H_∞Control for Continuous-Time Linear Systems with Infinite Markov Jump Parameters via Semigroup [1]

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Abstract

We examine the $H_\infty-$control problem for a class of continuous time linear systems subject to Markovian jumps in the parameters(LSMJP). We extend the results in [9] in two directions: we give *necessary* and sufficient conditions for the existence of a feedback controller which stochastically stabilizes the LSMJP and ensures that a certain L_2 induced norm be less than a prespecified value. In addition, we consider the case in which the state-space of the Markov chain takes value in a *countably infinity set*. The solution here is given in terms of a countably infinite set of coupled algebraic Riccati equations(ICARE).

1 Introduction

A significant research thrust in linear systems with Markovian jump parameters (LSMJP) has occurred during the past two decades. This can be confirmed by the fairly amount of literature on this subject (see, e.g., [3], [5]-[12], [15]-[18], and references therein). In addition, further insights has been gained through these years in applications, generally dealing with systems that are subject to abrupt changes in their structures, which encompasses fault tolerant and high integrity systems such as aircraft control systems, robotic manipulator systems, space stations structures such as solar arrays and antennas, government policy changes(see, e.g. [4], [14] and [22]).

Since the pioneer work of Zames (see [24]), interest in the $\mathcal{H}_\infty$ control problem has increased steadily, with emphasis in the linear case(see, e.g., [21] and references therein).

Regarding the LSMJP, the $\mathcal{H}_\infty$ control was previously studied in [9] and [19] for the continuous time and in [12] for the discrete time problem. The technique in [9] and [12] provides sufficient conditions for a solution whereas in [19] the differential game interpretation for the problem is employed. In [7] the H_∞ problem refers to the discrete-time case with a countably infinite state space for the Markov chain and, unlike previous works with this setting, necessary and sufficient conditions are obtained.

In this paper we consider the combined situation of *continuous-time* LSMJP with *countably infinite state space* for the Markov chain, and obtain necessary and sufficient condition for the existence of a feedback controller that stochastically stabilizes the LSMJP while ensuring that the L_2 induced norm is less than a prespecified value.

In its content, this paper is closely related to [7]. However, technically they are rather different. For instance, we use heavily the tools from operator theory in infinite dimensional spaces, making particular use of the theory of semigroup from which we provide an essential equivalence result that bounds up the concept of stochastic stabilizability (SS) and stochastic detectability (SD) with the spectrum of certain infinite dimensional Banach space operators, a result adapted from [11]. Moreover, we introduce a natural adaptation of the decomplexification concept, described in [1, Section 18], for nonlinear complex functions with range in $\mathbb{R}$, which allows us to focus differentiability on certain decomplexifications ${}^{\mathrm{R}}g$ *in lieu* of the original functions g (see Section 2).

The paper content is as follows. In Section 2 we present the essential definitions and notations. In Section 3 we state the H_∞ problem, its probabilistic structure and some preliminaries. The main results are exhibited in the Section 4

2 Notations and Assumptions

We shall denote by $\mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$, the normed linear space of all n by m complex matrices and, for simplicity, we shall write $\mathbb{M}(\mathbb{C}^n)$ whenever $n = m$. We shall write $^-$, $'$ and * for complex conjugate, transpose and conjugate transpose, respectively. We shall use the notation $L \geq 0$ and $L > 0$ if a self adjoint matrix is positive semidefinite or positive definite, respectively. We denote $\mathbb{M}(\mathbb{C}^n)^+ = \{L \in \mathbb{M}(\mathbb{C}^n); L = L^* \geq 0\}$. As usual, $\|\cdot\|$ will stand for the euclidean norm in $\mathbb{C}^n$, in $\mathbb{R}^m$ and in the product space $\mathbb{R}^m \times \mathbb{C}^n$, as well as for

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the spectral induced norm in $\mathbb{M}(\mathbb{C}^n)$. In addition, we denote by ι the pure imaginary complex number with unitary modulus.

Let $\mathcal{H}_1^{m,n}$ (resp. $\mathcal{H}_\infty^{m,n}$) be the linear space made up of all infinite sequences of complex matrices $H = (H_1, H_2, ...)$, $H_i \in \mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$ such that $\sum_{i=1}^\infty \|H_i\| < \infty$ (resp. $\sup\{\|H_i\|,\ i = 1, 2, ...\} < \infty$). For $H \in \mathcal{H}_1^{m,n}$ (resp. $H \in \mathcal{H}_\infty^{m,n}$) we define a norm in $\mathcal{H}_1^{m,n}$ (resp. $\mathcal{H}_\infty^{m,n}$) by $\|H\|_1 = \sum_{i=1}^\infty \|H_i\| < \infty$ (resp. $\|H\|_\infty = \sup\{\|H_i\|,\ i = 1, 2, ...\}$). We shall write $\mathcal{H}_1^n$ and $\mathcal{H}_\infty^n$ whenever $n = m$, $\mathcal{H}_1^{n^+} = \{H \in \mathcal{H}_1^n,\ H_i \in \mathbb{M}(\mathbb{C}^n)^+,\ i = 1, 2, ...\}$and $\mathcal{H}_\infty^{n^+} = \{H \in \mathcal{H}_\infty^n,\ H_i \in \mathbb{M}(\mathbb{C}^n)^+,\ i = 1, 2, ...\}$. For $H = (H_1, H_2, ...)$ and $L = (L_1, L_2, ...)$ in $\mathcal{H}_1^{n^+}$ we shall use the notation $H \leqslant L$ to indicate that $H_i \leqslant L_i$ for each i in $\mathbb{N}$. We have that $H \leqslant L \Rightarrow \|H\|_1 \leqslant \|L\|_1$.

