Filtering for Linear Systems Driven by Fractional Brownian Motion

N.U. Ahmed * and C.D. Charalambous †

Abstract In this paper we study continuous time filtering for linear systems driven by fractional Brownian motion processes. We present the derivation of the optimum linear filter equations which involve a pair of functional-differential equations giving the error co-variance (matrix-valued) function and the filter. These equations are the appropriate substitutes of the matrix-Riccati differential equation arising in classical Kalman filtering. However the optimum filter has the classical appearance and, as usual, it is driven by the increments of the observed process. Our derivation is based on the same general principles as used in [5,6,7].

1 Introduction

In recent years, there has been renewed interest in fractional Brownian motion, originally introduced by Mandlebrot and Van Nees [1], to model phenomena which exhibit the so-called self similarity which is a form of time invariance of the fundamental structure of the process (fractal), and long range dependence. The long range dependence which is absent in the regular Brownian motion or more precisely, its (distributional) derivative, the white noise, is historically observed in the study of water accumulation in hydrology [1], Ethernet and ATM traffic in telecommunication systems [8,9], and stock prices in mathematical finance. Since fractional Brownian motion is not so popular as the standard Brownian motion, we state some of its fundamental properties and refer the reader to the literature for details [1-4].

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $H \in (0,1)$. A parameterized family of random process $\{B_H(t), t \geq 0\}$ based on this probability space is said to be a fractional Brownian motion if

(i): $P\{B_H(0) = 0\} = 1$,

(ii): for each $t \in R_+ \equiv [0, \infty)$, $B_H(t)$ is an $\mathcal{F}$-measurable random variable having Gaussian distribution with $E\{B_H(t)\} = 0$.

(iii): for $t, s \in R_+$, $E\{B_H(t)B_H(s)\} = (1/2)\{t^{2H} + s^{2H} - |t-s|^{2H}\}$.

It follows from the property (iii) and the well-known Kolmogorov's criterion for continuity that, for $H > (1/2)$,

(iv): the sample paths of B_H are continuous with probability one but no where differentiable. Further it follows from (iii) again that the variance of $B_H(t)$ is t^{2H}, and for $H = 1/2$, $E\{B_{1/2}(t)B_{1/2}(s)\} = t \wedge s$. That is, $B_{1/2}$ is the standard Brownian motion.

In fact fractional Brownian motion can be constructed from the classical Brownian motion by a linear transformation of the form

$$B_H(t) \equiv \int_0^t K_H(t,s)dB(s) \quad (1)$$

where the process $\{B(t), t \geq 0\}$, is the classical Brownian motion and K_H is a kernel dependent on the parameter H known as the Hurst parameter (see [10]).

It follows from this construction that

(iv): B_H is self similar in the sense that, for any $\alpha > 0$, the probability laws of $B_H(\alpha t)$ and $\alpha^H B_H(t)$ coincide.

For other choices, and more general fractional Brownian motions and their properties see Mandlebrot [1] and Duncan et al [2-4]. It is reported in these papers that random processes arising form hydrological and economic time series exhibit long range interdependence and self similarity.

A function that plays an important role in the construction of stochastic integrals based on fractional Brownian motion is given by

$$\varphi_H(t) \equiv H(2H-1)|t|^{2H-2}, t \in R. \quad (2)$$

It can be easily shown [2,3,4] that, for all $t, s \in R_+$, we have

$$E\{B_H(t)B_H(s)\} = \int_0^t \int_0^s \varphi_H(\tau - \theta)d\theta d\tau. \quad (3)$$

Duncan et al [2,3,4] introduced the class of functions $L^2_\varphi(R_+)$ which consist of all Borel measurable real valued functions $\{f\}$ defined on R_+ satisfying

$$\| f \|^2_\varphi \equiv \int_0^\infty \int_0^\infty \varphi_H(t-s)f(s)f(t)dsdt < \infty. \quad (4)$$

*Professor, School of Information Technology and Engineering, University of Ottawa

†Associate professor, School of Information Technology and Engineering, University of Ottawa, Adjunct Professor with Department of Electrical and Computer Engineering and Associate Member of the Centre for Intelligent Machines, McGill University. This Work was Supported by the Natural Science and Engineering Research Council of Canada under Grant OGP0183720.

