1)-(2) we mean functions  $(x, y, u)$  satisfying (1)-(2). Since  $u$  is continuous

and globally Lipschitz,  $Ax + B \max\{0, Cx - d\}$  is also globally Lipschitz. Thus, the OFS has a unique solution for any initial state.

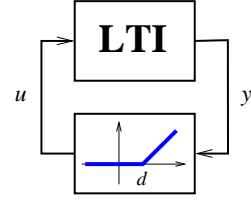


Figure 1: On/Off System

In the state space, the on/off controller introduces a switching surface composed of an hyperplane of dimension  $n - 1$

$$S = \{x \in \mathbb{R}^n : Cx = d\}$$

On one side of the switching surface ( $Cx < d$ ), the system is given by  $\dot{x} = Ax$ . On the other side ( $Cx > d$ ) the system is given by  $\dot{x} = Ax + B(Cx - d) = A_1x + B_1$ , where  $A_1 = A + BC$  and  $B_1 = -Bd$ . Note that the vector field is continuous along the switching surface since for any  $x \in S$ ,  $A_1x + B_1 = Ax$ .

An OFS has either zero, one, or two equilibrium points. We are interested in those cases where the system has a unique locally stable equilibrium point. Only here can an OFS have a globally stable equilibrium point. Next, we give necessary conditions for the existence of a single locally stable equilibrium point for different values of  $d$ .

If  $d > 0$  there is at least one equilibrium point at the origin. In this case, it is necessary that  $A$  is Hurwitz to guarantee the origin is locally stable. It is also necessary that  $A_1$  is invertible or otherwise there would exist a continuum of equilibrium points. The affine linear system  $\dot{x} = A_1x + B_1$  has an equilibrium point at  $-A_1^{-1}B_1$ . In order to guarantee an OFS has only the origin as an equilibrium point, it is necessary that  $-CA_1^{-1}B_1 < d$ . It is also necessary that  $A_1$  has no real unstable eigenvalues or, otherwise, the system will have trajectories that grow unbounded<sup>1</sup>. To see this, let  $\lambda$  be a real unstable eigenvalue of  $A_1$  with associated eigenvalue  $v$ . Let  $x_0 = \alpha v - A_1^{-1}B_1$ , where  $\alpha$  is chosen such that  $Cx_0 > d$ . The trajectory starting at  $x_0$  is given by  $x(t) = \alpha e^{\lambda t} v - A_1^{-1}B_1$ . Hence, the trajectory will grow unbounded without switching since  $Cx(t) = \alpha e^{\lambda t} Cv - CA_1^{-1}B_1 \geq \alpha Cv - CA_1^{-1}B_1 > d$ , for all  $t \geq 0$ . Note that  $\alpha Cv > d + CA_1^{-1}B_1 > d - d = 0$ .

When  $d = 0$ , the origin is the only equilibrium point. For the same reasons as above, it is necessary that both  $A$  and  $A_1$  do not have real unstable eigenvalues. Note that in this case, there is no “easy” way to check if the origin is locally stable or not.

<sup>1</sup>Possible exceptions occur when the eigenvector associated with the unstable real eigenvalue is perpendicular to  $C$ .

When  $d < 0$ , it must be true that  $-CA_1^{-1}B_1 > d$  or otherwise the system will have no equilibrium point. It is also necessary that  $A_1$  is Hurwitz and  $A$  has no real unstable poles. We can assume without loss of generality that  $d \geq 0$ . If  $d < 0$  and all necessary conditions are met, with an appropriate change of variables ( $x_{new} = -(x + A_1^{-1}B_1)$ ), the problem can be transformed to one of analyzing the origin with  $d_{new} \geq 0$ . In this case,  $A_{new} = A_1$ ,  $A_{1new} = A$ ,  $B_{1new} = AA_1^{-1}B_1$ , and  $d_{new} = -d - CA_1^{-1}B_1 \geq 0$ .