For any complex Banach or Hilbert space X, we denote by $\mathbb{B}(X)$ the Banach space of all bounded linear transformations of X into X with the uniform induced norm represented by $\|\cdot\|$, and for $L \in \mathbb{B}(X)$ we denote $\sigma(L)$ the spectrum of L.

We denote by $1_A\ \{.\}$ the Dirac measure and write $\{\eta\}$and sometimes η for any process $\{\eta(t), 0 \leqslant t \leqslant T\}$, whenever it is clear whether T is finite or not, and $E[\cdot]$ denotes the usual mathematical expectation.

We define the Hilbert space $C_D^m = L_2([0,\infty); \mathbb{C}^m)$ made up of all functions $f : [0,\infty) \to \mathbb{C}^m$ with the inner product $\langle f, g\rangle = \int_0^\infty f(t)^* g(t) dt < \infty$ and norm denoted by $\|f\|_2 = (\int_0^\infty \|f(t)\|^2 dt)^{1/2} < \infty$, for every $f, g \in C_D^m$.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space carrying a right continuous filtration $\{\mathcal{F}_t : t \in [0,\infty)\}$ with $\mathcal{F}_t$ being the σ- field generated by the observed process $\{(x(s), \theta(s)),\ 0 \leqslant s \leqslant t\}$. With such an underlying space, let us define the Hilbert space $C^m = L_2(\Omega, \mathcal{F}, \mathcal{P}; \mathbb{C}^m)$ made up of all second order $\mathbb{C}^m$-valued random variables (r.vs.) with inner product given by $\langle x, y\rangle = E[x^* y]$ and norm denoted by $\|x\|_2 = E\left[\|x\|^2\right]^{1/2}$ for every $x, y \in C^m$, the Hilbert space $L_2^m = L_2([0,\infty); \mathbb{C}^m)$ made up of all processes $\{y(t), t \geq 0\}$ equipped with the inner product $\langle x, y\rangle = \int_0^\infty E[x(t)^* y(t)]dt < \infty$ and norm denoted again by $\|x\|_2 = (\int_0^\infty E[\|x(t)\|^2]dt)^{1/2} < \infty$, for every $x, y \in L_2^m$, and the set $\mathcal{C}^m \subset L_2^m$ made up of all processes $x \in L_2^n$ adapted to $\{\mathcal{F}_t : t \in [0,\infty)\}$ i.e., such that $x(t) \in L_2(\Omega, \mathcal{F}_t, \mathcal{P}; \mathbb{C}^n)$. We have that $\mathcal{C}^m$ is a closed linear subspace of L_2^m and therefore a Hilbert space. We also define $\mathcal{C}_T^m$ the set made up of all processes $\{y(t),\ 0 \leqslant t \leqslant T\}$, $y(t) \in L_2(\Omega, \mathcal{F}_t, \mathcal{P}; \mathbb{C}^m)$ and denote Θ_0 the set of all $\mathcal{F}_0$-measurable r.vs. taking values in $\mathbb{N}$.

3 Problem Statement and Preliminaries

We shall be interested in the following class of stochastic differential equations.

$$\left\{ \begin{array}{l} \dot{x}(t) = A_{\theta(t)} x(t) + B_{\theta(t)} u(t) + D_{\theta(t)} w(t), \quad t > 0 \\ x(0) = x_0 \in \mathcal{C}_0^n, \quad \theta(0) = \theta_0 \in \Theta_0 \end{array} \right. \tag{1}$$

where $x_0 \in \mathcal{C}_0^n$, $w \in C_D^p$ and $u \in \mathcal{C}^m$. The vector $x(t) \in L_2(\Omega, \mathcal{F}_t, \mathcal{P}, \mathbb{C}^n)$ denotes the state, $u(t) \in L_2(\Omega, \mathcal{F}_t, \mathcal{P}, \mathbb{C}^m)$ and $w(t) \in L_2(\mathbb{C}^p)$, $t \geq 0$, denotes the system's control input and the noise, respectively. The system's parameters are fed by a Markov process $\{\theta(t), t \geq 0\}$ with right continuous trajectories and an infinite countable state space $\mathbb{N} = \{1, 2, ...\}$. We assume that $\{\theta(t), t \geq 0\}$ has stationary standard transition probability matrix function $\{P_s(i,j)\}_{i,j\in\mathbb{N}}$ in that, for $s \geq 0$,

$$\begin{aligned} P_s(i,j) =:\ & \mathcal{P}\{\theta(t+s) = j \mid \theta(t) = i\} \\ =\ & \left\{ \begin{array}{ll} \lambda_{ij} s + o(s) & i \neq j \\ 1 + \lambda_{ii} s + o(s) & i = j \end{array} \right. \end{aligned}$$