With respect to the scalar product

$$(f,g)_\varphi \equiv \int_0^\infty \int_0^\infty \varphi_H(t-s)f(s)g(t)dsdt, \qquad (5)$$

$L^2_\varphi(R_+)$ is a Hilbert space. Stochastic integrals, with respect to the fractional Brownian motion B_H, of deterministic integrands from the class L^2_φ are well defined. More precisely, for each $f \in L^2_\varphi$, the element X given by

$$X \equiv \int_0^\infty f(t)dB_H(t) \qquad (6)$$

is a well defined random variable (real valued $\mathcal{F}$ measurable). Since f is deterministic and B_H is Gaussian, the random variable X is also Gaussian and it is easy to see that

$$(a): E\{X\} = 0; \quad (b): E|X|^2 = \ \| f \|^2_\varphi . \qquad (7)$$

Since we are interested in the filtering problem for multidimensional processes, we modify the preceding results to suit this requirement. Again we use $(\Omega, \mathcal{F}, P)$ to denote the basic probability space on which all the random processes to be defined below are supported. For any integer n one may construct the fractional Brownian by the following expression

$$B_H(t) \equiv \int_0^t K_H(t,\theta)dB(\theta) \qquad (8)$$

where B is an n dimensional Brownian motion with covariance, say, $Q \in M^+_s(n\times n)$. Here $M^+_s(n\times n)$ denotes the class of real symmetric positive definite matrices and K_H is the scalar kernel as introduced above. In view of the previous results, B_H is an R^n-valued Gaussian random process having mean and covariance given by

$$(B1): \ E\{B_H(t)\} = 0$$

$$(B2): \ E\{(B_H(t),\xi)(B_H(s),\eta)\} = \int_0^t \int_0^s \varphi_H(\tau-\theta)(Q\xi,\eta)d\tau d\theta, \text{ for all } \xi,\eta \in R^n.$$

Clearly it follows from (B2) that

$$E\{(B_H(t),\xi)^2\} = t^{2H}(Q\xi,\xi), \xi \in R^n, t \in R_+. \qquad (9)$$

Now we can define stochastic Wiener integrals with respect to the FBM(fractional Brownian motion). For simplicity we consider finite intervals $I \equiv [0,T], T < \infty$. Let $M(k\times n)$ denote the vector space of $k \times n$ matrices with entries real. For any $H \in (0,1)$, let $L^2_H(I, M(k\times n))$ denote the Hilbert space with the scalar product defined by

$$(\sigma,\beta)_H \equiv \int_I \int_I \varphi_H(u-v)Tr\{\sigma(u)Q\beta^{'}(v)\}dudv, \qquad (10)$$

and the norm by

$$\| \sigma \|_H \equiv \left(\int_I \int_I \varphi_H(u-v)Tr\{\sigma(u)Q\sigma^{'}(v)\}dudv \right)^{1/2}, \qquad (11)$$

for $\sigma \in L^2_H(I, M(k\times n))$. Clearly this Hilbert space is related to the FBM B_H.

For $\sigma \in L^2_H(I, M(k\times n))$, define

$$Z \equiv \int_I \sigma(t)dB_H(t) \equiv L(\sigma). \qquad (12)$$

This is a well defined random variable with values in R^k. The following result is useful in the sequel.

