

Convex Method for Localized Control Design in Spatially Invariant Systems

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Abstract

A method is presented to impose localization in controller design for distributed arrays with underlying spatial invariance. The method applies to problems where the performance objective (e.g., stabilization, \mathcal{H}_2 or \mathcal{H}_∞ control) can be stated in terms of the search for a suitable Lyapunov matrix over spatial frequency. By restricting this matrix to be constant across frequency, controller localization can be naturally imposed. Thus we obtain sufficient conditions for the existence of a controller with the desired localization and performance, which take the form of linear matrix inequalities (LMIs) over spatial frequency. For one-dimensional arrays, we further show how to convert these conditions exactly to finite dimensional LMIs by means of the KYP lemma.

1 Introduction

Recently there has been renewed interest in the control of *arrays* of distributed units with sensing, actuation and control capabilities. This situation, considered long ago in the context of vehicle platoons [7], is now stimulated by the possibility of building spatially distributed control arrays based on micro electro-mechanical systems (MEMS) for sensing and actuation together with embedded intelligence, for applications such as control of fluid flow [6]. A schematic of such array is depicted in Figure 1. Inherent in the above control paradigm

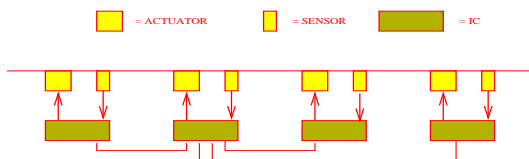


Figure 1: Distributed control array

is a notion of *localization*. Informally, a controller is *localized* if the control action at any given position is a function of information coming from a finite number of neighboring positions. When the control is based only on information coming from the position where it actuates, the controller is said to be *decentralized*. Finally, a controller which requires

information from all the infinite array to actuate on a specific position is referred to as *centralized*.

It is well-known that decentralized control is a difficult problem: particularly, no tractable general methods exist to optimize controllers with this additional constraint. The same holds for localized control. The main hope for success in the control array situation lies in the consideration of special structures, and/or the relaxation of optimality.

One such structure considered in recent work [1, 2, 3] is the *spatial invariance* in the underlying dynamics. In particular spatial Fourier transforms can be applied to quadratic-based optimization criteria (\mathcal{H}_2 , \mathcal{H}_∞), reducing the design to frequency-dependent Riccati equations [2, 9]. Also in [3, 4], LMI methods which extend those in LPV control [8] are used to design controllers with rational multi-dimensional structure and \mathcal{H}_∞ norm guarantees. The above papers have used spatial invariance to indirectly attack the problem of localization; in [9], it is shown how quadratically optimal controllers exhibit exponential decay rates in space, thus providing some degree of localization. The controller structure on [3] can be interpreted as each unit connected exclusively with its neighbors, however in principle with instantaneous feed-through terms. These observations have led to algorithms which combine fast communication with control, and studies of their performance [3, 10].

In this paper we take a different approach; we *directly* impose localization in the design, and for this we pay some price in terms of conservatism. Namely, we find conditions for the existence of localized controllers that satisfy certain performance specifications (stability, \mathcal{H}_2 , \mathcal{H}_∞), but as opposed to [2], these conditions will only be sufficient.

The results apply to feedback control problems that in the standard finite-dimensional case can be reduced to LMI computation by means of a certain change of variables; the distributed versions of these problems are reviewed in Section 2. In Section 3 we introduce the notion of localization for controllers. In Section 4 we present design methods in terms of LMIs across spatial frequency. In Section 5 we seek finite dimensional versions of these conditions; A numerical example is presented in

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Section 6. Some concluding remarks are given in Section 7.

2 Background

In this section we review the essential background for this paper; for a more detailed treatment see [2].

We will be concerned with distributed parameter linear time-invariant dynamical systems where signals are indexed by time t and by a discrete spatial coordinate $k \in \mathbb{Z}^d$, where \mathbb{Z} denotes the set of integers. Consider the following state space representation for a plant:

$$\begin{aligned} \frac{d}{dt}\psi(k, t) &= A\psi(k, t) + B_1w(k, t) + B_2u(k, t), \quad (1) \\ z(k, t) &= C_1\psi(k, t) + D_{11}w(k, t) + D_{12}u(k, t), \\ y(k, t) &= C_2\psi(k, t) + D_{21}w(k, t) + D_{22}u(k, t) \end{aligned}$$

where the operators A, B_j, C_i, D_{ij} for $i = 1, 2$ and $j = 1, 2$ act on $\ell_2 := \{f : \mathbb{Z}^d \rightarrow \mathbb{C}^r \mid \|f\|_2^2 := \sum_{k \in \mathbb{Z}^d} |f(k)|^2 < \infty\}$, for suitable dimensions d, r . At a fixed t the state is $\psi(\cdot, t)$, the disturbance $w(\cdot, t)$, the control is $u(\cdot, t)$ and the performance specifications variable $z(\cdot, t)$ are all functions in ℓ_2 .