with transition rates $\lambda_{ij} \geq 0$ for $i \neq j$. The Markov process $\{\theta(t), t \geq 0\}$ is conservative and stable in that

$$\sum_{j=1, j\neq i}^\infty \lambda_{ij} = -\lambda_{ii} \leq c < \infty\ , \quad i \in \mathbb{N}$$

where c does not depends on i. We assume that $A_{(\cdot)}$, $B_{(\cdot)}$, $D_{(\cdot)}$, are such that for each $i \in \mathbb{N}$, $\theta(t) = i$, $A_{\theta(t)} = A_i$, $B_{\theta(t)} = B_i$, $D_{\theta(t)} = D_i$, defined on the spaces $\mathbb{M}(\mathbb{C}^n)$, $\mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$, $\mathbb{M}(\mathbb{C}^p, \mathbb{C}^n)$, respectively, are constant matrices. In addition, the parameters are supposed norm bounded in that $A = (A_1, A_2, ...) \in \mathcal{H}_\infty^n$, $B = (B_1, B_2, ...) \in \mathcal{H}_\infty^{m,n}$, $D = (D_1, D_2, ...) \in \mathcal{H}_\infty^{p,n}$. We consider x_0 a second order $\mathbb{C}^n$ valued r.v. which may depend on the random variable θ_0 and we shall denote $\mathcal{V}_0$ the joint distribution of (x_0, θ_0). We assume the random variables $\theta(t+s)$, $t \geq 0$, $s > 0$, conditionally independent of x_0, given $\theta(t)$. Whenever necessary we shall write $E_{\mathcal{V}_0}(.)$ to indicate the expected value of some function of $x(t)$ given by (1) with assigned joint distribution $\mathcal{V}_0$.

The following definitions are essential to solve the $\mathcal{H}_\infty$ control problem.

Definition 1 *(Stochastic stabilizability) We say that the system (A, B, Λ) is stochastically stabilizable (SS) if there exists $K \in \mathcal{H}_\infty^{n,m}$ such that, for any joint distribution $\mathcal{V}_0$ of the initial conditions (θ_0, x_0) we have that*

$$\int_0^\infty E\left[\|x(t)\|^2\right] dt < \infty \tag{2}$$

where $x(t)$ is given by (1) with $u(t) = -K_{\theta(t)} x(t)$ and $w(t) = 0$, i.e,

$$\dot{x}(t) = F_{\theta(t)} x(t)\ , \quad t > 0 \tag{3}$$

with $F_{\theta(t)} = A_{\theta(t)} - B_{\theta(t)} K_{\theta(t)}$. In this case we say that (3) is stochastically stable and K stabilizes(A, B, Λ).

Definition 2 *(Stochastic detectability) Set* $C = (C_1, C_2, ...) \in \mathcal{H}_\infty^{n,r}$. *We say that the system* (C, A, Λ) *is stochastically detectable (SD) if there exists* $H \in \mathcal{H}_\infty^{r,n}$ *such that, for any joint distribution* $\mathcal{V}_0$ *of the initial conditions* (θ_0, x_0) *we have that*

$$\int_0^\infty E\left[\|x(t)\|^2\right] dt < \infty \tag{4}$$

where x(t) is given by

$$\dot{x}\ (t) = F_{\theta(t)} x(t)\ ,\quad t > 0 \tag{5}$$

with $F_{\theta(t)} = A_{\theta(t)} - H_{\theta(t)} C_{\theta(t)}$. *In this case we say that (5) is stochastically detectable and H stabilizes* (C, A, Λ).

For arbitrary initial condition (θ_0, x_0) let us consider the homogeneous dynamic system

$$\begin{cases} \dot{x}\ (t) = F_{\theta(t)} x(t) & t > 0 \\ x(0) = 0, \quad \theta(0) = \theta_0 \end{cases} \tag{6}$$

where $F = (F_1, F_2, ...) \in \mathcal{H}_\infty^n$, and define $Q(t) = (Q_1(t), Q_2(t), ...),\ t > 0$,where

$$Q_i(t) = E[x(t)x(t)^* 1_{\{\theta(t)=i\}}] \in \mathbb{M}(\mathbb{C}^n)^+,\ \ i = 1, 2, ... \tag{7}$$

In addition, for $H = (H_1, H_2, ...) \in \mathcal{H}_1^n$, we define the operator $\mathcal{D}$ such that $\mathcal{D}(H) = (\mathcal{D}_1(H), \mathcal{D}_2(H), ...)$ where

$$\mathcal{D}_i(H) = H_i F_i^* + F_i H_i + \sum_{j=1}^{\infty} \lambda_{ji} H_j \qquad i = 1, 2, ... \tag{8}$$

The next proposition is due to Fragoso and Baczynski [11].

Proposition 1 *Let* $x(t)$ *be given by (6) with arbitrary* $F \in \mathcal{H}_\infty^n$. *Then* $Q(t), t \geq 0$, *defined by (7), belongs to* $\mathcal{H}_1^{n+}$ *and satisfies the Banach space Riccati differential equation*

$$\begin{cases} \dot{Q}\ (t) = \mathcal{D}(Q(t)), \quad t > 0 \\ Q(0) = Q^0 \in \mathcal{H}_1^{n+}, Q_i^0 = E[x_0 x_0^* 1_{\{\theta_0 = i\}}], \quad i = 1, 2, ... \end{cases} \tag{9}$$

where $\mathcal{D}$ *is given by* (8) .

