Robust Output Feedback Control of Incompletely Observable Nonlinear Systems Without Input Dynamic Extension[1]

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Abstract

We introduce a new output feedback controller for general MIMO nonlinear systems which are only observable on regions of the state and input spaces. Unlike previous approaches, we do not add integrators at the input side of the system, and thus avoid the need to design a stabilizing control law for a higher order system. Robustness with respect to a class of time-varying disturbances is guaranteed, and the performance of any state feedback controller designed to achieve closed-loop stability with respect to a set (e.g., a robust controller) is recovered.

1 Introduction

The past decade has witnessed a growing interest in the output feedback control problem and the generalization and exploitation of a nonlinear separation principle. The main ingredients of the approach leading to the most recent results (see, e.g., [1, 2, 3]) can be summarized as follows: 1) A chain of integrators is added at the input side of the system, and the state feedback control law is redesigned to stabilize the "extended system." 2) A high-gain observer is designed to estimate the derivatives of the system output and, together with the integrator states, these are used to obtain an estimate of the state of the system. 3) This estimate is employed in the state feedback control law and control saturation is used to guarantee its global boundedness. Unfortunately, this approach cannot be used when the system is not uniformly completely observable (UCO).

In this work we extend the theory developed in [4, 5] (where a modification to points 2 and 3 above was proposed) to deal with MIMO nonlinear systems affected by time-varying disturbances, and for which a state feedback control law can be designed that robustly stabilizes the system (hence, a suitable compact set $\mathcal{N}$ is made attractive and positively invariant by the state feedback control law; notice the similarity with the work in [6]). A major difference between this work and other approaches, including our previous results, resides in a modification of point 1 above which entails employing a high-gain (or, alternatively, sliding mode) observer for the estimation of the control input derivatives (or, equivalently, the states of the integrators used in the input dynamic extension in point 1). This modification results in a simplification of the control design problem in that the original state feedback controller can be *directly* employed by the output feedback controller (i.e., , we eliminate the need to design a control law for the higher-order "extended system" in point 1 above). We prove that, in the presence of disturbances, this approach guarantees uniform ultimate boundedness (UUB) of the closed-loop system trajectories with respect to $\mathcal{N}$ (in complete analogy with the result of Theorem 4 in [6], but here we do not investigate conditions needed on the disturbance so that asymptotic stability is recovered). When no disturbance affects the system, it is shown that a design that is based on a sliding mode observer recovers the asymptotic stability of the closed-loop system with respect to $\mathcal{N}$, while the high-gain observer achieves UUB.

Due to space constraints, in what follows all the proofs are omitted.

2 Problem Formulation

Consider the following dynamical system,

$$\begin{aligned} \dot{x} &= f(x,u,d) \\ y &= h(x,u) \end{aligned} \qquad (1)$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$, f and h are sufficiently smooth functions, and $d \in \Delta \subset \mathbb{R}^q$ is an unknown time-varying disturbance contained in a compact set Δ. When no disturbance acts on the plant we replace $d(t)$ by $d^0 \in \Delta$, a known constant "nominal value" of $d(t)$. Our objective is to construct a robust output

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feedback controller which makes a known compact set uniformly asymptotically stable with a desired region of attraction. The observability properties of system (1) are characterized by the observability mapping

$$y_e \triangleq \left[y_1, \ldots, y_1^{(k_1-1)}, \ldots, y_p, \ldots, y_p^{(k_p-1)}\right] = \begin{bmatrix} h_1(x,u) \\ \vdots \\ \varphi_1^{k_1-1}\left(x, u, \ldots, u^{(k_1-1)}, d, \ldots, d^{(k_1-2)}\right) \\ \vdots \\ h_p(x,u) \\ \vdots \\ \varphi_p^{k_p-1}\left(x, u, \ldots, u^{(k_p-1)}, d, \ldots, d^{(k_p-2)}\right) \end{bmatrix} \triangleq \mathcal{H}\left(x, z, d, \ldots, d^{(n_d-1)}\right) \quad (2)$$

where $z \triangleq [u_1, \ldots, u_1^{(n_1-1)}, \ldots, u_m, \ldots, u_m^{(n_m-1)}]^\top \in \mathbb{R}^{n_u}$, $\sum_{i=1}^{p} k_i = n$, $n_u \triangleq n_1 + \ldots + n_m$, $0 \le n_i \le \max\{k_1, \ldots, k_p\}$, $0 \le n_d \le \max\{k_1, \ldots, k_p\} - 1$ (when $\mathcal{H}$ does not depend on u_i or d then, respectively, we set $n_i = 0$, or $n_d = 0$). Note that the vector z contains only the derivatives of u that end up appearing in the mapping $\mathcal{H}$ for the application at hand. The functions φ_i^j are defined as follows,