We denote the Fourier transform by: $f(\theta) = \mathcal{F}(f) = \sum_{k \in \mathbb{Z}^d} f(k)e^{j\langle k, \theta \rangle}$, where $\theta \in [-\pi, \pi]^d$ is the Fourier frequency, $\langle \cdot, \cdot \rangle$ denotes the scalar product and $[-\pi, \pi]^d$ denotes $[-\pi, \pi] \times \dots \times [-\pi, \pi]$. An operator A is *shift invariant* if it commutes with the bilateral shift operator S in ℓ_2 , i.e. $SA = AS$. The Fourier transform (\mathcal{F}) of a shift invariant operator A , defined by $A_{\mathcal{F}} := \mathcal{F}A\mathcal{F}^{-1}$, is a *multiplication operator*, i.e., there is a function $A(\theta) : [-\pi, \pi]^d \rightarrow \mathbb{C}^{n \times n}$ such that $[A_{\mathcal{F}}f](\theta) = A(\theta)f(\theta)$.

We assume the dynamics (1) are *spatially invariant* [2, 9], which amounts to requiring the operators A, B_j, C_i, D_{ij} to be shift invariant. The key property of spatially invariant dynamics is that it is *diagonalized* by the Fourier transform over the spatial domain.

Applying spatial Fourier transform on (1) with the assumption of spatial invariance, results in:

$$\begin{aligned} \frac{\partial \psi}{\partial t}(\theta, t) &= A(\theta)\psi(\theta, t) + B_1(\theta)w(\theta, t) + B_2(\theta)u(\theta, t), \quad (2) \\ z(\theta, t) &= C_1(\theta)\psi(\theta, t) + D_{11}(\theta)w(\theta, t) + D_{12}(\theta)u(\theta, t), \\ y(\theta, t) &= C_2(\theta)\psi(\theta, t) + D_{21}(\theta)w(\theta, t) + D_{22}(\theta)u(\theta, t), \end{aligned}$$

where $A(\theta), B_j(\theta), C_i(\theta), D_{ij}(\theta)$ are multiplication operators; thus the dynamics amount to a decoupled family of systems over spatial frequency.

For performance specifications on the map G_{zw} from w to z , the Plancherel theorem implies that for quadratic criteria ($\mathcal{H}_2, \mathcal{H}_\infty$) the problem also decouples over spatial frequency. This means that optimal control problems can also be solved in a decoupled fashion over θ , thus reducing the complexity of optimization.

Controllers resulting from optimal syntheses are spatially invariant, and can be represented on the space frequency domain as:

$$\begin{aligned} \frac{d\psi_c}{dt}(\theta, t) &= A_c(\theta)\psi_c(\theta, t) + B_c(\theta)y(\theta, t) \\ u(\theta, t) &= C_c(\theta)\psi_c(\theta, t) + D_c(\theta)y(\theta, t), \quad (3) \end{aligned}$$

where $A_c(\theta), B_c(\theta), C_c(\theta)$ and $D_c(\theta)$ are matrix valued functions of θ .

We restrict our performance requirements to \mathcal{H}_2 , since with only minor changes all the results presented here can be easily applied to any quadratic optimization criteria such as \mathcal{H}_∞ .

The objective is then to design feedback control laws to satisfy \mathcal{H}_2 performance requirements. From here on, we assume that the plant (1) has $D_{11} = D_{22} = 0$, and that D_{12} and D_{21}^* have full column rank. These assumptions are not essential but simplify the resulting formulae.

We start reviewing the problem of state-feedback, i.e., when $C_2 = I$ and $D_{21} = 0$. In this case, for the relevant performance criteria, there is no loss of generality if we assume the controllers to be static and spatially invariant, and we express it in the spatial transform domain as $u(\theta, t) = F(\theta)\psi(\theta, t)$.