The following lemma is due to Fragoso and Baczynski [11] and is a fundamental tool for the homogeneous case.

Lemma 2 *Consider the operator* $\mathcal{D}$ *given by (8) for some* $K = (K_1, K_2, ...) \in \mathcal{H}_\infty^{n,m}$, $C = (C_1, C_2, ...) \in \mathcal{H}_\infty^{n,r}$, *and* $H = (H_1, H_2, ...) \in \mathcal{H}_\infty^{r,n}$. *The system* (A, B, Λ) *is SS with stabilizing* K *if and only if*

$$sup\{\mathrm{Re}\,\lambda\ ; \lambda \in \sigma(\mathcal{D})\} < 0 \tag{10}$$

where $F_i = A_i - B_i K_i \quad i = 1, 2, ...$
Similarly the system (C, A, Λ) *is SD with stabilizing H if and only if*

$$sup\{\mathrm{Re}\,\lambda\ ; \lambda \in \sigma(\mathcal{D})\} < 0 \tag{11}$$

where $F_i = A_i - H_i C_i \quad i = 1, 2, ...$

Consider now (1) with $u(t) = -K_{\theta(t)} x(t)$ and the operator $\mathcal{D}$ as defined in (8) with $F_i = A_i - B_i K_i$.

Lemma 3 *For arbitrary initial condition* (θ_0, x_0) *let us consider the inhomogeneous dynamic system*

$$\begin{cases} \dot{x}\ (t) = F_{\theta(t)} x(t) + D_{\theta(t)} w(t) & t > 0 \\ x(0) = x_0, \quad \theta(0) = \theta_0 \end{cases} \tag{12}$$

where $F = (F_1, F_2, ...) \in \mathcal{H}_\infty^n$.*For every* $t \geq 0$ *let's define* $t_0 = 0$ *and the random variable* $\xi = (\theta(t_{m-1}), ..., \theta(t_1),\ t_{m-1}, ..., t_1)$. *Then, the trajectory of the state process* $\{x\}$ *with jump times* $t_1 < t_2 < ... < t_m,\ \ m = 1, 2, ...,$ *is represented by*

$$\begin{aligned} x(t) &= \exp(F_{\theta(t_{m-1})}(t - t_{m-1})) \exp(F_{\theta(t_{m-2})}(t_{m-1} \\ &-t_{m-2}))... \exp(F_{\theta(t_1)}(t_2 - t_1)) \exp(F_{\theta_0} t_1) + \textstyle\int_{t_{m-1}}^{t} \\ &\exp(F_{\theta(t_{m-1})}(t - \tau)) D_{\theta(\tau)} w(\tau) d\tau + \sum_{j=1}^{m-1} \textstyle\int_{t_{m-j-1}}^{t_{m-j}} \exp(\\ &F_{\theta(t_{m-1})}(t - t_{m-1})) \exp(F_{\theta(t_{m-2})}(t_{m-1} - t_{m-2}))... \\ &... \exp(F_{\theta(t_{m-j})}(t_{m-j+1} - t_{m-j})) \exp(F_{\theta(t_{m-j-1})}(t_{m-j} \\ &-\tau)) D_{\theta(\tau)} w(\tau) d\tau \end{aligned}$$

Proof: See [11], or [18]. ∎

The next proposition is a key result in this work.

Proposition 4 *Consider the operator given in (8). Then* $sup\{\mathrm{Re}\,\lambda\ ; \lambda \in \sigma(\mathcal{D})\} < 0$ *if and only if* $x \in \mathcal{C}^n$ *for every* $w \in C_D^p$, $x_0 \in \mathcal{C}_0^n$, *and* $\theta_0 \in \Theta_0$, *where x is defined as in (1).*

Proof: ($\Rightarrow$) *A consequence of Lemma 3 and Proposition 1.* ($\Leftarrow$) *A consequence of Lemma 2.* ∎

For $C = (C_1, C_2, ...) \in \mathcal{H}_\infty^{n,r}$, set $M_i = C_i^* C_i$, $i = 1, 2, ...,$ $M = (M_1, M_2, ...) \in \mathcal{H}_\infty^{n+}$. Let's define for $H = (H_1, H_2, ...) \in \mathcal{H}_\infty^{n+}$ the linear operators $\mathcal{E}(H) = (\mathcal{E}_1(H), \mathcal{E}_2(H), ...)$, $\mathcal{G}(H) = (\mathcal{G}_1(H), \mathcal{G}_2(H), ...)$ and the non linear operator $\overline{\mathcal{T}}(H) = (\overline{\mathcal{T}}_1(H), \overline{\mathcal{T}}_2(H), ...)$, where

$$\mathcal{E}_i(H) = \sum_{j=1, i\neq j}^{\infty} \lambda_{ij} H_j; \qquad \mathcal{G}_i(H) = B_i^* H_i \tag{13}$$

$$\begin{aligned} \overline{\mathcal{T}}_i(H) &= M_i + A_i^* H_i + H_i A_i - H_i B_i B_i^* H_i + \mathcal{E}_i(H) + \lambda_{ii} H_i \\ &= M_i + (A_i - B_i \mathcal{G}_i(H))^* H_i + H_i (A_i - B_i \mathcal{G}_i(H)) \\ &+ \quad \mathcal{G}_i(H)^* \mathcal{G}_i(H) + \sum_{j=1}^{\infty} \lambda_{ij} H_j. \end{aligned}$$

The following result is a straightforward modification of theorem 37 in [11].