$$\varphi_i^j(x, u, \ldots, u^{(j)}, d, \ldots, d^{(j-1)}) = \frac{\partial \varphi_i^{j-1}}{\partial x} f(x, u, d(t)) + \sum_{k=0}^{j-1} \frac{\partial \varphi_i^{j-1}}{\partial u^{(k)}} u^{(k+1)} + \sum_{k=0}^{j-2} \frac{\partial \varphi_i^{j-1}}{\partial d^{(k)}} d^{(k+1)},$$
$$\varphi_i^0(x,u) = h_i(x,u), \text{ for } i = 1, \ldots, p,\ j = 1, \ldots, k_i - 1$$

Assumption A1 (*Disturbance*). The vector fields f, h have the property that

$$\frac{\partial \varphi_i^j}{\partial d} = 0, \text{ for } i = 1, \ldots, p,\ j = 1, \ldots, k_i - 1, \quad (3)$$

on the domain of definition of φ_i^j. Therefore, (2) can be written as $y_e = \mathcal{H}(x, z)$.

Assumption A2 (*Observability*). System (1) is observable over the set $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$, i.e., there exists a set of indices $\{k_1, \ldots, k_p\}$ such that the mapping $y_e = \mathcal{H}(x, z)$ is invertible with respect to x and its inverse is smooth, for all $x \in \mathcal{X}$, $z \in \mathcal{U}$.

This assumption is a direct generalization of the observability assumption found in [4, 5] to the case of MIMO nonlinear systems. Notice that A2 relaxes analogous assumptions found in the literature (e.g., [7, 1, 3]), where the authors require UCO, i.e., $\mathcal{X} \times \mathcal{U} = \mathbb{R}^n \times \mathbb{R}^{n_u}$.

Assumption A3 (*Robust Dynamic Stabilizability*). There exists a dynamic state feedback controller

$$\begin{aligned} \dot{x}_c &= f_c(x_c, x) \\ u &= h_c(x_c, x) \end{aligned} \quad (4)$$

where $x_c \in \mathbb{R}^{n_c}$, f_c, h_c are sufficiently smooth, and such that $\mathcal{N} \subset \mathbb{R}^{n+n_c}$ is a positively invariant, uniformly asymptotically stable compact set for the closed-loop system

$$\begin{aligned} \dot{x}_c &= f_c(x_c, x) \\ \dot{x} &= f(x, h_c(x_c, x), d) \end{aligned} \quad (5)$$

for all $d(t) \in \Delta$. Let $\mathcal{D}$ be the domain of attraction of $\mathcal{N}$.

In [6] the authors use a similar assumption, with the additional requirement that f_c and h_c be globally bounded functions of x, a requirement that is not restrictive since it can be fulfilled by using saturation functions. On the other hand, owing to the structure of our output feedback controller, this restriction is not needed in our framework. Notice that, while in [6] $d(t)$ is a measurable exogenous signal, we assume $d(t)$ to be an unmeasurable disturbance (indeed we do not include $d(t)$ as an input to the control law (4)), and hence our control objective is somewhat different.

Next, define the closed-loop system

$$\begin{aligned} \dot{\chi} &= F(\chi, u, d(t)) \\ y &= H(\chi) \end{aligned} \quad (6)$$

where $\chi = [x_c^\top, x^\top]^\top$, the $\dot{\chi}$ equation is defined by (5), and $H(\chi) = h(x, h_c(x_c, x))$. Assumption A3 implies that $\mathcal{N}$ is positively invariant and a uniformly asymptotically stable set of (6) when $u = h_c(\chi)$.