The resulting closed-loop system has the form

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= [A(\theta) + B_2(\theta)F(\theta)]\psi + B_1(\theta)w \quad (4) \\ z &= [C_1(\theta) + D_{12}(\theta)F(\theta)]\psi + D_{11}(\theta)w. \end{aligned}$$

To gain some insight on the main result, we go through the simple task of state feedback stabilization: a feedback law $u(\theta, t) = F(\theta)\psi(\theta, t)$ is stabilizing if and only if for every $\theta \in [-\pi, \pi]^d$ there exists a solution $X(\theta) > 0$ to the Lyapunov inequality $[A(\theta) + B_2(\theta)F(\theta)]X(\theta) + X(\theta)[A(\theta) + B_2(\theta)F(\theta)]^* < 0$. That is not an LMI, but is easily turned into one by adopting the change of variables $Y(\theta) = F(\theta)X(\theta)$, an idea which goes back to [5]. With this change, the search for a stabilizer amounts to find a $X(\theta) > 0$ that solves the LMI

$$A(\theta)X(\theta) + B_2(\theta)Y(\theta) + X(\theta)A(\theta)^* + Y(\theta)^*B_2(\theta)^* < 0, \quad (5)$$

over θ , and then computing $F(\theta) = Y(\theta)X(\theta)^{-1}$.

A completely analogous procedure applies to the state-feedback \mathcal{H}_2 problem. We state the following result (see e.g. [12] for the finite dimensional versions):

Proposition 1 (State Feedback) *Given $\gamma > 0$, there exists a feedback gain $F(\theta)$ that internally stabilizes the closed loop system (4) and satisfies $\|G_{zw}\|_{\mathcal{H}_2}^2 < \gamma$ if and only if for each $\theta \in [-\pi, \pi]^d$, there exist $X(\theta) = X(\theta)^*$, $Z(\theta) = Z(\theta)^*$ and $Y(\theta)$ such that the following inequalities are satisfied for*

each θ :

$$\begin{aligned} [A(\theta) \ B_2(\theta)] \begin{bmatrix} X(\theta) \\ Y(\theta) \end{bmatrix} + [X(\theta) \ Y(\theta)^T] \begin{bmatrix} A(\theta)^T \\ B_2(\theta)^T \end{bmatrix} + \\ + B_1(\theta) B_1(\theta)^T < 0 \end{aligned} \quad (6)$$

$$\begin{bmatrix} X(\theta) & (\Omega(\theta))^T \\ \Omega(\theta) & Z(\theta) \end{bmatrix} > 0, \quad (7)$$

$$\frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \text{Tr}[Z(\theta)] d\theta < \gamma, \quad (8)$$

where $\Omega(\theta) = C_1(\theta)X(\theta) + D_{12}(\theta)Y(\theta)$. A suitable feedback is given by $F(\theta) = Y(\theta)[X(\theta)]^{-1}$.

Another result we will need is the following proposition on output feedback synthesis, which is a straightforward derivation from the finite dimensional version (see [12], for the finite dimensional version).

Proposition 2 (Output Feedback) *There exists a controller that renders $\|G_{zw}\|_{\mathcal{H}_2} < \gamma$ if and only if for each $\theta \in [-\pi, \pi]^d$ there exist $X(\theta)$, $Y(\theta)$, $K(\theta)$, $L(\theta)$, $M(\theta)$, and $Z(\theta)$ that satisfy*

$$\begin{aligned} \begin{bmatrix} A(\theta)^* + A(\theta) & B(\theta) \\ (B(\theta))^* & -\gamma I \end{bmatrix} < 0 \\ \begin{bmatrix} \mathcal{X}(\theta) & \mathcal{C}(\theta)^T \\ \mathcal{C}(\theta) & Z(\theta) \end{bmatrix} > 0 \\ \frac{1}{(2\pi)^d} \int_{\theta \in [-\pi, \pi]^d} \text{trace}(Z(\theta)) d\theta < \gamma, \end{aligned} \quad (9)$$

where we have used the notation

$$\begin{aligned} A(\theta) &= \begin{bmatrix} A(\theta)Y(\theta) + B_2(\theta)M(\theta) & A(\theta) \\ K(\theta) & A(\theta)X(\theta) + L(\theta)C_2(\theta) \end{bmatrix}, \\ B(\theta) &= \begin{bmatrix} B_1(\theta) \\ X(\theta)B_1(\theta) + L(\theta)D_{21}(\theta) \end{bmatrix}, \\ C(\theta) &= [C_1(\theta)Y(\theta) + D_{12}(\theta)M(\theta) \ C_1(\theta)], \\ X(\theta) &= \begin{bmatrix} Y(\theta) & I \\ I & X(\theta) \end{bmatrix} \end{aligned} \quad (10)$$