Consider a subset  $S_+$  of  $S$  given by

$$S_+ = \{x \in S : CAx \geq 0\}$$

This set is important since it tells us which points in  $S$  can be reached by trajectories starting at any  $x_0$  such that  $Cx_0 < d$  (see figure 2). Similarly, define  $S_- \subset S$  as

$$S_- = \{x \in S : CAx \leq 0\}$$

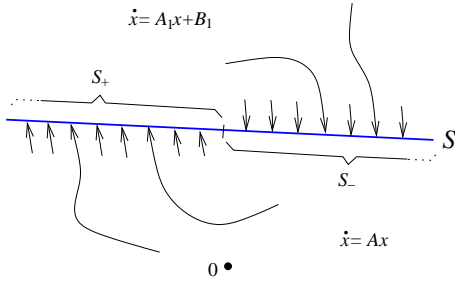


Figure 2: Both sets  $S_+$  and  $S_-$  in  $S$

Note that  $S = S_+ \cup S_-$  and  $S_+ \cap S_- = \{x \in S : CAx = 0\}$ . From here on, we assume  $d > 0$ . In terms of stability analysis,  $d = 0$  is a special case of when  $d > 0$ , and will be considered separately in section 5.

Since  $A$  must be Hurwitz, there is a set of points in  $S_-$  such that any trajectory starting in that set will never switch again and will converge asymptotically to the origin. In other words, let  $S^* \subset S_-$  be the set of points  $x_0$  such that the following equation

$$Ce^{At}x_0 = d$$

does not have a solution for any  $t > 0$ . Note that this set  $S^*$  is not empty. To see this, let  $P > 0$  satisfy  $PA + A'P = -I$ . Then, an obvious point in  $S^*$  is the point  $x_1^*$ , obtained from the intersection of  $S$  with the level set  $x'Px = k$ , where  $k \geq 0$  is chosen such that the ellipse  $x'Px = k$  is tangent to  $S$  (see figure 3).

The problem we propose to solve here is to give sufficient conditions that, when satisfied, prove the origin of an OFS is globally asymptotically stable. The strategy of this proof is as follows. Consider a trajectory starting at some point  $x_0 \in S_+$  (see figure 4). Since

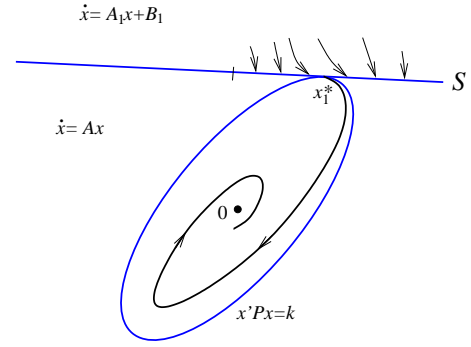


Figure 3: How to obtain  $x_1^*$

$A_1$  has no unstable real poles, the trajectory  $x(t)$  will eventually switch at some time  $t_1 > 0$ , i.e.,  $Cx(t_1) = d$  and  $Cx(t) \geq d$  for  $t \in [0, t_1]$ . Let  $x_1 = x(t_1) \in S_-$ . If  $x_1 \in S^*$ , the trajectory will not switch again and converge asymptotically to the origin. Since we already know  $S^*$  is a stable set, we need to concentrate on those points in  $S_- \setminus S^*$  since those are the ones that may lead to potentially unstable trajectories. So, assume the trajectory switches again at time  $t_2 > t_1$ , and let  $x_2 = x(t_2) \in S_+$ . Again, we would switch at  $x(t_3) = x_3$  and so on. The idea is to check if  $x_3$  is closer in some sense to  $S^*$  than  $x_1$ . If so, this would mean that eventually  $x(t_{2N-1}) \in S^*$ , for some positive integer  $N$ , and prove that the origin is globally asymptotically stable. This is the idea behind the results in the next section.