3 Cascade Observer Designs

The output feedback controller design for UCO systems involves adding integrators at the input side and redesigning a stabilizing control law for the extended system. This technique has the advantage of providing the exact knowledge of the input derivatives which are needed for observer design. However, such a dynamic extension may introduce a significant complication in the design of the control law for the extended system (e.g., when using integrator backstepping on u, one may observe an explosion of nonlinear terms in v, particularly when n_u is high). In [10] two dynamically decoupled high-gain observers are employed to estimate the derivatives of y (vector y_e) *and* those of u (vector z). By assuming the *explicit* knowledge of $\mathcal{H}^{-1}$ an estimate of x is then calculated as $\hat{x} = \mathcal{H}^{-1}(\hat{y}_e, \hat{z})$ (the hats are used to denote the estimates); see Figure 1. The dynamic decoupling between the two observers allows for a separate convergence analysis, and it is shown that the state estimation error converges with arbitrarily fast rate to a small neighborhood of the origin. In [10] the output feedback control problem was not taken in consideration and, to the best of our knowledge, this idea has not been further investigated in the literature. In previous work [4, 5] we showed how to avoid the need for the knowledge of $\mathcal{H}^{-1}$ by introducing a nonlinear observer for the state x. Furthermore, by projecting the observer estimates onto an appropriate compact set, we showed how one can achieve a separation principle for incompletely observable nonlinear systems. By following the same methodology, and using an idea similar to the one found in [10], we now aim at simplifying the output feedback controller design, particularly the construction of the stabilizing state feedback control law, by avoiding the use of integrators at the input side of the system. In order to do that we have to deal with two problems. First, by working in the original state

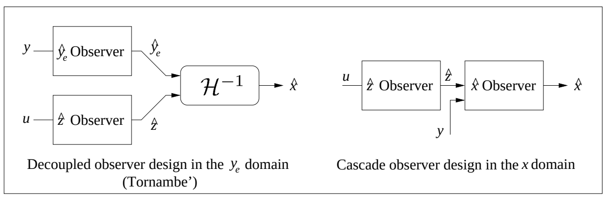

Figure 1: Observer structures of Tornambé and here.

domain (rather than the y_e domain) the two observers are in cascade, and therefore their dynamics, unlike in [10], are coupled (see Figure 1). Second, by employing a high-gain observer to estimate the input derivatives, we introduce a small asymptotic estimation error, whose effects on the closed-loop stability must be analyzed. To this end, we introduce two observer designs, both involving modifications to the original observer structure introduced in [4, 5].

3.1 Cascade Design 1: High-Gain Observer For The Input Derivatives

Consider the following nonlinear observer,

$$\dot{\hat{x}} = f(\hat{x}, \hat{u}, d^0) + \left[\frac{\partial \mathcal{H}(\hat{x}, \hat{z})}{\partial \hat{x}}\right]^{-1} \Big[(\mathcal{E}^x)^{-1} L\,(y - \hat{y}) - \frac{\partial \mathcal{H}(\hat{x}, \hat{z})}{\partial \hat{z}} (\mathcal{E}^z)^{-1} K C(u - \hat{u})\Big] \qquad (7)$$
$$\hat{y} = h(\hat{x}, \hat{u})$$

$$\dot{\hat{z}} = A\hat{z} + (\mathcal{E}^z)^{-1} K(u - \hat{u}), \qquad \hat{u} = C\hat{z} \qquad (8)$$

where $L = \textit{block-diag}[L^1, \ldots, L^p]$, $K = \textit{block-diag}[K^1, \ldots, K^m]$, and L^i, K^j are Hurwitz vectors of dimension $k_i \times 1$ and $n_j \times 1$, respectively, for $i = 1, \ldots, p$, $j = 1, \ldots, m$. Analogously, we let $A = \textit{block-diag}[A^1, \ldots, A^m]$, $B = \textit{block-diag}[B^1, \ldots, B^m]$, $C = \textit{block-diag}[C^1, \ldots, C^m]$, where A^i, B^i,and C^i are in contr./obs. canonical form and have dimensions $n_i \times n_i$, $n_i \times 1$, and $1 \times n_i$, respectively. Finally, $\mathcal{E}^x = \textit{block-diag}[\mathcal{E}^x_1, \ldots, \mathcal{E}^x_p]$, where $\mathcal{E}^x_i = \textit{diag}[\rho, \rho^2, \ldots, \rho^{k_i}]$ and $\rho \in \mathbb{R}$, and $\mathcal{E}^z = \textit{block-diag}[\mathcal{E}^z_1, \ldots, \mathcal{E}^z_p]$, where $\mathcal{E}^z_j = \textit{diag}[\epsilon, \epsilon^2, \ldots, \epsilon^{n_j}]$ and $\epsilon \in \mathbb{R}$. Let $U \triangleq [u_1^{(n_1)}, \ldots, u_m^{(n_m)}]^\top$, then the vector $\dot{z}$ can be expressed as $\dot{z} = Az + BU$, and $u = Cz$.