Moreover, if a solution to (9) exists, then there exist nonsingular $U(\theta)$, $V(\theta)$ with $I - XY = UV^T$ for each $\theta \in [-\pi, \pi]^d$. In this case, a suitable controller is given by:

$$\begin{aligned} A_c(\theta) &= U(\theta)^{-1}\Pi(\theta)V^{-T}(\theta), & B_c(\theta) &= U(\theta)^{-1}L(\theta), \\ C_c(\theta) &= M(\theta)V(\theta)^{-T}, & D_c(\theta) &= 0, \end{aligned} \quad (11)$$

where $V(\theta)^{-T} := (V(\theta)^{-1})^T$ and $\Pi(\theta) = [K(\theta) - X(\theta)(A(\theta)Y(\theta) + B_2(\theta)M(\theta)) - L(\theta)C_2(\theta)Y(\theta)]$.

Notice that since the formulas (10) are affine in the unknowns, the conditions (9) are LMIs, therefore define a convex optimization problem.

We note that the trace conditions on (9) and (8) only involve in effect the constant component of the Fourier expansion of $Z(\theta)$. For instance, in the one dimensional case, representing Z as the infinite series $Z(\theta) = \sum_{k=-\infty}^{\infty} Z_k e^{jk\theta}$, we have:

$\int_{-\pi}^{\pi} \text{Tr}[Z(\theta)] d\theta = \sum_{k=-\infty}^{\infty} \text{Tr}[Z_k] \int_{-\pi}^{\pi} e^{jk\theta} d\theta = 2\pi \text{Tr}[Z_0]$. Therefore in the one dimensional case the trace conditions in (9) and (8) reduce to

$$\text{Tr}[Z_0] < \gamma, \quad (12)$$

which will simplify the analysis later on. Similar arguments allow the same simplification for the multi-dimensional case ($x \in \mathbb{Z}^d$).

2.1 One dimensional KYP lemma

This section states a well known result: the discrete version of the Kalman-Yakubovich-Popov Lemma; following [11]:

Lemma 3 (KYP Lemma) *Given three matrices A , B and $\mathcal{M} = \mathcal{M}^*$, with $\det(e^{-j\theta}I - A) \neq 0$ for $\theta \in \mathbb{R}$, the following statements are equivalent: (i) $\begin{bmatrix} (e^{j\theta}I - A)^{-1}B \\ I \end{bmatrix}^* \mathcal{M} \begin{bmatrix} (e^{j\theta}I - A)^{-1}B \\ I \end{bmatrix} < 0$, $\forall \theta \in [-\pi, \pi]$; (ii) There exists a matrix $P = P^*$ such that $\mathcal{M} + \begin{bmatrix} A^*PA - P & A^*PB \\ B^*PA & B^*PB \end{bmatrix} < 0$.*

Given any LMI with FIR dependency on θ , say $\sum_{k=-N}^N M_k e^{jk\theta} < 0$ for all $\theta \in [-\pi, \pi]$, for arbitrary matrices M_k , it is possible to write it in the form of item (i) on the KYP Lemma. Then, by using the KYP Lemma, this θ dependent LMI can be made of the form (ii) and therefore finite dimensional.

3 The Problem of Localization

We will frequently expand a multiplication operator $A(\theta)$ in a series of the form: $A(\theta) = \sum_{k \in \mathbb{Z}^d} A_k e^{j(k, \theta)}$, where A_k are matrices indexed by the discrete spatial coordinate k . That expansion embeds the spatial structure of the operator A . To understand that, consider an arbitrary operator A and the application $g(\theta) = A(\theta)f(\theta)$ for the one dimensional case $\theta \in [-\pi, \pi]$. Transforming back to the spatial coordinate $k \in \mathbb{Z}$ we get the convolution in space: $g(n) = \sum_{k=-\infty}^{\infty} A_k f(n-k)$. Note that in general the value of $g(n)$ will depend on the value of f at an infinite number of positions.

