Optimal Initial Value Compensation for Fast Settling Times in Mode-Switching Control Systems

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Abstract

This paper presents an approach to state initialization of heterogeneous mode-switching controllers. It is shown how the controller initialization that achieves the minimal settling time for the closed-loop system can be computed via convex optimization. Algorithms are given that minimize the worst-case settling time for all initial values in a polytope or an ellipsoid. The approach can also account for state and control constraints. Special attention is given to systems with bias disturbance and it is shown how the worst-case settling time can be minimized also when the exact value of the bias is not known. Finally, we show how the settling times can be improved further by adding an impulsive control at the switching instant. The impulsive control can be optimized jointly with the initial value compensator to achieve a minimal settling time.

1. Introduction

As performance demands on modern control systems continue to increase, limitations of linear controllers become more and more evident. This is particularly true in multi-objective control problems that require fast set-point responses and good disturbance rejection properties while imposing hard state and control constraints. The optimal control strategies are then nonlinear and optimal feedback solutions are very hard to obtain. In many applications, economical or practical constraints prohibit the use of advanced computational techniques such as model predictive control.

The current state-of-the-art in control applications is then to do separate control designs for different operating conditions and combine these using some kind of high-level switching. The separate controllers might not be of the same order and the combined controller might not admit an interpretation as a gain-scheduled observer/controller-structure. We will call these controllers heterogeneous mode-switching controllers. Traditionally, such controllers have relied on repeated re-designs and extensive simulations in order to predict the performance of the resulting control strategy. The recent interest in the area of hybrid control systems has resulted in a number of interesting tools, but much work remains to be done.

This paper deals with state initialization after switching and we how the controller initialization that achieves the minimal settling time can be computed via convex optimization. The motivation for this work comes from the particular task of designing controllers that have both time-optimal set-point responses and excellent disturbance rejection properties. The state-of-the-art solution, used in disk-drives [1, 8, 5] and process control systems [2], is to design a time-optimal controller for handling set-point responses and a dynamic controller for regulation around a fixed set-point. A simple approach is to switch from the tracking controller to the regulating controller when the output error is sufficiently small, see Figure 1.

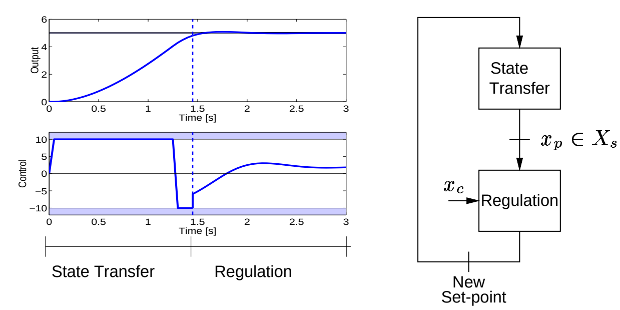

Figure 1: Combining an optimal state-transfer with a tightly tuned regulating controller allows separation of conflicting loop goals. Good transient response relies on proper initialization of the controller states.

This controller will always have a transient phase after the regulating controller has been switched into operation. The high performance demands make it unacceptable to de-tune the regulating controller to achieve better transient response after the switching. The transient behavior of the hybrid control can then be improved in essentially three ways; by modifying the

the switching conditions, by proper initialization of the controller states at the switching instant, or by introducing an intermediate "hand-over" controller for shaping the initial value response. This paper focuses on the state initialization problem. A particular class of hand-over controllers that use an impulsive action at the switching instant are also treated.

Initial value compensation appears to be well-known in practice but has received fairly little attention from academia. A common engineering approach is to initialize the integral state of the controller to its stationary value (see, *e.g.*, [2]), or so to achieve a continuous control signal at the switching instant (so-called bumpless transfer [4]). However, these initializations do not necessarily give a short settling time. Theoretical approaches to initial value compensation have appeared mostly in the disk drive community where servo systems since long need to meet very high performance demands. Early work on initial value compensation [8] considered the transfer function from initial values to output and derived design methods based on pole placement ideas and linear-quadratic optimization. More recently, time-optimal initial value compensation has been considered in [5]. However, [5] only considers fixed (in contrast to state-dependent) initialization of a single controller state (rather than the general case). Moreover, the optimization procedure uses a scalar search and is only optimal if the switching always occurs at the same known plant state.