Theorem 1 *Consider system (1) and suppose A2 is satisfied for $\mathcal{X} = \mathbb{R}^n, \mathcal{U} = \mathbb{R}^{n_u}$, x and z are confined to within a compact invariant set Ω, and U is bounded for all $t \geq 0$. Then, the cascaded observers (7) and (8) guarantee that $\hat{x}$ and $\hat{z}$ are bounded for all $t \geq 0$, and the estimation error converges arbitrarily fast to an arbitrarily small neighborhood of the origin, i.e., for all $\delta, T > 0$, there exist $\bar{\rho}, \bar{\epsilon}$, $0 < \bar{\rho}, \bar{\epsilon} \leq 1$, such that $\|\hat{x} - x\| \leq \delta$, $\|\hat{z} - z\| \leq \delta$, for all $t \geq T$, whenever $\rho \in (0, \bar{\rho}), \epsilon \in (0, \bar{\epsilon})$.*

Note that even when $d(t) = d^0$, i.e., no disturbance acts on the system, the observers (7), (8) achieve UUB of the estimation error, only. In other words, in order to simplify the controller design by avoiding the need for a dynamic extension at the input side, we pay the price of loosing the asymptotic stability of the estimation error relative to the result in [4, 5]. On the other hand, however, when $d(t) \neq d^0$, we achieve the same result one would get by adding integrators at the input side of the system (see [3]).

3.2 Cascade Design 2: Sliding Mode Observer For The Input Derivatives

As we mentioned earlier, using an high-gain observer to estimate the input derivatives destroys the asymptotic properties of the ideal case when z is perfectly known and $d(t) = d^0$. One way to recover these properties is to use a sliding mode observer based on the equivalent control method (see [11, 12]) to estimate z because of its peculiar property of providing the derivatives of a signal in arbitrarily small finite time. In order to do so, we will have to estimate not only z, but also the vector U. To this end, we let $\bar{z} \triangleq [u_1, \dot{u}_1, \ldots, u_1^{(n_1)}, \ldots, u_m, \ldots, u_m^{(n_m)}]^\top$, and we build a sliding mode observer for $\bar{z}$, rather than z. The nonlinear observer has the form

$$\dot{\hat{x}} = f(\hat{x}, \hat{u}, d^0) + \left[\frac{\partial \mathcal{H}(\hat{x}, \hat{z})}{\partial \hat{x}}\right]^{-1} \Big[(\mathcal{E}^x)^{-1} L\,(y(t) - \hat{y}(t)) + \frac{\partial \mathcal{H}(\hat{x}, \hat{z})}{\partial \hat{z}} \mathrm{sgn}(\mathcal{V}) K\Big]$$
$$\hat{y}(t) = h(\hat{x}, \hat{u}) \qquad (9)$$

$$\dot{\hat{\bar{z}}} = \bar{A}\hat{\bar{z}} - \mathrm{sgn}(\mathcal{V}) K, \qquad \hat{u} = \bar{C}\hat{\bar{z}} \qquad (10)$$

where $\hat{z}$ is obtained from $\hat{\bar{z}}$ in an obvious way, $\mathcal{E}^x$ and L are defined as before, $K = [K^{1\top}, \ldots, K^{m\top}]^\top$, $K^i = [K_1^i, \ldots, K_{n_i+1}^i]^\top$, for $i = 1, \ldots, m$. Similarly, $\mathcal{V} = block\text{-}diag[\mathcal{V}^1, \ldots, \mathcal{V}^m]$, $\mathcal{V}^i = diag[\mathcal{V}_1^i, \ldots, \mathcal{V}_{n_i+1}^i]$, where $\mathcal{V}_1^i = \hat{u}_i - u_i$, $\mathcal{V}_{j+1}^i = [K_j^i \operatorname{sgn}(\mathcal{V}_j^i)]_{\text{eq}}$, for $j = 1, \ldots, n_i$, $i = 1, \ldots, m$. Finally, $\bar{A} = block\text{-}diag[\bar{A}^1, \ldots, \bar{A}^m]$, $\bar{C} = block\text{-}diag[\bar{C}^1, \ldots, \bar{C}^m]$, with $\bar{A}^i \in \mathbb{R}^{n_i+1 \times n_i+1}$, $\bar{C}^i \in \mathbb{R}^{n_i+1}$, and $(\bar{A}^i, \bar{C}^i)$ in canonical observable form. Moreover, $\operatorname{sgn}(diag[x_1, \ldots, x_n]) \triangleq diag[\operatorname{sgn}(x_1), \ldots, \operatorname{sgn}(x_n)]$, where $\operatorname{sgn}(x)$ denotes the discontinuous sign function (i.e., if $x > 0$ $\operatorname{sgn}(x) = 1$, if $x < 0$ $\operatorname{sgn}(x) = -1$). The notation $[\ldots]_{\text{eq}}$ is the standard way to denote the equivalent control value of the bracketed term (see [13] for details about the equivalent control method).