Now assume $A_k = 0$ for $|k| > N$. Then $g(n)$ will depend on $f(n-k)$ for $|k| \leq N$ only, i.e, $g(n) = A_{-N}f(n-N) + \dots + A_0f(n) + \dots + A_Nf(n+N)$. We define such A to be a *localized operator* (of order N). In other words, in one-dimensional case, a localized operator has the finite impulse response (FIR) form in space: $A(\theta) = \sum_{k=-N}^N A_k e^{jk\theta}$.

The definition extends naturally to the multi dimensional case: an operator A with is said to be a localized operator when its associated multiplication operator $A(\theta)$ has the FIR form in space: $A(\theta) = \sum_{k \in \{-N, \dots, N\}^d} A_k e^{j(k, \theta)}$, for some $N < \infty$. The extreme case happens when $N = 0$, i.e, $g(n) = A_0f(n)$, then A is said to be a *decentralized operator*. An operator for which no $N < \infty$ can be found is said to be a *centralized operator*.

From now on we will assume that the operators on the system (1) are localized for some order $N_p < \infty$, in other words, we assume that $A(\theta)$, $B_j(\theta)$, $C_i(\theta)$, $D_{ij}(\theta)$ have FIR dependency on the space of order N_p . This is a very natural assumption for open loop models obtained by discretizing spatially distributed dynamics; in fact typically $N = 1$, meaning the state space description of a particular discretized element is only directly influenced by its immediate neighbors. This issue is less obvious for the operators C_1 , D_{11} and D_{12} which embed our performance specifications, but we make this assumption as well which will simplify the following discussion.

We now study the characterization of controllers with respect to localization. Consider an arbitrary controller of the form (3). Such a controller requires input from $\psi_c(k, t)$ and $y(k, t)$ from several neighboring controllers and units of the plant. The operators $A_c(\theta)$ and $C_c(\theta)$ rule the form that the states ψ_c from neighboring locations enter the controller at a specific location. For example, consider the one dimensional case. If A_c or C_c are centralized operators, then the controller needs to have access to the state $\psi_c(k, t)$ from an infinite number of neighbors. Conversely, if $A_c(\theta)$ and $C_c(\theta)$ are localized (e.g. in the one dimensional case: $A_c(\theta) = \sum_{k=-N}^N A_k e^{jk\theta}$ and $C_c(\theta) = \sum_{k=-N}^N C_k e^{jk\theta}$), then the controller at a certain location only needs to have access to the state of the $2N$ closest neighbors.

The same holds for the operators acting on $y(k, t)$. Therefore requiring localization from a controller (3) amounts to require localization both on the transmission of the state $\psi_c(k, t)$ and output $y(k, t)$. We generalize this idea for the multi-dimensional case: the controller (3) is *localized controller (decentralized)* when $A_c(\theta)$, $B_c(\theta)$, $C_c(\theta)$ and $D_c(\theta)$ are localized (respectively, decentralized) operators. A *static* controller, $u(\theta, t) = F(\theta)y(\theta, t)$, is called localized (decentralized) if $F(\theta)$ is localized (respectively, decentralized).

In general, controllers found by Propositions 1 and 2 require information from an infinite number neighboring locations, i.e. are centralized.

In the next section, at the price of some conservatism, we will suggest a method to preserve convexity while including localization on the synthesis problems.

4 Convex Synthesis of Localized Control

Consider the following restriction on the synthesis: if in the LMIs of Proposition 1, the Lyapunov-like matrix $X(\theta)$ were chosen to be constant across θ , then imposing that $F(\theta)$ is constant amounts to requiring $Y(\theta)$ to be constant as well. In other words, solving those LMIs for constant X and Y provides a sufficient condition for the resulting feed-

back to be decentralized. This idea is reminiscent of the techniques proposed in [12] to treat multi-objective problems, the difference here is that the search for a constant Lyapunov matrix for all the systems is not essential to the solution but simply a tool to force decentralization in the controller.

To illustrate the idea, we look at feedback stabilization alone. Setting $X(\theta) = X_0 > 0$ and $Y(\theta) = Y_0$ in (5) results in $A(\theta)X_0 + B_2(\theta)Y_0 + X_0A(\theta)^* + Y_0^*B_2(\theta)^* < 0$, which replacing $F_0 = Y_0X_0^{-1}$ gives $[A(\theta) + B_2(\theta)F_0]X_0 + X_0[A(\theta)^* + F_0^*B_2(\theta)^*] < 0$. This is of course the Lyapunov inequality for the closed loop system. So we see that besides requiring a decentralized $F(\theta) = F_0$, the above condition is restricting the search to those controllers F_0 that admit a constant Lyapunov function $X(\theta) = X_0$. In the original space coordinates, this amounts to a Lyapunov function $V(\psi) = \sum_{k \in \mathbb{Z}^d} \psi(k)^* X_0 \psi(k)$, a “decentralized” Lyapunov function in the sense that it involves no cross-terms between state variables at different locations. The same observation applies to the Lyapunov or storage functions that arise in \mathcal{H}_2 or \mathcal{H}_∞ control.