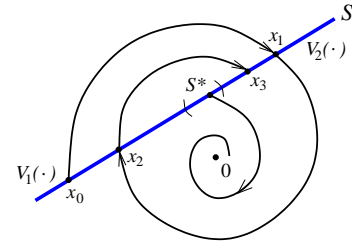


Figure 4: Trajectory of an OFS

Before we present the main results, it is convenient to notice that  $x_0, x_1, x_2 \in S$  can be parametrized. Let  $x_0 = x_0^* + \Delta_0$ ,  $x_1 = x_1^* + \Delta_1$ , and  $x_2 = x_0^* + \Delta_2$ , where  $x_0^*, x_1^* \in S$ , and  $C\Delta_0 = C\Delta_1 = C\Delta_2 = 0$ . Also, define  $x_0^*(t)$  ( $x_1^*(t)$ ) as the trajectory of  $\dot{x} = A_1x + B_1$  ( $\dot{x} = Ax$ ), starting at  $x_0^*$  ( $x_1^*$ ), for all  $t > 0$ . Since  $x_i^*$  can be any points in  $S$ , we chose them to be such that  $Cx_i^*(t) < d$  for all  $t > 0$ . This is always possible, even when  $A_1$  is unstable, as long as it has at least one stable eigenvalue with an associated eigenvector that is not perpendicular to  $C'$  (see [3] for details). The reason for this particular choice of  $x_0^*$  and  $x_1^*$  is so that  $Cx_i^*(t) - d \neq 0$  for all  $t > 0$ . This will be necessary in proposition 3.1.

As in RFS, impact maps associated with OFS are mul-

tivalued. Define the expected switching times  $\mathcal{T}_1$  and  $\mathcal{T}_2$  as the set of all possible switching times associated with the impact maps from  $S_+$  to  $S_-$  and from  $S_- \setminus S^*$  to  $S_+$ , respectively.

### 3 Global asymptotic stability of on/off systems

Before presenting the main result of this paper, we first show that each impact map associated with the OFS can be represented as a linear transformation analytically parametrized by the correspondent switching time. This result is similar to the main theorem in [5, 6] and, therefore, the proof is omitted here.

**Proposition 3.1** *Define*

$$w_1(t) = \frac{Ce^{A_1 t}}{d - Cx_0^*(t)}, \quad w_2(t) = \frac{Ce^{At}}{d - Cx_1^*(t)}$$

Let  $H_1(t) = e^{A_1 t} + (x_0^*(t) - x_1^*)w_1(t)$  and  $H_2(t) = e^{At} + (x_1^*(t) - x_0^*)w_2(t)$ . Then, for any  $\Delta_0 \in S_+ - x_0^*$  there exists a  $t_1 \in \mathcal{T}_1$  such that

$$\Delta_1 = H_1(t_1)\Delta_0$$

Such  $t_1$  is the switching time associated with  $\Delta_1$ . Similarly, for any  $\Delta_1 \in S_- \setminus S^* - x_1^*$  there exists a  $t_2 \in \mathcal{T}_2$  such that

$$\Delta_2 = H_2(t_2)\Delta_1$$

Such  $t_2$  is the switching time associated with  $\Delta_2$ .

Next, define two quadratic Lyapunov functions  $V_1$  and  $V_2$  on the switching surface  $S$

$$V_i(x) = x'P_i x - 2x'g_i + \alpha_i \quad (3)$$

where  $P_i > 0$ , for  $i = 1, 2$ . Global asymptotically stability of the origin follows if both impact maps are quadratically stable, i.e., if there exist  $P_i > 0$ ,  $g_i$ ,  $\alpha_i$  such that

$$V_2(\Delta_1) < V_1(\Delta_0) \quad \text{for all } \Delta_0 \in S_+ - x_0^* \quad (4)$$

$$V_1(\Delta_2) < V_2(\Delta_1) \quad \text{for all } \Delta_1 \in S_- \setminus S^* - x_1^* \quad (5)$$

The next theorem gives conditions that, when satisfied, guarantee both impact maps are quadratically stable. Let  $P > 0$  on  $S$  stand for  $x'Px > 0$  for all  $x \in S$ . As a short hand, in the following result we use  $H_{it} = H_i(t)$  and  $w_{it} = w_i(t)$ .