Lemma 1.1 For each $\sigma \in L^2_H(I, M(k\times n))$, the element Z is a well defined Gaussian random variable with values in R^k satisfying the following properties

$$(p1): \ EZ = 0,$$

$$(p2): \ E(Z,\xi)^2 = \int_I \int_I \varphi_H(u-v)(Q\sigma^{'}(u)\xi, \sigma^{'}(v)\xi)dudv,$$

$$(p3): \ E\{\| Z \|^2\} = \int_I \int_I \varphi_H(u-v)Tr\{\sigma(u)Q\sigma^{'}(v)\}dudv.$$

$$(p4): \ \text{for } H \geq 1/2, \quad L^2(I, M(k\times n)) \hookrightarrow L^2_H(I, M(k\times n))$$

for each $\xi \in R^k$.

2 System and Measurement Dynamics and Filtering Problem

Since long term inter dependence is encountered only for Hurst parameter $H > 1/2$, from now one we consider only this case. The system is governed by the following linear stochastic differential equation,

$$dx(t) = A(t)x(t)dt + \sigma(t)dB_H(t), \qquad x(0) = x_0, \qquad (13)$$

and the measurement dynamics is given by

$$dy(t) = H(t)x(t)dt + \sigma_0(t)dV_H(t), t \geq 0, \qquad y(0) = 0. \ (14)$$

We consider the processes $\{x, y\}$ taking values from R^n and R^m respectively. The noise processes $\{B_H(t), V_H(t), t \geq 0\}$ are fractional Brownian motions taking values from R^d and R^m respectively. For compatibility, it is clear that the matrices $\{A, \sigma, H, \sigma_0\}$, which are deterministic, must take values from $M(n\times n), M(n\times d), M(m\times n), M(m\times m)$ respectively. Let $\mathcal{F}^y_t, t \geq 0$, be an increasing family of subsigma algebras of the sigma algebra $\mathcal{F}$ induced by the random process $\{y(t), t \geq 0\}$. In other words this is the filtration associated with the process y. The basic filtering problem is to find a process z so that for each $t \geq 0$, $z(t)$ is $\mathcal{F}^y_t$ adapted (measurable) satisfying

(1): $E\{z(t)\} = E\{x(t)\}, t \geq 0,$

(2): $E \| x(t) - z(t) \|^2 \longrightarrow$ is minimum for $t \geq 0$.

That is, we want an unbiased minimum variance filter. Define

$$\hat{x}(t) \equiv E\{x(t)|\mathcal{F}^y_t\}. \qquad (15)$$

It is clear that the process $\{\hat{x}(t), t \geq 0\}$, satisfies the two requirements. Our objective here is to find the best (unbiased-minimum variance = UMV) linear filter driven by the observed process y as described by the following stochastic differential equation

$$dz(t) = B(t)z(t)dt + \Gamma(t)dy(t), \; z(0) = \hat{x}_0 \equiv Ex_0, \quad (16)$$

where B and Γ are suitable matrix valued functions to be determined.

3 Reformulation of the Filtering Problem as a Control Problem.

We introduce the following basic assumptions:

(A1): there exist matrices $Q \in M_s^+(d \times d)$ and $Q_0 \in M_s^+(m \times m)$ such that

$$\begin{aligned} &E\{(B_H(t), \xi)(B_H(s), \eta)\} \\ &= \int_0^t \int_0^s \varphi_H(\theta - \tau)(Q\xi, \eta) d\tau d\theta, \xi, \eta \in R^d, \\ &E\{(V_H(t), \xi)(V_H(s), \eta)\} \\ &= \int_0^t \int_0^s \varphi_H(\theta - \tau)(Q_0\xi, \eta) d\tau d\theta, \xi, \eta \in R^m. \end{aligned}$$

(A2): The matrices A and H are locally integrable while $\sigma \in L_H^2(R_+, M(n \times d))$ and $\sigma_0 \in L_H^2(R_+, M(m \times m))$.

(A3) The random elements $\{x_0, B_H(t), V_H(t), t \geq 0\}$ are mutually statistically independent.