The approach taken in this paper improves many of these shortcomings. An efficient computational procedure for computing the time-optimal initial value compensation for general linear controllers is given. The initial controller states are allowed to be affine functions of the plant states. The approach optimizes the worst-case settling time over *all* initial values from a given set. Algorithms are given for three classes of initial sets; singletons, polytopes and ellipsoids. The method can explicitly handle uncertainties in the bias, account for actuator limitations, and restrict the initial state to a pre-specified set. This last feature is important when stable switching schemes are constructed based on invariant-set ideas. Finally, we show how a simple type of hand-over controller, disclosed in the patent [7], can be treated in the same framework.

2. A Motivating Example

To illustrate the benefits of initial value compensation, consider the system

$$\ddot{y}(t) = -\frac{1}{T}\dot{y}(t) + \frac{K}{T}u(t) + w_0.$$

This could be a simple model of a disk-drive servo where w_0 is a constant bias due to forces from the flex cable, gravity, and other sources. For sake of simplicity we let $K = 1$, $T = 1$ and $w_0 = 1$. To counteract the disturbance, we employ a PID controller

$$u(t) = k_p\{r(t) - y(t)\} + k_i \int_0^T r(s) - y(s)\, ds - k_d\dot{y}(t).$$

The choice $k_p = 12, k_i = 8, k_d = 5$ places the closed loop poles at $s = -2$. Assume that $r(t) = 0$ and the PID controller is switched into operation at $t = 0$ when $y(0) = -1$ and $\dot{y}(0) = 0$.

If the integral state is initialized to zero, the transient response has a pronounced overshoot and a long settling time, see Figure 2. An idea that comes naturally is to initialize the integral state to its stationary value. As shown in Figure 2 (dotted), however, this gives an even worse transient response. Using the methodology developed in this paper we compute the controller initialization that gives the minimal achievable settling time. The simulation shown in Figure 2 reveals a large improvement in settling time compared to the more naive controller initializations (from 3.2 to 1.2 seconds).

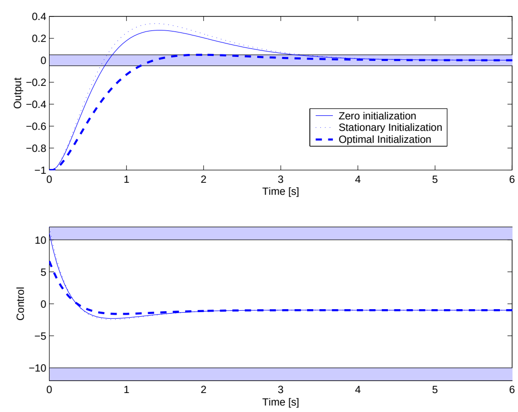

Figure 2: Initial value compensation reduces the settling time from 3.2 seconds to 1.2 seconds without detuning the controller parameters.

Note that the three responses all use the same PID controller parameters - the only difference is in the treatment of controller state initialization. Hence, improved transient response is achieved without sacrificing regulatory performance in any way.

Initial Value Compensation

System Model

Let the plant dynamics within the operating regime of the settling controller be linear and time-invariant

$$\begin{aligned} x_p(k+1) &= A_p x_p(k) + B_{pu}[u_{fb}(k) + u_{ff}(k)] + B_{pw} w_0 \\ y(k) &= C_{py} x_p(k) \\ z(k) &= c_{pz}^T x_p(k). \end{aligned}$$

Here, w_0 is a constant bias disturbance while u_{ff} and u_{fb} are feed-forward and feedback control signals respectively. The feedback control is generated by the linear controller

$$x_c(k+1) = A_c x_c(k) + B_{cr} r(k) + B_{cy} y(k)$$
$$u_{fb}(k) = C_{cu} x_c(k) + D_{cr} r(k) + D_{cy} y(k).$$

We assume that the reference signal is constant and (without loss of generality) let $r(k) = 0$. Introducing

$$x(k) = \left[\begin{array}{cc} x_p^T(k) & x_c^T(k) \end{array} \right]^T$$

we can write the closed loop dynamics on the form

$$x(k+1) = Ax(k) + B_w w_0 + B_u u_{ff}(k)$$
$$z(k) = c_z^T x(k).$$

We will initially let $u_{ff}(k) = 0$.