Theorem 2 *Consider system (1) and suppose A2 is satisfied for $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^{n_u}$, x and z are confined to within a compact invariant set Ω, and U is bounded for all $t \geq 0$. Then, observers (9), (10) guarantee that the following properties hold*

(i) *Existence and uniqueness of the observer estimates: The differential equations (9), (10) have bounded and unique solutions $\hat{x}(t)$ and $\hat{\bar{z}}(t)$, respectively, defined for all $t \geq 0$.*

(ii) *Arbitrarily fast convergence of the estimation error to an arbitrarily small neighborhood of the origin: For all $\delta, T > 0$, there exist $\bar{\rho}$, $0 < \bar{\rho} \leq 1$, and $\bar{K} > 0$ (component-wise) such that $\|\hat{x} - x\| \leq \delta$, $\|\hat{z} - z\| = 0$, for all $t \geq T$, whenever $\rho \in (0, \bar{\rho})$, $K > \bar{K}$ (component-wise).*

(iii) *Asymptotic convergence when $d(t) = d^0$: For all $\delta, T > 0$, there exist ρ^*, $0 < \rho^* \leq 1$, and $K^* > 0$ (component-wise) such that $\|\hat{x} - x\| \leq \delta$, $\|\hat{z} - z\| = 0$, for all $t \geq T$, and $\hat{x} \to x$ as $t \to \infty$, whenever $\rho \in (0, \rho^*)$, $K > K^*$ (component-wise).*

In practical implementation, the values of the equivalent controls may be obtained by low-pass filtering the discontinuous terms in the observer (10). Such a filtering may, in general, introduce small estimation errors (see [13] for a complete analysis of this problem), with a consequent impact on the observer error $\tilde{z}$, which then may not become *identically* zero, but rather remain small. In such a case, result (iii) of Theorem 2 would be lost. Hence, apart from its theoretical interest, the practical relevance of Theorem 2 remains limited. Moreover, the lack of Lyapunov analysis in the literature for the convergence of this observer will force us, in what follows, to restrict the observability Assumption A2 when this observer is employed in closed-loop control (see Section 4.2).

4 Robust Output Feedback Stabilizing Control

Our main objective in this section is to extend the projection idea found in [4, 5] to the case under consideration. Recall that the observers (7), (9) exhibit peaking; hence, when A2 is not satisfied globally, the observer states $\hat{x}$, $\hat{z}$ may exit the observable region $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$, thus rendering the observer equations (7), (9) undefined. Here, we will modify the $\dot{\hat{x}}, \dot{\hat{z}}$ equations to guarantee that $\hat{x}$ and $\hat{z}$ are confined to within $\mathcal{X} \times \mathcal{U}$, while preserving their convergence properties.

Considering system (6) and using Assumption A3 we conclude that $u = h_c(\chi)$ makes the compact set $\mathcal{N}$ positively invariant and uniformly asymptotically stable with domain of attraction $\mathcal{D}$, for all $d(t) \in \Delta$. If $\mathcal{D}$ is not all of $\mathbb{R}^{n+n_c}$, we can use the converse Lyapunov theorem introduced in [6] (Theorem 1) to conclude that there exists a smooth Lyapunov function in $\mathcal{D}$ with respect to $\mathcal{N}$, i.e., there exist two $\mathcal{K}_\infty$ functions α_1 and α_2 and a positive function α_3 such that,

$$\alpha_1(\omega_{\mathcal{N}}(\chi)) \leq V(\chi) \leq \alpha_2(\omega_{\mathcal{N}}(\chi)), \quad \forall \chi \in \mathcal{D} \tag{11}$$

$$\dot{V}(\chi) \leq -\alpha_3(\omega_{\mathcal{N}}(\chi)), \quad \forall \chi \in \mathcal{D}/\mathcal{N} \tag{12}$$