Proposition 5 *Suppose* (A, B, Λ) *stochastically stabilizable and* (C, A, Λ) *stochastically detectable. Then there exists only one solution* $L = (L_1, L_2, ...) \in \mathcal{H}_\infty^{n+}$ *such that for each* $i \in \mathbb{N}$,

$$\begin{aligned} \overline{\mathcal{T}}_i(L) &= M_i + A_i^* L_i + L_i A_i - L_i B_i B_i^* L_i + \mathcal{E}_i(L) + \lambda_{ii} L_i \\ &= M_i + (A_i - B_i J_i)^* L_i + L_i (A_i - B_i J_i) + J_i^* J_i \\ &+ \quad \sum_{j=1}^{\infty} \lambda_{ij} L_j = 0 \end{aligned}$$

where $J_i = B_i^* L_i$. Furthermore, $J = (J_1, J_2, ...)$ stabilizes (A, B, Λ).

For $(x_0, w, q) \in (\mathcal{C}_0^n \times C_D^p \times \mathcal{C}^m)$, $K \in \mathcal{H}_\infty^{n,m}$, we define the linear operator $X_K(\theta_0, \cdot)$ from $\mathcal{C}_0^n \times C_D^p \times \mathcal{C}^m$ to $\mathcal{C}^n$ as

$$X_K(\theta_0, x_0, w, q) = x \tag{14}$$

where x is defined as in (1) above with $u(t) = -K_{\theta(t)}x(t) + q(t)$, $q(t) \in \mathcal{C}^m$. From proposition 4 it is clear that if $sup\{\mathrm{Re}\,\lambda\ ; \lambda \in \sigma(\mathcal{D})\} < 0$, then $x \in \mathcal{C}_0^n$ and $X_K(\theta_0, \cdot) \in \mathbb{B}[\mathcal{C}_0^n \times C_D^p \times \mathcal{C}^m\ , \mathcal{C}^n]$.

Define now $z : [0, \infty) \ni t \mapsto z(t) \in \mathcal{C}^r$ as

$$z(t) = C_{\theta(t)}x(t) + N_{\theta(t)}u(t) \tag{15}$$

where $C_{(\cdot)}$, $N_{(\cdot)}$, are such that for each $i \in \mathbb{N}$, $\theta(t) = i$, $C_{\theta(t)} = C_i$, $N_{\theta(t)} = N_i$, which are constant matrices. In addition, $C = (C_1, C_2, ...) \in \mathcal{H}_\infty^{n,r}$, $N = (N_1, N_2, ...) \in \mathcal{H}_\infty^{m,r}$ satisfy:

$$C^*_{(\theta(t)=i)}N_{(\theta(t)=i)} = 0; \quad N^*_{(\theta(t)=i)}N_{(\theta(t)=i)} = I, \tag{16}$$

for each $i \in \mathbb{N}$. We have that $z \in \mathcal{C}^r$ and the operator $Z_K(\theta_0, \cdot) \in \mathbb{B}[\mathcal{C}_0^n \times C_D^p \times \mathcal{C}^m\ , \mathcal{C}^r]$ is defined as $Z_K(\theta_0, x_0, w, q) = z$, where z is defined as in (15). Finally let us define the operators $X_K^0(\theta_0, \cdot)$ and $Z_K^0(\theta_0, \cdot)$, in $\mathbb{B}[C_D^p\ , \mathcal{C}^n]$ and $\mathbb{B}[C_D^p\ , \mathcal{C}^r]$ respectively, as

$$X_K^0(\theta_0, \cdot) = X_K(\theta_0, 0, w, 0); \quad Z_K^0(\theta_0, \cdot) = Z_K(\theta_0, 0, w, 0)$$

Consider the system (1) and let $\delta > 0$ be arbitrarily chosen. Our H_∞control problem is to find necessary and sufficient conditions to assure existence of a control $K = (K_1, K_2, ...) \in \mathcal{H}_\infty^{n,m}$ that stabilizes (A, B, Λ) according to the definition (1) and is such that

$$\sup_{\theta_0 \in \Theta_0} \left\| Z_K^0(\theta_0, \cdot) \right\| < \delta. \tag{17}$$

for $Z_K^0(\theta_0, \cdot)$ defined as in (15) and (17).