The Initial Value Compensation Problem

We consider controller initializations that are affine mappings of the plant state at the switching instant

$$x_c(0) = G(\mathbf{L}Fx_p(0) + \mathbf{l}) \qquad (1)$$

Here G and F are fixed matrices, while $\mathbf{L}$ and $\mathbf{l}$ are the adjustable parameters ("the gains") of the initial value mapping. The matrix F reflects the fact that only a subset of the plant states might be available to the initial value compensator, while the matrix G models the situation where only a subset of the controller states are adjustable at the switching.

The design problem is to find matrices $\mathbf{L} \in \mathbb{R}^{m\times p}$ and $\mathbf{l} \in \mathbb{R}^m$ so that the initialization (1) gives a desirable transient. In particular, we will compute the initial value mappings that achieve a minimal settling time. We will define the settling time of the system to be the minimal K_s such that

$$|z(k)| \leq \epsilon \qquad \text{for all } k \geq K_s.$$

Optimizing the Initial Value Mapping

The system state at time k is given by

$$x(k) = A^k x(0) + \sum_{t=1}^{k} A^{t-1} B w_0.$$

For convenient notation, we introduce

$$\widetilde{a} = \sum_{t=1}^{k} A^{t-1} B w_0$$

and partition A^k as

$$A^k = \left[\begin{array}{c|c} \widetilde{A}_p & \widetilde{A}_c \end{array} \right].$$

Applying the initial value mapping (1) we obtain

$$x(k) = (\widetilde{A}_p + \widetilde{A}_c G\mathbf{L}F)x_p(0) + \widetilde{A}_c G\mathbf{l} + \widetilde{a}.$$

We note that for a given initial state $x_p(0)$ this expression is affine in the parameters $\mathbf{L}$ and $\mathbf{l}$. The same holds true for the output bounds

$$c_z^T x(k) \leq \epsilon, \qquad -c_z^T x(k) \leq \epsilon. \qquad (2)$$

In order to guarantee that the output has settled at a given time K, however, we need to find an initial value mapping that enforces the above constraints for all $k \geq K$. A simple approach to this problem is to enforce the output bounds over a sufficiently long time horizon. In other words, we look for parameters $\mathbf{L}$ and $\mathbf{l}$ such that the inequalities (2) hold for all k in the set $\{K, K+1, \ldots, K+H-1\}$.

In the next sections we will show that if $x_p(0)$ is known, or known to belong to a given polytope or ellipsoid, the problem of finding parameters $\mathbf{L}$ and $\mathbf{l}$ so that the settling time bounds (2) are satisfied for a given K can be formulated as a convex optimization problem. Finding the settling time K_s can then be done by bisection over K. In each step we solve a convex optimization problem that either returns the parameters $\mathbf{L}$ and $\mathbf{l}$, or proves that the output cannot be made to settle within the requested time.

Initial Plant State in Singleton

If the switching always occurs at a specific plant state, it is natural to find the initial value mapping that minimizes the settling time from this particular initial condition. The output at time K satisfies the settling limit $z(K) \leq \epsilon$ if and only if

$$c_z^T(\widetilde{A}_p + \widetilde{A}_c G\mathbf{L}F)x_p(0) + c_z^T \widetilde{A}_c G\mathbf{l} + c_z^T \widetilde{a} \leq \epsilon. \qquad (3)$$

This is a linear inequality in $\mathbf{L}$ and $\mathbf{l}$ for a fixed $x_p(0)$.

Let $\operatorname{vec}(X)$ denote the vector obtained by stacking the columns of the matrix X, and let $A \otimes B$ denote the Kronecker product of matrices A and B. We can transform (3) into standard form by introducing

$$\bar{\mathbf{l}} = \begin{bmatrix} \operatorname{vec}(\mathbf{L}) \\ \mathbf{l} \end{bmatrix}.$$

The inequality (3) can be re-written as

$$a^T \bar{\mathbf{l}} \leq b$$

with

$$a^T = \left[(Fx_p(0))^T \otimes c_z^T \widetilde{A}_c G \quad c_z^T \widetilde{A}_c G \right],$$
$$b = \epsilon - c_z^T \widetilde{A}_p x_p(0) - c_z^T \widetilde{a}.$$

The condition $z(K) \geq -\epsilon$ follows similarly. Hence, the problem of finding gains $\mathbf{L}$ and $\mathbf{l}$ so that the output lies within the settling time limits at a particular time K is a linear programming problem.