Define

$$e(t) \equiv x(t) - z(t), t \geq 0. \quad (17)$$

For unbiased estimate, it follows from this that B must satisfy the identity $B = A - \Gamma H$. Clearly, for this choice of B, the filter equation becomes

$$dz(t) = (A(t) - \Gamma(t)H(t))z(t)dt + \Gamma(t)dy(t), \quad (18)$$

Lemma 3.1 Suppose the assumptions (A1)-(A3) hold. Then, for each $\Gamma \in L^\infty(I, M(n \times m))$, the error covariance K, satisfies the following functional differential equation

$$\begin{aligned} &\dot{K}(t) = A_\Gamma(t)K + KA_\Gamma^{'}(t) \\ &+ \int_0^t \varphi_H(t-s)\big\{\Phi_\Gamma(t,s)\tilde{Q}(s,t) + \tilde{Q}^{'}(s,t)\Phi_\Gamma^{'}(t,s)\big\} ds \\ &+ \int_0^t \varphi_H(t-s)\big\{\Phi_\Gamma(t,s)\Gamma(s)\tilde{Q}_0(s,t)\Gamma^{'}(t) \\ &+ \Gamma(t)\tilde{Q}_0^{'}(s,t)\Gamma^{'}(s)\Phi_\Gamma^{'}(t,s)\big\} ds, \quad K(0) = K_0, \quad (19) \end{aligned}$$

where $A_\Gamma(t) = A(t) - \Gamma(t)H(t)$, and $\tilde{Q}(s,t) \equiv \sigma(s)Q\sigma^{'}(t)$, $\tilde{Q}_0(s,t) \equiv \sigma_0(s)Q_0\sigma_0^{'}(t)$.

Now we are prepared to formulate the filtering problem as a control problem. First we recall that equation (18) with Γ to be determined, gives an unbiased estimate of x. For minimum variance estimate, we must now choose Γ so that $TrK(t)$ is minimum. We consider a more general problem which covers the filtering problem. Let T be any arbitrary but finite time with $I \equiv [0,T]$, and Σ any real positive definite symmetric matrix valued function, for example, $\Sigma \in L^1(I, M_s^+(n \times n))$. Define

$$J(\Gamma) = \int_0^T Tr(\Sigma(t)K(t))dt. \quad (20)$$

For compactness of notation set $\mathcal{G} \equiv L^\infty(I, M(n \times m))$. Then the optimum filtering problem is equivalent to the problem : find $\Gamma \in \mathcal{G}$ that imparts a minimum to the functional J subject to the dynamic constraint (19).

The first question that must be settled is, does the problem, as stated, have a solution?. This is answered in the following corollary.

Theorem 3.2. Suppose the assumptions (A1)-(A3) hold and $\Sigma \in L^1(I, M_s^+((n \times n))$. Then, the optimal control problem as stated above has a solution.

4 Optimal Filter

We have seen in the preceding section, that an optimum linear filter exists and that it can be determined by solving the control problem

$$J(\Gamma) = \int_0^T Tr(\Sigma(t)K(t))dt \longrightarrow min. \quad (21)$$

subject to the dynamic constraint (19).

To obtain the necessary conditions of optimality and finally the optimum filter we use variational technique as in [5, 6]. For this we shall need the Gateaux differential of K with respect to Γ on $\mathcal{G}$. In general we may assume that $\mathcal{G}$ is any closed convex subset of $L^\infty(I, M(n \times m))$. Let $\Gamma_o \in \mathcal{G}$ be the optimal control and $\Gamma \in \mathcal{G}$ any other arbitrary element. We show that the Gateaux differential of K at Γ_o in the direction $(\Gamma - \Gamma_o)$ is given by a functional differential equation. For this we must show that the transition operator Φ_Γ is Gateaux differentiable. This is stated in the following result.