So far, the conservatism of the method comes from the fact that not all decentralized controllers need to admit a decentralized Lyapunov function. Indeed, an admissible decentralized controller for any of the above problems could allow a general $X(\theta)$, $Y(\theta)$, provided the product $F_0 = Y(\theta)[X(\theta)]^{-1}$ is constant. Unfortunately, this cancellation feature is not a convex condition, which explains the reasoning behind our assumption.

Besides decentralized control, we would also like to treat the problem where we allow some degree of localization on the controller, e.g. we may want to impose that each unit can use sensor measurements from immediate neighbors, in addition to its own, but is not allowed access to any sensors which are further away. In a two-dimensional array we could seek localized feedback laws of the form $F(\theta_1, \theta_2) = F_0 + F_1^1 e^{j\theta_1} + F_{-1}^1 e^{-j\theta_1} + F_1^2 e^{j\theta_2} + F_{-1}^2 e^{-j\theta_2}$. As long as the restriction of constant X remains in place, this change is easily accommodated without losing convexity, by choosing $Y(\theta)$ of the same form $Y(\theta_1, \theta_2) = Y_0 + Y_1^1 e^{j\theta_1} + Y_{-1}^1 e^{-j\theta_1} + Y_1^2 e^{j\theta_2} + Y_{-1}^2 e^{-j\theta_2}$. Then the condition (5) is an LMI in the unknowns $X_0, Y_0, Y_1^1, Y_{-1}^1, Y_1^2, Y_{-1}^2$, at each frequency θ , and the resulting control law $F(\theta) = Y(\theta)X_0^{-1}$ will have the desired spatial structure.

The same idea applies to the state feedback \mathcal{H}_2 synthesis, as follows:

Proposition 4 (Localized State Feedback)

Assume that the plant has localization of order N_p . Given an integer $N < \infty$ and a $\gamma > 0$, if there exist matrices X_0, Y_k for $k \in \{-N, \dots, N\}^d$,

and Z_r for $r \in \{-N_z, \dots, N_z\}^d$, such that γ , $X(\theta) = X_0$, $Y(\theta) = \sum_{k \in \{-N, \dots, N\}^d} Y_k e^{j\langle k, \theta \rangle}$ and $Z(\theta) = \sum_{k \in \{-N_z, \dots, N_z\}^d} Z_k e^{j\langle k, \theta \rangle}$ satisfy inequalities (6), (7), and (8), then the feedback gain $F(\theta) = \sum_{k \in \{-N, \dots, N\}^d} Y_k e^{j\langle k, \theta \rangle} [X_0]^{-1}$ internally stabilizes the closed loop system (4), satisfies $\|G_{zw}\|_{\mathcal{H}_2}^2 < \gamma$, with localization of order N .

We note that Proposition 4 follows directly by application of Proposition 1. By inspection, the degree of decentralization of the resulting controller is found to be N .

We remark that Proposition 4 has the extra assumption that $Z(\theta)$ is a localized operator of order N_z . This assumption adds some conservativeness to the method, but the choice of large values of N_z can reduce the conservativeness added on this step.

The extension to output feedback is more elaborate, but the basic idea is the same. It can be shown that \mathcal{X} plays a role similar to a Lyapunov matrix in Proposition 2, then our strategy becomes forcing \mathcal{X} to be constant with respect to θ . This is done without losing convexity by forcing $X(\theta) = X_0$ and $Y(\theta) = Y_0$ both to be constant matrices.