Initial Plant State in Polytope

Mode switching rarely occurs at the same plant state every time. More oftenly, the switching from one controller to the other is performed when the plant states enter a pre-specified set. We might for example switch to the regulating controller when the output error is sufficiently small regardless of the values of the other states. It is then natural to find the initial value mapping that minimizes the worst-case settling time over all values in the switching set.

Let the switching set be a polytope in vertex form

$$X_s = \overline{\text{conv}}\,(\nu_1, \nu_2, \ldots, \nu_N)$$

where ν_i are the vertices of the polytope, and $\overline{\text{conv}}$ denotes the closed convex hull. Note that the conditions (2) are linear in $x_p(0)$. Hence, to verify that the settling time bounds are satisfied at time K for all initial values in the polytope X_s we simply have to ensure that the bounds are simultaneously satisfied with $x_p(0) = \nu_i$ for all vertices ν_i of X_s. Minimizing the guaranteed settling time is therefore a linear programming problem with $2H \cdot N$ inequalities.

EXAMPLE 1—GUARANTEED SETTLING TIMES

Optimizing the initial value compensator for one particular initial value will typically lead to poor performance for other plant states. This is illustrated in Figure 3, where an initial value mapping for the system in Section 2 has been optimized for the initial plant state

$$y(0) = -1, \qquad \dot{y}(0) = 0.$$

The optimization suggests that the integral state of the PID controller should be initialized to $x_c(0) = -0.6754$. This initialization achieves a settling time of 1.25 seconds. However, when the initial value compensator is applied for a number of other plant states in the switch set $\|x_p(0)\|_\infty \leq 1$ settling times as long as 3.6 second result. By optimizing the guaranteed settling-time from all initial values in the switch set while restricting the controller initialization to use only the measured output $y(0)$, we find the initial value mapping

$$x_c(0) = 0.6864y(0) - 0.1246.$$

This initialization guarantees a worst-case settling time of 1.75 seconds, see Figure 3. If we allow the controller initialization to use the full state vector we find an initial value mapping with a worst-case settling time of 1.3 seconds. □

This example shows that considerable improvements in guaranteed settling times can be achieved with only a slight increase in computations.

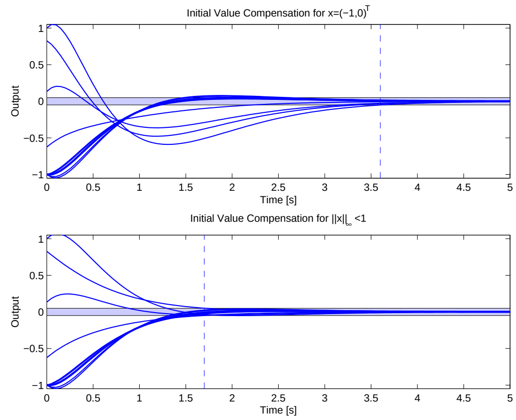

Figure 3: By optimizing the worst-case settling time for all $x_p(0) \in X_s$ (lower plot), considerable improvements in guaranteed settling times are obtained.

Initial Plant State in Ellipsoid

In some situations it is more natural to use ellipsoidal switching sets. One example of this is when stable switching schemes are derived using quadratic Lyapunov functions and invariant-set arguments (see, *e.g.*, [2]). To this end, let the plant state at the switching instant belong to the ellipsoid

$$X_s = \{x = Eu + e \,:\, \|u\| \leq 1\}\,. \tag{4}$$

Recall that the ellipsoid (4) belongs to the halfspace $c^T x_p \leq \epsilon$ if and only if

$$\|E^T c\| \leq \epsilon - c^T e.$$

Hence, if $x_p(0)$ belongs to the ellipsoid (4) then $x(K)$ satisfies the bound $c_z^T x(K) \leq \epsilon$ if and only if

$$\|(\widetilde{A}_c G \mathbf{L} F E)^T c_z + (\widetilde{A}_p E)^T c_z\| \leq \\ \epsilon - c_z^T(\widetilde{A}_p + \widetilde{A}_c G \mathbf{L} F)e - c_z^T(\widetilde{A}_c G \mathbf{l} + \widetilde{a}) \tag{5}$$