Lemma 4.1 The Gateaux differential of the map $\Gamma \longrightarrow \Phi_\Gamma$ at Γ_o in the direction $\Gamma - \Gamma_o$, denoted by $\tilde{\Phi}$, is given by

$$\tilde{\Phi}(t,\theta) = -\int_\theta^t ds \; \Phi_{\Gamma_o}(t,s)(\Gamma(s) - \Gamma_o(s))H(s)\Phi_{\Gamma_o}(s,\theta) \quad (22)$$

which satisfies the following differential equation

$$\begin{aligned} &(\partial/\partial t)\tilde{\Phi}(t,\theta) = (A(t) - \Gamma_o(t)H(t))\tilde{\Phi}(t,\theta) \\ &-(\Gamma(t) - \Gamma_o(t))H(t)\Phi_{\Gamma_o}(t,\theta), 0 \leq \theta \leq t, \quad (23) \\ &\tilde{\Phi}(\theta,\theta) = 0, \theta \geq 0, \quad (24) \end{aligned}$$

and

$$(\partial/\partial t)\Phi(t,s) = A_\Gamma\Phi_\Gamma(t,s) = (A(t) -\Gamma(t)H(t))\Phi(t,s), \quad (25)$$

$$0 \le \theta \le t, \quad \Phi(t,t) = I_d. \quad (26)$$

Further, as $\Gamma \to \Gamma_o$, $\tilde{\Phi}(t,\theta) \to 0$ uniformly on $I \times I$.

Lemma 4.2 Let $\mathcal{G}$ be any closed convex subset of $L^\infty(I, M(n\times m))$. Then, for each pair of $\Gamma_o, \Gamma \in \mathcal{G}$, the Gateaux differential of K at $\Gamma_o \in \mathcal{G}$ in the direction $\Gamma - \Gamma_o$, denoted by $\tilde{K}$, is the solution of the functional differential equation

$$\begin{aligned}\dot{\tilde{K}}(t) &= A_{\Gamma o}(t)\tilde{K}(t) + \tilde{K}(t)A^{'}_{\Gamma o}(t)\\ &-(\Gamma(t)-\Gamma_o(t))H(t)K_o(t) - K_o(t)H^{'}(t)(\Gamma(t)-\Gamma_o(t))^{'}\\ &+\int_0^t \varphi_H(t-s)\big\{\tilde{\Phi}(t,s)\tilde{Q}(s,t) + \tilde{Q}^{'}(s,t)\tilde{\Phi}^{'}(t,s)\big\}ds\\ &+\int_0^t \varphi_H(t-s)\big\{\tilde{\Phi}(t,s)\Gamma_o(s)\tilde{Q}_0(s,t)\Gamma_o^{'}(t)\\ &+\Gamma_o(t)\tilde{Q}_0^{'}(s,t)\Gamma_o^{'}(s)\tilde{\Phi}^{'}(t,s)\big\}ds\\ &+\int_0^t \varphi_H(t-s)\big\{\Phi_{\Gamma_o}(t,s)(\Gamma-\Gamma_o)(s)\tilde{Q}_0(s,t)\Gamma_o^{'}(t)\\ &+\Gamma_o(t)\tilde{Q}_0^{'}(s,t)(\Gamma-\Gamma_o)^{'}(s)\Phi^{'}_{\Gamma_o}(t,s)\big\}ds\\ &+\int_0^t \varphi_H(t-s)\big\{\Phi_{\Gamma_o}(t,s)\Gamma_o(s)\tilde{Q}_0(s,t)(\Gamma-\Gamma_o)^{'}(t)\\ &+(\Gamma-\Gamma_o)(t)\tilde{Q}_0^{'}(s,t)\Gamma_o^{'}(s)\Phi^{'}_{\Gamma_o}(t,s)\big\}ds, \quad (27)\end{aligned}$$

with $\tilde{K}(0) = 0$.