4 Main results

Theorem 6 *Suppose that (C, A, Λ) is SD and consider $\delta > 0$ fixed. Then there exists $K = (K_1, K_2, ...) \in \mathcal{H}_\infty^{n,m}$, that stabilizes (A, B, Λ) and is such that $\sup_{\theta_0 \in \Theta} \left\| Z_K^0(\theta_0, \cdot) \right\| < \delta$, if and only if there exists $P = (P_1, P_2, ...) \in \mathcal{H}_\infty^{n+}$ satisfying:*

$$(i) M_i + (A_i - B_iK_i + \frac{1}{\delta}D_iG_i)^* P_i + P_i(A_i - B_iK_i + \frac{1}{\delta}D_iG_i)$$
$$+ K_i^*K_i - G_i^*G_i + \sum_{j=1}^{\infty} \lambda_{ij}P_j = 0 \quad forall\ i \in \mathbb{N}.$$

where

$$K_i = B_i^*P_i; \quad G_i = \frac{1}{\delta}D_i^*P_i,\ for\ all\ i \in \mathbb{N}.$$

(ii) $sup\{\mathrm{Re}\,\lambda\ ; \lambda \in \sigma(\mathcal{D})\} < 0$ where $\mathcal{D}$ is defined as in and (8), with $F_i = A_i - B_iK_i + \frac{1}{\delta}D_iG_i$, for all $i \in \mathbb{N}$.

Proof: $\Leftarrow$ (Sufficiency): Is a straightforward consequence of Propositions 7, 8 and 9 below, which is proved in [18].

Proposition 7 *Suppose that (i) and (ii) of the theorem hold. Then $K = (K_1, K_2...)$ stabilizes (A, B, Λ).*

For the remainder of this section we consider, for arbitrary $\theta_0 \in \Theta_0$, $w \in C_D^p$,

$x = X_K^0(\theta_0, w)$, $\tilde{z} = Z_K^0(\theta_0, w)$.

Proposition 8 *Suppose that (i) and (ii) of the theorem 6 hold. Then*

$$\|z\|_2^2 = \delta^2(\|w\|_2^2 - \|r\|_2^2) \tag{18}$$

where $r \in C_D^p$ is defined as

$$r(t) = \frac{1}{\delta}G_\theta(t)x(t) - w(t) \tag{19}$$

Define now the operator $\widetilde{W}(\theta_0, \cdot) \in \mathbb{B}[C_D^p]$ as $\widetilde{W}(\theta_0, \cdot) = \frac{1}{\delta}GX_K^0(\theta_0, \cdot) - I$, that is, for $w \in C_D^p$, $\widetilde{W}(\theta_0, w) = \widetilde{w}$, where $w(t) = \frac{1}{\delta}G_{\theta(t)}x(t) - w(t)$, for any $t \in [0, \infty)$.

Proposition 9 *Suppose that (i) and (ii) of the theorem are satisfied. Then the operator $\widetilde{W}$ defined above is invertible.*

$\Rightarrow$ (Necessity):

The necessity part is a consequence of the following propositions, which is proved in [18].

Consider $J = (J_1, J_2, ...)$ and $L = (L_1, L_2, ...) \in \mathcal{H}_\infty^{n+}$ as in proposition 5. For any $(x_0, w, q) \in \mathcal{C}_0^n \times C_D^p \times \mathcal{C}^m$, $\theta_0 \in \Theta_0$ where w is such that $w : t \in [0, \infty) \mapsto w(t) \in L_2(\mathbb{C}^p)$, $q : t \in [0, \infty) \mapsto q(t) \in L_2(\Omega, \mathcal{F}_t, \mathcal{P}; \mathbb{C}^m)$. Set $x = X_J(\theta_0, x_0, w, q)$ and $z = Z_J(\theta_0, x_0, w, q)$ as defined in (14) and (15) respectively, replacing K for J. In this case $u(t) = -J_{\theta(t)}x(t) + q(t)$. For each $\theta_0 \in \Theta_0$ we define the cost given by the operator $\mathcal{J}(\theta_0, x_0, w, q) : \mathcal{C}_0^n \times C_D^p \times \mathcal{C}^m \to \mathbb{R}$ as

$$\mathcal{J}(\theta_0, x_0, w, q) = \|Z_J(\theta_0, x_0, w, q)\|_2^2 - \delta^2 \|w\|_2^2$$
$$= \int_0^\infty E_{\theta_0, x_0} \left(\left\|C_{\theta(t)}x(t)\right\|^2 \right.$$
$$+ \quad \left\|-J_{\theta(t)}x(t) + q(t)\right\|^2 - \delta^2 \|w(t)\|^2 dt.$$

We want to solve the minimax problem $\widehat{\mathcal{J}}(\theta_0, x_0) = \sup_{w \in C_D^p} \inf_{q \in \mathcal{C}^m} \mathcal{J}(\theta_0, x_0, w, q)$ We shall first solve the minimization problem

Proposition 10 *Consider $\theta_0 \in \Theta_0$ fixed and $x_0 \in \mathcal{C}_0^n$. Suppose that (C, A, Λ) is SD and there exists $\overline{K} = (\overline{K}_1, \overline{K}_2, ...) \in \mathcal{H}_\infty^{n,m}$ such that stabilizes (A, B, Λ) and $\sup_{\theta_0 \in \Theta_0} \left\| Z_{\overline{K}}^0(\theta_0, \cdot) \right\| < \delta$ for some $\delta > 0$. Then, there exists $Q : C_D^p \to \mathcal{C}^m$ such that $Q \in \mathbb{B}[C_D^p, \mathcal{C}^m]$. Moreover,*

$$\widetilde{\mathcal{J}}(\theta_0, x_0, w) = \inf_{q \in \mathcal{C}^m} \mathcal{J}(\theta_0, x_0, w, q) = \mathcal{J}(\theta_0, x_0, w, \widetilde{q})$$

where $\widetilde{q} = Q(w)$.