**Theorem 3.1** *Define*

$$R_1(t) = P_1 - H_{1t}'P_2H_{1t} - 2(g_1 - H_{1t}'g_2)w_{1t} + w_{1t}'\alpha w_{1t}$$

$$R_2(t) = P_2 - H_{2t}'P_1H_{2t} - 2(g_2 - H_{2t}'g_1)w_{2t} - w_{2t}'\alpha w_{2t}$$

where  $\alpha = \alpha_1 - \alpha_2$ . The origin of the OFS is globally asymptotically stable if there exist  $P_1, P_2 > 0$  and  $g_1, g_2, \alpha$  such that

$$\begin{cases} R_1(t_1) > 0 & \text{on } S_+ - x_0^* \\ R_2(t_2) > 0 & \text{on } S_- \setminus S^* - x_1^* \end{cases}$$

for all expected switching times  $t_1 \in \mathcal{T}_1$  and  $t_2 \in \mathcal{T}_2$ .

A relaxation of the constraints on  $\Delta_0$  and  $\Delta_1$  in the previous theorem results in computationally efficient conditions.

**Corollary 3.1** *The origin of the OFS is globally asymptotically stable if there exist  $P_1, P_2 > 0$  and  $g_1, g_2, \alpha$  such that*

$$\begin{cases} R_1(t_1) > 0 & \text{on } S - x_0^* \\ R_2(t_1) > 0 & \text{on } S - x_1^* \end{cases} \quad (6)$$

for all expected switching times  $t_1 \in \mathcal{T}_1$  and  $t_2 \in \mathcal{T}_2$ .

For each  $t_1, t_2$  these conditions are LMIs for which we can solve for  $P_1, P_2 > 0$  and  $g_1, g_2, \alpha$  using efficient available software. As we will see in the next section, although these conditions are more conservative than the ones in theorem 3.1, they are already enough to prove global asymptotic stability of many important OFS.

Other less conservative conditions are considered and discussed in [6, 3]. These are based on the fact that, for each impact map, the set of points in  $S$  with the same switching time is a convex subset of a linear manifold of dimension  $n - 2$ .

**Proof of theorem 3.1:** From (4) and using proposition 3.1, we have

$$\Delta_0'(P_1 - H_{1t}'P_2H_{1t})\Delta_0 - 2\Delta_0'(g_1 - H_{1t}'g_2) + \alpha > 0$$

Finally, using the fact that  $w_{1t}\Delta_0 = 1$  we have

$$\begin{aligned} \Delta_0'(P_1 - H_{1t}'P_2H_{1t})\Delta_0 - 2\Delta_0'(g_1 - H_{1t}'g_2)w_{1t}\Delta_0 \\ + \Delta_0'w_{1t}'\alpha w_{1t}\Delta_0 > 0 \end{aligned}$$

which is the desired result.  $R_2(t)$  can be obtained in a similar way. ■

## 4 Examples

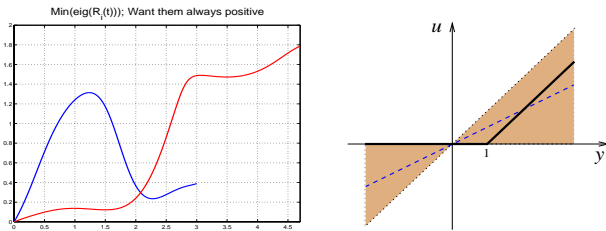
The following examples were processed in `matlab` code. The latest version of this software is either available at [7] or upon request. Before we present the examples, we briefly explain the `matlab` function we developed. The inputs to this function are a transfer function of an LTI system together with the displacement of the nonlinearity switch  $d$ . If the OFS is proven to be globally stable, the function returns the values of the parameters of the Lyapunov functions (3). The `matlab` function also returns a graphic showing the minimum eigenvalues of each  $R_i(t_i)$  in (6), which must be positive for all expected switching times  $t_i$ . Note that for most OFS, the expected switching times include  $t_i = 0$  and large values of  $t_i$ . Technical details on how to guarantee the impact maps are quadratically stable for all  $t \geq 0$  once conditions (6) are satisfied on some finite intervals ( $t_{imin}, t_{imax}$ ) can be found in [3, chapter 6].