Now we are prepared to present the optimal filter equations. Define the following functionals

$$\begin{aligned}F_1(t,K,\Gamma) &\equiv A_\Gamma K + KA^{'}_\Gamma\\ &+\int_0^t \varphi_H(t-s)\big\{\Phi_\Gamma(t,s)\tilde{Q}(s,t)\\ &+\tilde{Q}^{'}(s,t)\Phi^{'}_\Gamma(t,s)\big\}ds\\ &+\int_0^t \varphi_H(t-s)\big\{\Phi_\Gamma(t,s)\Gamma(s)\tilde{Q}_0(s,t)\Gamma^{'}(t)\\ &+\Gamma(t)\tilde{Q}_0^{'}(s,t)\Gamma^{'}(s)\Phi^{'}_\Gamma(t,s)\big\}ds, \quad (28)\end{aligned}$$

and

$$\begin{aligned}F_2(t,K,\Gamma) \equiv -HK\int_0^t \varphi_H(t-s)\Big\{&\tilde{Q}_0(s,t)\Gamma^{'}(t)\\ &+\tilde{Q}_0^{'}(s,t)\Gamma^{'}(s)\Phi^{'}_\Gamma(t,s)\\ &-C_\Gamma(t,s)\Gamma(s)\tilde{Q}_0(s,t)\Gamma^{'}(t) - C_\Gamma(t,s)\tilde{Q}(s,t)\Big\}ds\end{aligned}$$

where C_Γ is given by

$$C_\Gamma(t,s) \equiv \int_s^t H(\theta)\Phi_\Gamma(\theta,s)d\theta. \quad (29)$$

Theorem 4.3 Suppose the assumptions of Theorem 3.2 hold. Then, the optimum linear filter is given by the stochastic differential equation

$$dz = (A - \Gamma_o H)zdt + \Gamma_o dy, \quad z(0) = \hat{x}_0, \quad (30)$$

where the pair $\{\Gamma_o, K_o\}$ satisfies the following functional differential equations

$$\begin{aligned}&\dot{K}_o = F_1(t,K_o,\Gamma_o), K_o(0) = K_0, t\in I,\\ &0 = F_2(t,K_o,\Gamma_o), t \in I \quad (31)\end{aligned}$$

Remark. If $\sigma_0(t)$ is nonsingular, the filter equation (30) can be rewritten as

$$dz = A(t)zdt + \Gamma_o\ \sigma_0\ d\nu_H(t), \quad (32)$$

where ν_H, given by

$$\nu_H(t) = \int_0^t \sigma_0^{-1}(s)\{dy(s) - H(s)z(s)ds\}, t \ge 0, \quad (33)$$

is an $\mathcal{F}_t^y$ measurable Gaussian process. As pointed out in [11] ν_H is not the innovation process.

Some Special Cases. Here we present some special cases.

(C1): We show here that if the Hurst parameter $H \to (1/2)$, our filter equations reduce to the classical Kalman filter equations. We need the following Lemma.

Lemma 4.4 As $H \to (1/2)$, $\varphi_H(t) \to (1/2)\delta(t)$.

Corollary 4.5 Suppose $Q_0(t)$, as defined by

$$Q_o(t) = \tilde{Q}_o(t,t) = \sigma_o(t)Q_o\sigma_o^{'}(t)$$

is nonsingular. Then the Kalman filter is given by the error covariance equation,

$$\begin{aligned}\dot{K}_o(t) &= A(t)K_o(t) + K_o(t)A^{'}(t) + Q(t)\\ &-K_o(t)H^{'}(t)Q_0^{-1}(t)H(t)K_o(t), \quad K(0) = K_0, \quad (34)\end{aligned}$$

and the filter equation,

$$\begin{aligned}dz(t) &= A(t)z(t)dt + K_o(t)H^{'}(t)Q_0^{-1}(t)(dy(t)\\ &-H(t)z(t)dt), \quad z(0) = \hat{x}_0. \quad (35)\end{aligned}$$