Proposition 5 (Output Feedback) *Assume the plant has localization of order N_p . Given an integer N and a $\gamma > 0$, if there exist $X_0, Y_0, K_k, L_k, M_k, k \in \{-N, \dots, N\}^p$, and $Z_r, r \in \{-N_z, \dots, N_z\}^p$, such that $X(\theta) = X_0, Y(\theta) = Y_0, K(\theta) = \sum_{k \in \{-N, \dots, N\}^d} K_k e^{j\langle k, \theta \rangle}$, $L(\theta) = \sum_{k \in \{-N, \dots, N\}^d} L_k e^{j\langle k, \theta \rangle}$, $M(\theta) = \sum_{k \in \{-N, \dots, N\}^d} M_k e^{j\langle k, \theta \rangle}$, and $Z(\theta) = \sum_{k \in \{-N_z, \dots, N_z\}^d} Z_k e^{j\langle k, \theta \rangle}$ satisfy the conditions on Proposition 2, then there exists a controller with localization of order $N_p + N$ that renders $\|G_{zw}\|_{\mathcal{H}_2} < \gamma$. Moreover, there exist nonsingular U_0, V_0 with $I - X_0 Y_0 = U_0 V_0^T$. A suitable controller is given by the formula (11).*

Similarly to the state feedback case, we have added the conservative restriction on the localization of $Z(\theta)$. It is easy to see from (11) that Proposition 5 produce controllers of localization not greater than $N_p + N$.

In computational terms, the restrictions in the variables narrow down the variable spaces to finite dimensions. However we have a coupled family of LMIs. If we grid the frequency θ , this gives rise to a large problem, which is a costly approach. This coupling can be overcome exactly for the one dimensional case, as we see in the next section.

5 Finite Dimensional Conditions for Localized Synthesis

In this section we use the ideas in Section 4 to obtain finite dimensional conditions for localized synthesis. For simplicity, the presentation is carried

out with a one dimensional ($k \in \mathbb{Z}$) state feedback control.

Assume we are to synthesize a state feedback control law $F = F_0$ with no spatial dependence with respect to \mathcal{H}_2 performance. Using the assumption of X constant, this amounts to solving the required LMIs by searching for a solution pair $X = X_0$ and $Y = Y_0$ with no dependency on the spatial frequency θ . Then, $F_0 = Y_0 X_0^{-1}$ is our decentralized control law.

Assume the plant is given by (2) with

$$A(\theta) = A_{-1} e^{-j\theta} + A_0 + A_1 e^{j\theta}, \quad (13)$$

and arbitrary constant matrices B_1, B_2, C_1 and D_{12} . Now we look at Proposition 1 under the current assumptions and use the KYP Lemma to eliminate the frequency dependency of the \mathcal{H}_2 synthesis conditions.

For this particular case we can take $N_z = 0$, without adding conservativeness to the method. To note that this extra assumption does not add conservativeness in this case, assume that a solution to Proposition 1 is found such that $X(\theta) = X_0$ and $Y(\theta) = Y_0$. Then substituting the expressions for X and Y in inequality (7), if we perform a Schur complement on the left-hand-side of (7), we arrive at: $(W)^* X_0^{-1} W < Z(\theta)$, where $W = C_1 X_0 + D_{12} Y_0$. Since W and X are a constant matrices; $Z(\theta)$ can be taken to be a constant $Z = Z_0$ without loss of generality. Then (7) is automatically freed of any frequency dependency. Condition (8) can be replaced by (12). We are left with the task of reducing (6) to a finite dimensional condition.

In view of (13) and the spatial structure assumed for X and Y , the inequality (6) becomes:

$$\begin{aligned} & [A_{-1} e^{-j\theta} + A_0 + A_1 e^{j\theta} \ B_2] \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} + \\ & \begin{bmatrix} X_0 & Y_0^* \end{bmatrix} \begin{bmatrix} (A_{-1} e^{-j\theta} + A_0 + A_1 e^{j\theta})^* \\ B_2^* \end{bmatrix} + B_1 B_1^* < 0. \end{aligned} \quad (14)$$

If we define: $\mathcal{A} = 0, \mathcal{B} = I, \mathcal{M} = \begin{bmatrix} 0 & \mathcal{M}_{21} \\ \mathcal{M}_{21}^* & \mathcal{M}_{22} \end{bmatrix}$, with $\mathcal{M}_{21} = A_1 X_0 + X_0 A_{-1}^*$ and $\mathcal{M}_{22} = [A_0 \ B_2] \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} + [X_0 \ Y_0^*] \begin{bmatrix} A_0 \\ B_2^* \end{bmatrix} + B_1 B_1^*$, then it is possible to write the inequality (14) in the form (i) of the KYP Lemma. We then apply the KYP Lemma to get an equivalent finite dimensional condition of the form (ii). With this result, we re-state Proposition 1 for the particular case studied here.