Let us now solve the maximization problem.

Proposition 11 *Consider $\theta_0 \in \Theta_0$ fixed. Suppose that (C, A, Λ) is SD and there exists $\overline{K} = (\overline{K}_1, \overline{K}_2, ...) \in \mathcal{H}_\infty^{n,m}$ such that stabilizes (A, B, Λ) and $\sup_{\theta_0 \in \Theta_0} \left\| Z_{\overline{K}}^0(\theta_0, \cdot) \right\| < \delta$ for some $\delta > 0$. For fixed $x_0 \in \mathcal{C}_0^n$ there exists a unique element $\widehat{w} \in C_D^p$ such that*

$$\widehat{\mathcal{J}}(\theta_0, x_0) = \sup_{w \in C_D^p} \widetilde{\mathcal{J}}(\theta_0, x_0, w) = \widetilde{\mathcal{J}}(\theta_0, x_0, \widehat{w})$$

Furthermore, for some $\mathcal{W}(\theta_0, .) \in \mathbb{B}[\mathcal{C}_0^n, C_D^p]$, $\widehat{w} = \mathcal{W}(\theta_0, x_0)$.

From propositions 10 and 11 we have that for each $\theta_0 \in \Theta_0$, $x_0 \in \mathcal{C}_0^n$ there exists $\widehat{w} \in C_D^p$ and $\widehat{q} \in \mathcal{C}^m$ such that

$$\begin{aligned} \widehat{\mathcal{J}}(\theta_0, x_0) &= \sup_{w \in C_D^p} \inf_{q \in \mathcal{C}^m} \mathcal{J}(\theta_0, x_0, w, q) \\ &= \mathcal{J}(\theta_0, x_0, \widehat{w}, \widehat{q}) \end{aligned} \quad (20)$$

For each $i \in \mathbb{N}$, $x_0 \in \mathbb{C}^n$, we can define now, $P = (P_1, P_2, ...)$ with $P_i \in \mathbb{B}[\mathbb{C}^n]$, and $P_i \geq 0$ as

$$\widehat{J}(i, x_0) = x_0^* P_i x_0 \geq 0 \quad (21)$$

In order to prove that $P = (P_1, P_2, ...)$ as defined above satisfies the conditions (i) and (ii) of Theorem 6, we shall consider the truncated minimax problem.

Let us turn attention to the Banach space Riccati differential equation with finite time horizon.

For $C = (C_1, C_2, ...) \in \mathcal{H}_\infty^{n,r}$, set $M_i = C_i^* C_i$, i = 1,2,..., $M = (M_1, M_2, ...) \in \mathcal{H}_\infty^{n+}$. For $H = (H_1, H_2, ...) \in \mathcal{H}_\infty^{n+}$ let us define the non linear operator $\mathcal{T}(H) = (\mathcal{T}_1(H), \mathcal{T}_2(H), ...)$, where

$$\begin{aligned} \mathcal{T}_i(H) = M_i + A_i^* H_i + H_i A_i - H_i B_i B_i^* H_i \\ + \mathcal{E}_i(H) + \lambda_{ii} H_i + \tfrac{1}{\delta^2} H_i D_i D_i^* H_i \end{aligned} \quad (22)$$

For finite T and $L \in \mathcal{H}_\infty^{n+}$ arbitrarily fixed, the Banach space Riccati differential equation is written as

$$\left\{ \begin{array}{l} \dot{P}^T(t) + \mathcal{T}\left(P^T(t)\right) = 0, \quad t \in (0, T) \\ P^T(T) = L \end{array} \right. \quad (23)$$

where

$$P^T(t) = (P_1^T(t), P_2^T(t), ...) \quad (24)$$

The proposition below can be found in [11] and guarantees the existence of solution to the Riccati differential equation (23).

Proposition 12 *For $T \in [0, \infty)$ and terminal data $L \in \mathcal{H}_\infty^{n+}$ arbitrarily fixed there exists a solution $P^T(.) : [0, \infty) \to \mathcal{H}_\infty^{n+}$ for (23), continuous for $t \in [0, T]$ and continuous differentiable for $t \in (0, T)$. Moreover, this solution is unique (within the class of solutions with these properties).*

By its turn, the Riccati differential equation (23) can be written as (see [11])

$$\left\{ \begin{array}{l} \frac{d}{dt} P^{T-t}(0) + \mathcal{T}\left(P^{T-t}(0)\right) = 0, \quad t \in (0, T) \\ P^0(0) = L \end{array} \right. \quad (25)$$