This is a second-order cone constraint. It can be transformed into standard form using the decision variable $\bar{\mathbf{l}}$ defined previously. The condition (5) is equivalent to

$$\|P_i \bar{\mathbf{l}} + p_i\| \leq q_i^T \bar{\mathbf{l}} + r_i$$

with

$$\begin{aligned} P_i &= \begin{bmatrix} (FE)^T \otimes c_z^T \widetilde{A}_c G & 0 \end{bmatrix}, \\ p_i &= E^T \widetilde{A}_p^T c_z, \\ q_i^T &= \begin{bmatrix} -(Fe)^T \otimes c_z^T \widetilde{A}_c G & -c_z^T \widetilde{A}_c G \end{bmatrix}, \\ r_i &= \epsilon - c_z^T \widetilde{A}_p e - c_z^T \widetilde{a}. \end{aligned}$$

We conclude that finding the initial value mapping that minimizes the worst-case settling time when the initial plant state belongs to an ellipsoid is a second-order cone program with $2H$ inequalities.

3. Extensions

Many useful extensions can be made to the basic design algorithm without loosing convexity of the resulting optimization problem.

Initial Closed-Loop State in a Set

It is often desirable to constrain the initial value mapping so that the initial closed-loop state is contained in some set. We might for example want to constrain the state initialization so that the actuator limitations $|u_{fb}(0)| \leq u_{lim}$ are not violated. This is done by adding the constraints

$$c_u^T x(0) \leq u_{lim}, \qquad - c_u^T x(0) \leq u_{lim}.$$

In terms of the initial value mapping parameters $\mathbf{L}$ and $\mathbf{l}$, the first constraint takes the form

$$c_u^T \begin{bmatrix} 0 \\ G \end{bmatrix} \begin{bmatrix} \mathbf{L} & \mathbf{l} \end{bmatrix} \begin{bmatrix} F x_p(0) \\ 1 \end{bmatrix} + c_u^T \begin{bmatrix} I \\ 0 \end{bmatrix} x_p(0) \leq u_{lim}.$$

and the second constraint follows similarly. Guaranteeing that the constraint is met for all plant states in the switch set is done as above.

Another important example is when one wants to constrain the initial closed-loop state to an invariant set. The initial value compensator (1) has no effect on the closed loop stability. However, if the plant is unstable, the regulating controller will only provide a limited region of attraction. To avoid system failure it is then important to constrain the initial state to this set. Similarly, a careless application of initial value compensation in a hybrid system might provoke undesired changes in the discrete state. The constraint that the initial state must belong to a certain polytope or ellipsoid can easily be added to the optimization problem.

Uncertain Bias Value

The development so far has assumed that the bias disturbance is perfectly known. This is not always reasonable. It is, however, possible to extend the approach to the case when the constant bias is unknown but bounded

$$\underline{w} \leq w_0 \leq \overline{w}.$$

By simultaneously enforcing all conditions for both extreme values of the bias, the resulting initial value mapping minimizes the worst-case settling time for all values of the constant bias in the interval $[\underline{w}, \overline{w}]$.

General Envelope Constraints

Since all conditions are enforced point-wise in time it is straight-forward to add general envelope constraints on the settling response. This approach can be used for trading off the overshoot of the transient response against guaranteed settling time.

4. Impulsive Control at Switching Instant

A related method for fast settling time in mode-switching systems has been disclosed in the patent [7]. The patent suggests to improve the settling time by adding an impulsive feed-forward to the control signal at the switching instant. The patent describes a tuning procedure based on cancellation of modes, similar to the pole placement ideas in [8]. However, there is no guarantee that such a procedure would actually give a minimal settling time.