where $\omega_{\mathcal{N}}(\chi)$ is a positive definite function with respect to $\mathcal{N}$, continuous and proper in $\mathcal{D}$ (i.e., it tends to infinity on the boundary of $\mathcal{D}$). See [6] for more details. Given any $c > 0$, define the compact sets $\Omega_c \triangleq \{\chi \,|\, V \leq c_1\}$ and $\Omega_c^x \triangleq \{x \in \mathbb{R}^n \,|\, \chi \in \Omega_c\}$. Consider $c_2 > c_1 > 0$ and let Ω^z be the compact set which is invariant with respect to the z trajectories (its existence follows from the smoothness of $u = h_c(\chi)$ which implies the boundedness of its derivatives). Notice that for all $c_2 > 0$ we have $\mathcal{N} \subset \Omega_{c_2}^x \subset \mathcal{D}$. Now consider the following mapping,

$$\mathcal{F} : \mathbb{R}^n \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}^n \times \mathbb{R}^{n_u}, \quad \mathcal{F}(x, z) \triangleq \begin{bmatrix} \mathcal{H}(x, z) \\ z \end{bmatrix}$$

then, it is readily seen that $\mathcal{F}$ is a diffeomorphism on $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$. Next, the following assumption is needed.

Assumption A4 (*Topology of $\mathcal{X} \times \mathcal{U}$*). Assume that c_2 can be selected so that the following condition is satisfied for some convex compact set C_ξ,

$$\mathcal{F}\left(\Omega_{c_2}^x, \Omega^z\right) \subset C_\xi \subset \mathcal{F}\left(\mathcal{X}, \mathcal{U}\right), \tag{13}$$

4.1 Observer Estimates Projection: First Design

Recall the coordinate transformation used in the proofs of Theorem 1 and 2, and let

$$\xi = \mathcal{H}(x, z), \quad \hat{\xi} = \mathcal{H}(\hat{x}, z), \quad \tilde{\xi} = \hat{\xi} - \xi \tag{14}$$

Note that $\dot{\hat{\xi}} = \left[\frac{\partial \mathcal{H}(\hat{x},\hat{z})}{\partial \hat{x}}\right]\dot{\hat{x}} + \left[\frac{\partial \mathcal{H}(\hat{x},\hat{z})}{\partial \hat{z}}\right]\dot{\hat{z}}$, which is well-defined when $\hat{x} \in \mathcal{X}$, $\hat{z} \in \mathcal{U}$. Next, project

$$\dot{\hat{x}}^P = \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1}\left\{\mathcal{P}_1\left(\hat{\xi}, \dot{\hat{\xi}}, \hat{z}, \dot{\hat{z}}\right) - \frac{\partial \mathcal{H}}{\partial \hat{z}}\dot{\hat{z}}^P\right\} \tag{15}$$

$$\dot{\hat{z}}^P = \mathcal{P}_2\left(\hat{\xi}, \dot{\hat{\xi}}, \hat{z}, \dot{\hat{z}}\right) \tag{16}$$

where $\mathcal{P}_1\left(\hat{\xi}, \dot{\hat{\xi}}, \hat{z}, \dot{\hat{z}}\right) = \dot{\hat{\xi}} - \Gamma_1 \frac{N_\xi\left(N_\xi^\top \dot{\hat{\xi}} + N_z^\top \dot{\hat{z}}\right)}{N_\xi^\top \Gamma_1 N_\xi + N_z^\top \Gamma_2 N_z}$ if $N_\xi^\top \dot{\hat{\xi}} + N_z^\top \dot{\hat{z}} \geq 0$ and $[\hat{\xi}^\top, \hat{z}^\top]^\top \in \partial C_\xi$, while $\mathcal{P}_1 = \dot{\hat{\xi}}$, otherwise. Similarly, $\mathcal{P}_2\left(\hat{\xi}, \dot{\hat{\xi}}, \hat{z}, \dot{\hat{z}}\right) = \dot{\hat{z}} - \Gamma_2 \frac{N_z\left(N_\xi^\top \dot{\hat{\xi}} + N_z^\top \dot{\hat{z}}\right)}{N_\xi^\top \Gamma_1 N_\xi + N_z^\top \Gamma_2 N_z}$ if $N_\xi^\top \dot{\hat{\xi}} + N_z^\top \dot{\hat{z}} \geq 0$ and $[\hat{\xi}^\top, \hat{z}^\top]^\top \in \partial C_\xi$, while $\mathcal{P}_2 = \dot{\hat{\xi}}$ otherwise.