Corollary 4.6 Suppose the following conditions hold (C2):the signal process $\{x\}$ is perturbed by Q-Brownian motion and the measurement process $\{y\}$ is driven by Q_0-fractional Brownian motion. (C3): the signal process $\{x\}$ is perturbed by Q- fractional Brownian motion and the measurement process $\{y\}$ is driven by Q_0-Brownian motion. Then the filter equations for these cases are given by the same equation (30) while the covariance equations are given by

$$
\begin{aligned}
(C2): \quad & \dot{K} = A_\Gamma K + KA_\Gamma' + Q(t) \\
& + \int_0^t \varphi_H(t-s)\Big\{ \Phi_\Gamma(t,s)\Gamma(s)\tilde{Q}_0(s,t)\Gamma'(t) \\
& + \Gamma(t)\tilde{Q}_0'(s,t)\Gamma'(s)\Phi_\Gamma'(t,s) \Big\} ds \qquad (36) \\
& 0 = \int_0^t \varphi_H(t-s)\Big\{ \tilde{Q}_0(s,t)\Gamma'(t) \\
& + \tilde{Q}_0'(s,t)\Gamma'(s)\Phi_\Gamma'(t,s) \\
& - C_\Gamma(t,s)\Gamma(s)\tilde{Q}_0(s,t)\Gamma'(t) \Big\} ds - H(t)K(t).
\end{aligned}
$$

$$
\begin{aligned}
& (C3): \quad \dot{K} = A_\Gamma K + KA_\Gamma' \\
& + \int_0^t \varphi_H(t-s)\Big\{ \Phi_\Gamma(t,s)\tilde{Q}(s,t) + \tilde{Q}'(s,t)\Phi_\Gamma'(t,s) \Big\} ds \\
& + \Gamma(t)Q_0(t)\Gamma'(t), \\
& 0 = Q_0(t)\Gamma'(t) - \int_0^t \varphi_H(t-s)C_\Gamma(t,s)\tilde{Q}(s,t)ds \\
& - H(t)K(t). \qquad (37)
\end{aligned}
$$

5 Optimal Filter for Dynamically Coupled Systems

Consider

$$
\begin{aligned}
& dx(t) = A(t)x(t)dt + B(t)y(t)dt \\
& + \sigma(t)dB_H(t), \quad x(0) = x_0, \qquad (38) \\
& dy(t) = H(t)x(t)dt + C(t)y(t)dt \\
& + \sigma_0(t)dV_H(t), \quad y(0) = 0, \qquad (39)
\end{aligned}
$$

where B and C are the coupling matrices taking values from $M(n \times m)$ and $M(m \times m)$ respectively.

Theorem 5.1 Suppose the assumptions of Theorem 4.3 hold and further, B and C are locally integrable. Then the optimum linear filter for the system (38), (39) is

$$
\begin{aligned}
& dz = (A - \Gamma_o H)zdt + (B - \Gamma_o C)ydt + \Gamma_o dy, \\
& z(0) = \hat{x}_0.
\end{aligned}
$$

6 An Algorithm for Computation

Here we present briefly a conceptual algorithm for computing the optimum filter gain. This we do by using the result of theorem 4.3. Suppose the nth stage of iteration has been reached and Γ^n has been determined.

Step 1: Use $\Gamma = \Gamma^n$ to solve for K^n using the first equation of (31).

Step 2: Use K^n in the second equation of (31). If $F_2(t, K^n, \Gamma^n) \neq 0$, solve the corresponding functional equation given by $F_2(t, K^n, \Gamma) = 0$, and call the solution Γ^{n+1}.

Step 3: Use a suitable metric, for example the metric induced by the standard L^∞ norm, to compute

$$d(\Gamma^{n+1}, \Gamma^n) \equiv \| \Gamma^{n+1} - \Gamma^n \|_{L^\infty(I, M(n\times m))} \qquad (40)$$

and stop if a predetermined level of accuracy has been met, and print, if not, go to step 1.

It may be interesting to investigate if the functional equation (31) can be solved by use of singular perturbation techniques.

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