Proposition 6 (\mathcal{H}_2 Decentralized Synthesis) *Let the plant (2) be given by (13) and B_1, B_2, C_1 and D_{12} constant. Given a $\gamma > 0$, if there exist $X_0 = X_0^*, Z_0 = Z_0^*, P = P^*$ and Y_0 satisfying the*

conditions $\text{Tr}Z_0 < \gamma$, $\begin{bmatrix} -P & \mathcal{M}_{21}^* \\ \mathcal{M}_{21} & \mathcal{M}_{22} + P \end{bmatrix} < 0$, and $\begin{bmatrix} X_0 & (C_1 X_0 + D_{12} Y_0)^* \\ (C_1 X_0 + D_{12} Y_0) & Z_0 \end{bmatrix} > 0$, then there exists a constant feedback gain $F(\theta) = Y_0 X_0^{-1}$ that internally stabilizes the closed loop system (4) and satisfies $\|G_{zw}\|_{\mathcal{H}_2}^2 < \gamma$.

Whenever the underlying spatial coordinate is one dimensional, i.e $k \in \mathbb{Z}$, the same idea can be used to find finite dimensional conditions for the \mathcal{H}_2 synthesis of localized controllers, namely simply make use of Propositions 4 and 5 and apply the KYP Lemma on the resulting inequalities to reduce them to finite dimensional conditions.

For multi-dimensional cases $k \in \mathbb{Z}^d$, it is possible to extend the KYP lemma using a descriptor form. Although that extension is only a sufficient condition, we can use it the on the respective LMIs of Propositions 4 and 5 to generate finite dimensional conditions.

6 An Example

We illustrate the method with the problem of temperature regulation along an infinite rod. The space-discretized model for the dynamics of the temperature on the rod is given by $\frac{dT(k,t)}{dt} = C[T(k+1,t) - 2T(k,t) + T(k-1,t)] + w(k,t) + u(k,t)$, where C is the conductivity constant, $T(k,t)$ is the temperature, $u(k,t)$ the control and $w(k,t)$ is a disturbance. The position is $k \in \mathbb{Z}$. The spatial Fourier transform of the system is $\frac{dT(\theta,t)}{dt} = C(e^{j\theta} - 2 + e^{-j\theta})T(\theta,t) + w(\theta,t) + u(\theta,t)$.

The \mathcal{H}_2 performance signal is $z(k,t)^* := [qT(k,t) u(k,t)]^*$, where $q \in \mathbb{R}$ is a constant. For $C = 10$, we solved the following problems: **Truncated Centralized Optimal Solution**: found by truncating the optimal control law F (see [2]) to a decentralized law $u(k,t) = F_0 T(k,t)$; **Decentralized Synthesis Solution**: Found using Proposition 6; **Optimal Decentralized Solution**: For this simple example, the optimal decentralized controller can be found by searching over the space of all pure gain controllers. The plot in Figure 2 shows how the norms achieved by: the gain found by truncation (T), the decentralized synthesis gain (DS), and the optimal decentralized gain (OD) compare for $q = 1$. The controller found through Proposition 6 has shown to be a better option in this case than the controller obtained by simply truncating the optimal centralized control law.

7 Concluding Remarks

We have presented a method to synthesize localized and decentralized controllers for systems with spatial invariance. The basic assumption is that the closed loop system admits a decentralized Lyapunov function; this automatically gives convex, though infinite dimensional, synthesis. A reduction

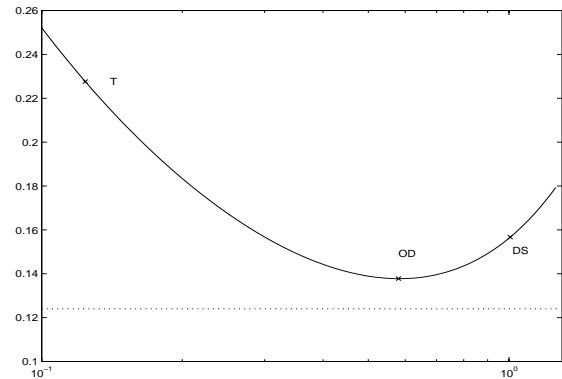


Figure 2: Closed Loop \mathcal{H}_2 Norm for $q = 1$ versus controller gain

to finite dimensional LMIs was given in terms of the KYP Lemma, which is exact for one dimensional arrays, but conservative in higher dimensions.

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