In the definitions above $\Gamma_1 = (S^1\bar{\mathcal{E}}^x)^{-1}(S^1\bar{\mathcal{E}}^x)^{-1}$, $\Gamma_2 = (S^2\bar{\mathcal{E}}^z)^{-1}(S^2\bar{\mathcal{E}}^z)^{-1}$, $S^1 = {S^1}^\top$, $S^2 = {S^2}^\top$ denote the matrix square roots of P_1, P_2, the positive definite solutions of the Lyapunov equations associated to the Hurwitz matrices $A - L_1C$ and $A - L_2C$, respectively; $\bar{\mathcal{E}}^x, \bar{\mathcal{E}}^z$ are the scalings introduced in the proof of Theorem 1, and N_ξ, N_z are the ξ and z components of the normal vector N to ∂C_ξ, the boundary of C_ξ, at $\hat{\xi}$, $\hat{z}$ i.e., $N(\hat{\xi}, \hat{z}) = \left[N_\xi^\top(\hat{\xi}, \hat{z}), N_z^\top(\hat{\xi}, \hat{z})\right]^\top$. The following lemma shows that (15), (16) guarantee boundedness and preserve convergence of the estimates $\hat{x}$, $\hat{z}$.

Lemma 1 *If A4 holds and (15), (16) are applied to observers (7), (8), respectively, then*
(i) *Boundedness:* $[{\hat{x}^P}^\top, {\hat{z}^P}^\top]^\top \in \mathcal{F}^{-1}(C_\xi) \subset \mathcal{X} \times \mathcal{U}$, *for all* $t \geq 0$.
If, in addition, $x \in \Omega^x_{c_2}, z \in \Omega^z$, *then the following is also true*
(ii) *Preservation of the original convergence characteristics: the result of Theorem 1 remains valid for* $\hat{x}^P$, $\hat{z}^P$.

4.2 Observer Estimates Projection: Second Design

When using a sliding mode observer to estimate z we do not have the tools to build a projection that preserves the convergence characteristics outlined in Theorem 2. This is due to the fact that we do not have a Lyapunov function for the sliding mode observer outside of the sliding manifold (to the best of our knowledge, the convergence analyses found in the literature do not provide it), and our projection design relies on this knowledge. In order to avoid this problem we need to restrict Assumption A2.

Assumption A2' (*Observability*). System (1) is observable over the set $\mathcal{X} \times \mathbb{R}^{n_u}$, where $\mathcal{X} \subset \mathbb{R}^n$, i.e., the mapping $y_e = \mathcal{H}(x,z)$ is invertible with respect to x and its inverse is smooth, for all $x \in \mathcal{X}$, $z \in \mathbb{R}^{n_u}$.

Assumption A2' requires the uniform observability with respect to z, but it is still less restrictive than the UCO assumption commonly found in the literature. Assumption A4 can be consequently relaxed as follows.

Assumption A4' (*Topology of* $\mathcal{X} \times \mathcal{U}$). Assume that c_2 can be selected so that the following condition is satisfied for some convex compact C_ξ

$$\mathcal{F}\left(\Omega^x_{c_2}, \hat{\Omega}^z\right) \subset C_\xi \subset \mathcal{F}\left(\mathcal{X}, \mathbb{R}^{n_u}\right), \tag{17}$$

where $\hat{\Omega}^z$ is defined in the proof of Theorem 2.

Using these assumptions, we do not need to project the estimate provided by the sliding mode observer, thus we only need to make sure that $\hat{x}(t) \in \mathcal{X}$ for all $t \geq 0$. We do this by modifying projection (15) as follows,

$$\dot{\hat{x}}^P = \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1}\left\{\mathcal{P}_1\left(\hat{\xi}, \dot{\hat{\xi}}, \hat{z}, \dot{\hat{z}}\right) - \frac{\partial \mathcal{H}}{\partial \hat{z}}\dot{\hat{z}}^P\right\} \tag{18}$$

where $\mathcal{P}_1\left(\hat{\xi}, \dot{\hat{\xi}}, \hat{z}, \dot{\hat{z}}\right) = \dot{\hat{\xi}} - \Gamma_1 \frac{N_\xi\left(N_\xi^\top \dot{\hat{\xi}} + N_z^\top \dot{\hat{z}}\right)}{N_\xi^\top \Gamma_1 N_\xi}$ if $N_\xi^\top \dot{\hat{\xi}} + N_z^\top \dot{\hat{z}} \geq 0$ and $[\hat{\xi}^\top, \hat{z}^\top]^\top \in \partial C_\xi$, while $\mathcal{P}_1 = \dot{\hat{\xi}}$ otherwise. Using this projection a result completely analogous to that found in Lemma 1 holds. Its statement is omitted for lack of space.