In this section we show how the very same ideas that were applied to the initial value compensation scheme can be applied to this method. This gives a novel tuning procedure for the impulsive mode switch compensator that guarantees optimal settling times. The initial value compensator and the impulsive mode switch compensator are in a sense complementary, and can be used together. We will show that it is possible to tune the gains of both compensators jointly, giving a tuning procedure for time-optimal settling responses also when the two approaches are combined.

Adding an impulsive control at the switching instant gives the control

$$u_{ff}(k) = \begin{cases} u_{iff} & k = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Let the value of the impulsive control be an affine function of $x_p(0)$, *i.e.*,

$$u_{iff} = \mathbf{M} F x_p(0) + \mathbf{m}. \tag{7}$$

Combining the initial value compensation with the impulsive mode switch compensator above, the system state at any time k is given by

$$\begin{aligned} x(k) = A^k \begin{bmatrix} I \\ G\mathbf{L}F \end{bmatrix} x_p(0) + A^k \begin{bmatrix} 0 \\ G\mathbf{l} \end{bmatrix} + \\ + A^{k-1} B_u (\mathbf{M} F x_p(0) + \mathbf{m}) + \\ + \sum_{k=1}^{K} A^{k-1} B w_0. \end{aligned}$$

Here, the first row describes the influence of the initial value compensator, the second row describes the impact of the impulsive mode switch compensation while the last row describes the influence of the bias disturbance. We note that the expression is affine in the tuning parameters $\mathbf{L}, \mathbf{l}, \mathbf{M}$ and $\mathbf{m}$ when $x_p(0)$ is fixed, and that it is affine in the initial plant state $x_p(0)$ for fixed compensator gains. Hence, all design procedures that we developed for the initial value compensator can be applied to the joint scheme.

The impact of the impulsive control is highly dependent on the sampling interval and the actuator limitations. The following example illustrates that the benefits of the impulsive control is limited if the sampling time is relatively short compared to the time constants of the closed-loop system.

EXAMPLE 2
Applying the joint scheme to the same set-up as in Example 1 with actuator bounds $|u(k)| \leq 20$ and sampling time $h = 0.05$ seconds, the joint scheme guarantees a settling time of 1.65 seconds, see Figure 4. This is only a slight improvement over the settling times achieved by the optimized initial value mapping in Example 1.

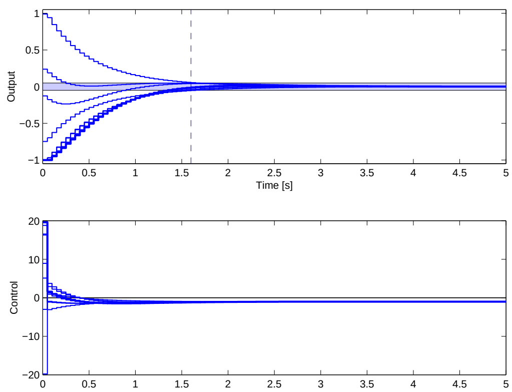

Figure 4: The joint scheme guarantees a settling time of 1.65 seconds, only a slight improvement over the optimized initial value mapping in Example 1.

It is possible to extend the design optimization procedure to the case where the feed-forward control is held constant and applied over a number of samples. □

5. Conclusions

We have considered the state initialization problem for heterogeneous mode-switching controllers and shown how the initialization that achieves the minimal settling time can be computed via convex optimization. Algorithms were given that minimize the worst-case settling time for all initial values in a polytope or an ellipsoid. The approach can account for uncertainties in the bias and deal with state and control constraints. Several examples were given to illustrate the benefits compared to more naive controller initializations.

A question that arises naturally is to what degree the method is sensitive to model uncertainties. The solution proposed in this paper assumes perfect process knowledge when predicting future states. Robust set propagation can be done using the methods in [6] for polytopic initial sets, and [3] for ellipsoidal initial sets. It would be interesting to see if these methods could be used also for design of robust initial value compensators.

The approach described in this paper gives the minimal settling times that can be achieved by proper initialization of the controller states. Admittedly, state initialization alone gives limited possibilities to shape the transient response. If the results are not sufficient one might have to introduce a hand-over controller or consider alternative controller structures. Design of robust hand-over controllers is a topic of current studies.

Acknowledgements

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