For simplicity, we shall note in the sequel $P^0(0) = P^0$, and $P^T(0) = P^T$. So, let us define the sequences:

$$\begin{array}{l} P^T = (P_1^T, P_2^T, ...), \quad P_i^T \in \mathbb{M}(\mathbb{C}^n) \\ P^0 = (P_1^0, P_2^0, ...) = (L_1, L_2, ...) \\ K^T = (K_1^T, K_2^T, ...), \quad K_i^T \in \mathbb{M}(\mathbb{C}^n, \mathbb{C}^m) \\ G^T = (G_1^T, G_2^T, ...), \quad G_i^T \in \mathbb{M}(\mathbb{C}^n, \mathbb{C}^p) \end{array}$$

where $K_i^T = B_i^* P_i^T$; $G_i^T = \frac{1}{\delta} D_i^* P_i^T$, $i \in \mathbb{N}$. We also define $\mathcal{J}^T$ as

$$\begin{aligned} \mathcal{J}^T(\theta_0, x_0, w, q) = \textstyle\int_0^T E_{\theta_0, x_0}(\left\| C_{\theta(r)} x(r) \right\|^2 \\ + \left\| -J_{\theta(r)} x(r) + q(r) \right\|^2 - \delta^2 \left\| w(r) \right\|^2) dr \\ + x(T)^* L_{\theta(T)} x(T) \end{aligned} \quad (26)$$

$$\widehat{\mathcal{J}}^T(\theta_0, x_0) = \sup_{w \in C_T^p} \inf_{q \in \mathcal{C}^m} \mathcal{J}^T(\theta_0, x_0, w, q) \quad (27)$$

Proposition 13 *For $u(t) = -J_{\theta(t)} x(t) + q(t)$, where J is defined as in Proposition 5, the cost defined in (26) reads as follows*

$$\begin{aligned} &\mathcal{J}^T(\theta_0, x_0, w, q) = \\ &E[\textstyle\int_0^T E_{\theta_0, x_0}(\left\| -\left(J_{\theta(r)} - K_{\theta(r)}^{T-r}\right) x(r) + q(r) \right\|^2 \\ &-\delta^2 \left\| w(r) - \tfrac{1}{\delta} G_{\theta(r)}^{T-r} x(r) \right\|^2) dr] + E\left(x_0^* P_{\theta_0}^T x_0\right) \end{aligned} \quad (28)$$

with $P_{\theta(t)}^{T-t} \in \mathcal{H}_\infty^{n+}$ satisfying (25).

Proposition 14 *Consider $\theta_0 \in \Theta_0$ fixed. Suppose that (C, A, Λ) is SD and there exists $\overline{K} = (\overline{K}_1, \overline{K}_2 ...) \in \mathcal{H}_\infty^{n,m}$ such that it stabilizes (A, B, Λ) and $\sup_{\theta_0 \in \Theta} \left\| Z_{\overline{K}}^0(\theta_0, \cdot) \right\| < \delta$, for some $\delta > 0$. Then for each $T > 0$ we have*

$$\widehat{J}^T(\theta_0, x_0) = J(\theta_0, x_0, \widehat{w}^T, \widehat{q}^T) = E\left(x_0^* P_{\theta_0}^T x_0\right) \quad (29)$$

where

$$\widehat{w}^T(t) = \left\{ \begin{array}{ll} \frac{1}{\delta} G_{\theta(t)}^{T-t} \, \widehat{x}^T(t) & t \in [0, T] \\ 0 & t > T \end{array} \right. \quad (30)$$

$$\widehat{q}^T(t) = \left\{ \begin{array}{ll} \left(J_{\theta(t)} - K_{\theta(t)}^{T-t}\right) \widehat{x}^T(t) & t \in [0, T] \\ 0 & t > T \end{array} \right. \quad (31)$$

and

$$\widehat{x}^T = X_J(\theta_0, x_0, \widehat{w}^T, \widehat{q}^T). \quad (32)$$

Proof: Follows imediatelly from Proposition 13. ∎

Proposition 15 $[0, T] \ni t \to P^T(t) = P \in \mathcal{H}_\infty^n$ *satisfies*

$$\left\{ \begin{array}{ll} \dot{P}^T(t) + \mathcal{T}\left(P^T(t)\right) = 0, & t \in (0, T) \\ P^T(T) = P \end{array} \right. \quad (33)$$

if and only if P satisfies $\mathcal{T}(P) = 0$.

Definition 3 *We say that $P = (P_1, P_2, ...)$ is a positive semidefinite solution of the ICARE if and only if $P \in \mathcal{H}_\infty^{n+}$ and satisfies the ICARE*

$$\mathcal{T}(P) = 0 \tag{34}$$

Proposition 16 *Suppose (A, B, Λ) is SS and (C, A, Λ) is SD. Then, for arbitrary terminal condition $L \in \mathcal{H}_\infty^{n+}$ the value $\widetilde{P}_i^T(0) = \widetilde{P}_i^T$ of the unique solution $\widetilde{P}^T(t) \in \mathcal{H}_\infty^{n+}$, $t \in [0, T]$ to (23), converges to $P_i \in \mathbb{M}(\mathbb{C}^n)^+$(defined in (21)), as $T \to \infty$, for each $i \in \mathbb{N}$. Furthermore $P(0) = P = (P_1, P_2, ...) \in \mathcal{H}_\infty^{n+}$ and satisfies the ICARE (34).*

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