**Example 4.1** Consider the OFS in figure 1 with the LTI system given by the transfer function

$$\frac{Y(s)}{U(s)} = -2 \frac{s^2 + s + 6}{s^3 + 2s^2 + 2s + 3}$$

and with  $d = 1$ . It is easy to see that the origin of this system is locally stable.

Using conditions (6), we show that the origin is in fact globally asymptotically stable. The left side of figure 5 illustrates this fact: the minimum eigenvalue of each condition (6) is positive on its respective expected switching times. The expected switching times for this example are  $\mathcal{T}_1 = (0, 1.85)$  and  $\mathcal{T}_2 = (0, 5.4)$ . For instance, if  $t_1 \geq 1.85$ , there is no point in  $S_+$  with switching time equal to  $t_1$ .



**Figure 5:** System with unstable nonlinearity sector

Note that this system has an unstable nonlinearity sector. If the on/off nonlinearity is replaced by a linear constant gain of  $1/2$ , the system becomes unstable (see the right side of figure 5). This is very interesting since it tells us that analysis tools like small gain theorem, Popov criterion, Zames-Falb criterion, and integral quadratic constraints, would all fail to analyze OFS of this nature. ■

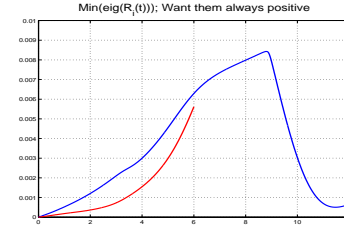
**Example 4.2** Consider the OFS in figure 1 with the LTI system given by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{k}{(s+1)^7}$$

and with  $k > 0$  and  $d = 1$ . Once again, it is easy to see that the origin of this system is locally stable for any  $k > 0$ .

Note that  $\|Ce^{At}B\|_{\mathcal{L}_1} = k$ . Thus, the small gain theorem can be applied whenever  $k \leq 1$ . When  $k > 1$ , however, the small gain theorem fails to analyze the system.

Let  $k = 2$ . Using conditions (6), we show the origin is globally asymptotically stable. Figure 6 shows how conditions (6) are satisfied in some intervals  $(0, t_{imax})$ ,  $i = 1, 2$ . The intervals  $(0, t_{imax})$  are bounds on the expected switching times, and can be found as in [3, chapter 6]. ■

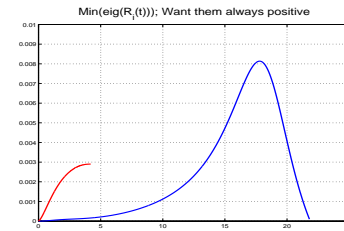


**Figure 6:** System with relative degree 7 (left)

**Example 4.3** Consider the OFS in figure 1 with the LTI system given by the transfer function

$$\frac{Y(s)}{U(s)} = -4 \frac{s^2 - s + 0.05}{s^3 + 2s^2 + 2s + 0.1}$$

and with  $d = 1$ . It is easy to see that the origin of this system is locally stable.  $A_1$ , however, is unstable.



**Figure 7:** System with unstable subsystem

Although  $A_1$  is unstable, since this is a  $3^{rd}$ -order system, it is easy to find bounds on the expected switching times for the subsystem  $\dot{x} = A_1x + B_1$ . In this case, no point in  $S_+$  has a switching time higher than 21.8. As for  $t_2$ , we use the same ideas as in the previous example, based on the results in [3, chapter 6]. Using conditions (6), we show that although  $A_1$  is unstable, the origin is globally asymptotically stable (see figure 7). ■

## 5 Special case: $d = 0$

When  $d = 0$  we can write stability conditions that are, in general, much less conservative than conditions (6). First, since the origin belongs to both systems  $\dot{x} = Ax$  and  $\dot{x} = (A + BC)x$ , it is only required that both systems do not have real unstable poles.  $d = 0$  also means  $x_0^* = x_1^* = g_1 = g_2 = 0$  and  $\alpha = 0$ . All we need to find is  $P_1, P_2 > 0$ .