4.3 Closed-Loop Stability

To perform output feedback control, we replace the dynamic state feedback compensator (4) by

$$\begin{aligned} \dot{x}_c &= f_c(x_c, \hat{x}^P) \\ \hat{u} &= h_c(x_c, \hat{x}^P). \end{aligned} \tag{19}$$

By the properness of $V(\chi)$ on $\mathcal{D}$, we have that for any compact set $\mathcal{D}' \subset \mathcal{D}$ there exist c_1, c_2 large enough so that $\mathcal{D}' \subset \Omega_{c_1} \subset \Omega_{c_2} \subset \mathcal{D}$. Taking in account the restriction on c_2 imposed by Assumption A4 (A4'), choose $\mathcal{D}'$ to be an arbitrary compact set contained in Ω_{c_1}. In the following we will state two closed-loop stability theorems for the case when $d(t) \neq d^0$ and $d(t) = d^0$, respectively. The first of the two theorems applies to both of the observer designs introduced in Section 3, while the other applies to the second cascade design, only; in the statement of Theorem 3 we will refer to the first design (see Section 3.1), pointing out the differences relative to the second one in square brackets, whenever needed.

Theorem 3 *Suppose that A1, A2, [A2'], A3, A4 [A4'] are satisfied and the initial condition* $\chi(0)$ *is contained in* $\mathcal{D}' \subset \Omega_{c_1}$*; define the set* $\Omega_\varepsilon \triangleq \{\chi : V(\chi) \leq d_\varepsilon\}$ $(\mathcal{N} \subset \Omega_\varepsilon)$*, where* $d_\varepsilon = \alpha_2 \circ \alpha_3^{-1}(2A\varepsilon)$ *and* A *is defined in the proof. Then, for any* $\varepsilon > 0$ *chosen so that* $d_\varepsilon < c_1$*, there exist positive scalars* ρ^*, $\epsilon^*[K^*]$ *such that, for all* $\rho \in (0, \bar{\rho})$, $\epsilon \in (0, \bar{\epsilon})$ *[*$K > K^*$ *component-wise], the output feedback control law (19) (where* $\hat{x}^P$ *is generated* *by (7), (8), (15), (16) [(9), (10), (18)]) renders Ω_ε a uniformly asymptotically stable positively invariant set for (6), with domain of attraction containing Ω_{c_1}, and it is reached in finite time; moreover all the internal variables are bounded for all $t \geq 0$.*

Note that, as $\varepsilon \to 0$, Ω_ε becomes arbitrarily close to $\mathcal{N}$. Thus, Theorem 3 implies that the original uniformly asymptotically stable positively invariant set under state feedback is recovered, in the limit as $\varepsilon \to 0$, under output feedback. Furthermore, note that the implication of Theorem 3 is essentially identical to one idea found in [6]. This indicates that using an input derivative observer is advantageous in that, besides simplifying the controller design, in the presence of disturbance it yields the same stability results as when dynamic extension is employed at the input side of the system.

Theorem 4 *Suppose that Assumptions A1, A2', A3, A4' are satisfied and the initial condition $\chi(0)$ is contained in $\mathcal{D}' \subset \Omega_{c_1}$. If $d(t) = d^0$, i.e., if no disturbance affects the system, then the output feedback control law (19) (where $\hat{x}^P$ is generated by (9), (10), (18)) renders $\mathcal{N}$ a uniformly asymptotically stable positively invariant set for (6), with domain of attraction containing Ω_{c_1}.*

5 Conclusions

Theorems 3 and 4 establish a separation principle for incompletely observable MIMO nonlinear systems with and without disturbances. These results extend and simplify our previous result in [4, 5], but are inspired by the same methodology: the output feedback controller design is entirely carried out in the original system coordinates (hence the analytical knowledge of $\mathcal{H}^{-1}$ is not needed) and an appropriate projection is designed to deal with the incomplete observability of the system and, at the same time, to guarantee separation between the state feedback controller and observer designs. When Assumptions A2 and A3 hold globally, Theorems 3 and 4 show that any compact set in $\mathbb{R}^n \times \mathbb{R}^{n_c}$ can be made a region of attraction for the sets Ω_ε and $\mathcal{N}$, respectively. In other words, the closed-loop system becomes semi-globally stable with respect to Ω_ε, $\mathcal{N}$, respectively. We did not investigate conditions under which $\mathcal{N}$ is asymptotically stable under output feedback even when $d(t) \neq d^0$. The reader may refer to the analysis in [6, 3] for more details on this issue.

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