In this case,  $\Delta_1 = e^{A_1 t_1} \Delta_0$  and  $\Delta_2 = e^{A_2 t_2} \Delta_1$ . Thus, the stability conditions come down to

$$\begin{aligned} \Delta'_1 P_2 \Delta_1 &< \Delta'_0 P_1 \Delta_0 \\ \Leftrightarrow \Delta'_0 \left( P_1 - e^{A_1 t_1} P_2 e^{A_1 t_1} \right) \Delta_0 &> 0 \end{aligned}$$

and

$$\begin{aligned} \Delta'_2 P_1 \Delta_2 &< \Delta'_1 P_2 \Delta_1 \\ \Leftrightarrow \Delta'_1 \left( P_2 - e^{A_2 t_2} P_1 e^{A_2 t_2} \right) \Delta_1 &> 0 \end{aligned}$$

for some  $P_1, P_2 > 0$ , all expected switching times  $t_1, t_2$ , and all  $\Delta_0 \in S_+$ ,  $\Delta_1 \in S_- \setminus S^*$ .

Notice that  $C\Delta_0 = 0$ ,  $C\Delta_1 = 0$ , and  $C\Delta_2 = 0$ . Therefore  $Ce^{A_1 t_1} \Delta_0 = 0$  and  $Ce^{A_2 t_2} \Delta_1 = 0$ . That is, for fixed values of  $t_1$  and  $t_2$ ,  $\Delta_1$  and  $\Delta_2$  are restricted to a subspace of dimension  $n-2$ . Let  $\Pi \in C^\perp$ , where  $C^\perp$  are the *orthogonal complements* to  $C$ , i.e., matrices with a maximal number of column vectors forming an orthonormal set such that  $CC^\perp = 0$ . Let also  $l_{t_1} \in (Ce^{A_1 t_1} \Pi)^\perp$  and  $l_{t_2} \in (Ce^{A_2 t_2} \Pi)^\perp$ . We have the following result.

**Theorem 5.1** *The origin of the OFS with  $d = 0$  is globally asymptotically stable if there exist  $P_1, P_2 > 0$  such that*

$$\begin{cases} l_{t_1}' \Pi' (P_1 - e^{A_1 t_1} P_2 e^{A_1 t_1}) \Pi l_{t_1} > 0 \\ l_{t_2}' \Pi' (P_2 - e^{A_2 t_2} P_1 e^{A_2 t_2}) \Pi l_{t_2} > 0 \end{cases} \quad (7)$$

for all expected switching times.

## 6 Conclusion

This paper confirms the idea that global stability analysis of equilibrium points and limit cycles of certain classes of piecewise linear systems can be done using impact maps and quadratic surface Lyapunov functions. The search for quadratic surface Lyapunov functions is efficiently done by solving a set of LMIs.

In [5, 6], we showed how this approach is powerful in globally analyzing limit cycles of relay feedback systems. Here, we demonstrated that similar ideas can be used to check if equilibrium points of on/off systems are globally asymptotically stable. Impact maps can be proven quadratically stable by constructing quadratic Lyapunov functions on the switching surface. The methodology can be applied even when the equilibrium point does not belong to the switching surface.

With this new results, a large number of examples was successfully proven globally stable. These include systems with an unstable affine linear subsystem, systems of relative degree larger than one and of high dimension, and systems with unstable nonlinearity sectors, for which all classical fail to analyze. In fact, existence of an example with a globally stable equilibrium point that could not be successfully analyzed with this new methodology is still an open problem.

There are still many open problems following this work. In a recent paper [4], we have showed that similar ideas to the ones presented here can be applied to globally analyze saturation systems. The question is how to use quadratic surface Lyapunov functions to systematically globally analyze larger and more complex classes of PLS.

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